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## Nonclassical effects in two coupled oscillators at non resonant region

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## PAPER

## Nonclassical effects in two coupled oscillators at non resonant region

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E-mail: [arjun4physics@gmail.com](mailto:arjun4physics@gmail.com)**Keywords:** coupled oscillator, squeezed states, sub-poissonian statistics, intermodal entanglement

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**Abstract**

We consider two coupled quantum harmonic oscillators with different free frequencies. Here the interaction between the two modes does not involve the Rotating Wave Approximation (RWA). The Heisenberg equations of motion are solved analytically using an approximate technique and the solutions are used to measure nonclassicalities associated with the system. The nonclassicality criteria chosen in this study are experimentally measurable. The analytical solutions are matched with numerical simulations with the help of a numerical toolbox. It is evident from the time developments of the operators that nonclassicalities, namely squeezing and quantum entanglement are present in the system. The counting statistics for the oscillator with lower free energy and the coupled mode are shown to be sub-Poissonian for a particular set of parameters. Consequently, the system may exhibit quantum mechanical antibunching.

**1. Introduction**

The systems of coupled harmonic oscillators have been drawing attentions for quite sometimes [1–9]. These systems raise considerable discussions due to their applications in quantum physics and quantum field theory [10–13]. Different biophysical processes, in particular, photosynthesis, can be explained with the help of these coupled oscillators [14, 15]. Two linearly coupled harmonic oscillators constitute a quantum system comprising of classical mass-spring components which can be solved exactly [16]. They provide efficient models to explore the fields like quantum measurements, quantum entanglement etc [17, 18].

The ubiquitous nature of nonclassical states is apparent in the rapid developments of the fields like quantum computation and communication. In particular, nonclassical states are essential in the studies of quantum cryptography [19, 20], dense coding [21, 22], quantum teleportation [23, 24]. In this paper, the nonclassical properties, namely squeezing, antibunching and quantum entanglement are studied in case of two coupled harmonic oscillators. For a nonclassical state, the Glauber-Sudarshan  $p$ -function is more singular than  $\delta$ -function. Although a single photon state is the most nonclassical of the quantum states, it may involve a number of photons [25]. The negativity of Wigner function, Mandel's  $Q$  parameter, etc., serve as the criteria for the measurement of nonclassicality. The present study involves some practically measurable nonclassicality criteria.

For light field and harmonic oscillator, theoretical studies on squeezed states are widespread [26–28]. They are accompanied by experimental works [29, 30]. Squeezed states are relevant for the preparation of Fock states of harmonic oscillator strongly coupled to a single two level atom [31]. These states attract wide interests and fundamental connections between the squeezed state and entanglement are studied [8, 32–35]. For producing antibunched photons, the single photon sources remain the essential component. Or reversibly, presence of antibunching confirms the single photon emitter in the system. The process like parametric down conversion are used to generate antibunched photons [36–38]. Quantum entanglement is the one of the

important-most key factors for the quantum information processing, information splitting or teleportation. Several inseparability criteria are proposed to examine whether a mixed state with two or more modes is separated or entangled [39, 40].

In this study the Heisenberg equations of motion is first derived from the model Hamiltonian corresponding to the system of two coupled oscillators. The approximate solutions of these equations are presented up to cubic order on the interaction constant using an approximate perturbative approach. This approximation technique is more general than short time approximation technique. Next the general dynamics is given in terms of mean particle numbers. The general solutions are then applied to study nonclassicalities using several criteria which have so far not been investigated collectively for this system. The expressions corresponding to squeezing, particle counting statistics and quantum entanglement are plotted. The nonclassicality criteria for each mode along with the coupled mode are investigated. The numerical simulations are also shown for the verification of the approximate analytical expressions.

## 2. The model Hamiltonian

For the system of N harmonic oscillators coupled by bilinear interaction, the Hamiltonian is given as [16]

$$H = \hbar \sum_{p,q} \omega_{pq} a_p^\dagger a_q + \frac{1}{2} \lambda_{pq} a_p^\dagger a_q^\dagger + \frac{1}{2} \lambda_{pq}^* a_p a_q \quad (1)$$

where  $a_p$  ( $a_p^\dagger$ ) is the annihilation (creation) operator corresponding to the  $p$  th oscillator and  $\omega_{pq}$  and  $\lambda_{pq}$  are the coupling constants. To estimate quantity such as mean particle number, one may resort to the approximation where the non energy conserving terms like  $a_p^\dagger a_q^\dagger$  or  $a_p a_q$  is neglected at resonance. This approximation is called resonance approximation or RWA. The present study considers the system of two oscillators connected with position-position coupling at region far from resonance without RWA. As the matter of fact, for the oscillators of unit mass, the Hamiltonian can be expressed as

$$H = \frac{p_1^2}{2} + \frac{1}{2} \omega_1^2 x_1^2 + \frac{p_2^2}{2} + \frac{1}{2} \omega_2^2 x_2^2 + 2\lambda \sqrt{\omega_1 \omega_2} x_1 x_2 \quad (2)$$

where  $\omega_1$  and  $\omega_2$  are the natural frequencies of the first and the second oscillator respectively. Here without any loss of generality, the energy of the first one is taken greater than the energy of the second one, i.e.,  $\omega_1 > \omega_2$ . Expressing the transformations of position ( $x$ ) and momentum ( $p$ ) in terms of dimensionless bosonic creation and annihilation operators as  $x_1 = \sqrt{\frac{\hbar}{2\omega_1}} (a^\dagger + a)$ ,  $p_1 = i\sqrt{\frac{\hbar\omega_1}{2}} (a^\dagger - a)$  and  $x_2 = \sqrt{\frac{\hbar}{2\omega_2}} (b^\dagger + b)$ ,  $p_2 = i\sqrt{\frac{\hbar\omega_2}{2}} (b^\dagger - b)$ , (2) can be rewritten as

$$H = \hbar\omega_1 \left( a^\dagger a + \frac{1}{2} \right) + \hbar\omega_2 \left( b^\dagger b + \frac{1}{2} \right) + \hbar\lambda (a^\dagger + a)(b^\dagger + b) \quad (3)$$

Neglecting the vacuum energy terms and taking  $\hbar = 1$  from here through the rest of the study, the Hamiltonian now becomes

$$H = \omega_1 a^\dagger a + \omega_2 b^\dagger b + \lambda (a^\dagger + a)(b^\dagger + b) \quad (4)$$

## 3. Solution

The Heisenberg equations of motion can be derived from the Hamiltonian (4)

$$\begin{aligned} \dot{a}(t) &= -i[\omega_1 a + \lambda(b^\dagger + b)] \\ \dot{b}(t) &= -i[\omega_2 b + \lambda(a^\dagger + a)] \end{aligned} \quad (5)$$

The above Heisenberg equations of motion were solved at resonant condition [16] and comparisons of dynamical behaviours of various observables have been made between the solutions with or without RWA. The disparity between two results increases with the dimensionless time as expected. In this

study the general solutions of equation (5) are achieved from an approximate perturbative approach resembling that in the references [41, 42]. The solution of the first of equation (5) may be written as

$$a(t) = \exp(iHt)a \exp(-iHt) = a + it[H, a] + \frac{(it)^2}{2!}[H, [H, a]] + \dots \quad (6)$$

and  $b(t)$  follows in the same manner as  $a(t)$ . The commutators in equation (6) are evaluated and the terms comprising of  $a, a^\dagger, b, b^\dagger$  or any combination of them are taken up to the desired order of the coupling constant along with their time dependent coefficients. The trial solutions of equation (6) assumes the following form taking up to  $\mathcal{O}(\lambda^3)$  terms

$$a(t) = f_1(t)a + f_2(t)b + f_3(t)b^\dagger + f_4(t)a + f_5(t)a^\dagger + f_6(t)b + f_7(t)b^\dagger \quad (7)$$

and

$$b(t) = h_1(t)b + h_2(t)a + h_3(t)a^\dagger + h_4(t)b + h_5(t)b^\dagger + h_6(t)a + h_7(t)a^\dagger \quad (8)$$

where  $f_1(h_1)$  is the term free of the interaction constant  $\lambda$ ,  $f_2(h_2)$  and  $f_3(h_3)$  are  $\mathcal{O}(\lambda)$  functions,  $f_4(h_4)$  and  $f_5(h_5)$  are the functions of  $\lambda^2$  whereas  $f_6(h_6)$  and  $f_7(h_7)$  depend upon  $\lambda^3$ . This is evident from the sequence of the terms in equation (7) that next two terms would contain only  $a$  and  $a^\dagger$  but are not considered since the coefficients would carry  $\lambda^4$ . The same reasoning goes with equation (8) also. The initial conditions for the functions  $f_i$ s and  $h_i$ s are  $f_1(0) = h_1(0) = 1$  whereas  $f_i(0) = h_i(0) = 0$  for  $i = 2, 3, 4, 5, 6, 7$  and the reason behind this choice is that if there is no coupling constant ( $\lambda = 0$ ), all the terms other than the first one would vanish. This is equally evident from equation (6) also. The corresponding solutions for  $f_i$ s and  $h_i$ s are

$$\begin{aligned} f_1 &= e^{-i\omega_1 t} \\ f_2 &= \frac{\lambda}{\Delta} f_1 \{1 - \alpha(t)\} \\ f_3 &= \frac{\lambda}{\Sigma} f_1 \{1 - \beta(t)\} \\ f_4 &= \lambda^2 f_1 \left[ \frac{1}{\Delta^2} \{ \alpha(t) - i\Delta t - 1 \} - \frac{1}{\Sigma^2} \{ \beta(t) - i\Sigma t - 1 \} \right] \\ f_5 &= \lambda^2 f_1 \left[ \frac{1}{\Sigma} \left\{ \frac{1}{2\omega_1} (e^{2i\omega_1 t} - 1) - \frac{\alpha(t)}{\Delta} \right\} \right. \\ &\quad \left. + \frac{1}{\Delta} \left\{ \frac{\beta(t)}{\Sigma} + \frac{1}{2\omega_1} (1 - e^{2i\omega_1 t}) \right\} \right] \\ f_6 &= \lambda^3 f_1 \left[ \frac{1}{\Delta^2} \left\{ \frac{2}{\Delta} (\alpha(t) - 1) - it(\alpha(t) + 1) \right\} \right. \\ &\quad \left. + \frac{1}{\Sigma^2} \left\{ \frac{1}{2\omega_1} (e^{2i\omega_1 t} - 1) - \frac{\alpha(t)}{\Delta} \left( \frac{\Sigma}{\Delta} - i\Sigma t - 1 \right) - \frac{\Sigma}{\Delta^2} + \frac{1}{\Delta} \right\} \right. \\ &\quad \left. - \frac{1}{\Sigma} \left\{ \frac{1}{2\Delta\omega_1} (e^{2i\omega_1 t} - 1) + \frac{1}{2\omega_2} \left( \frac{\alpha(t)}{\Delta} - \frac{\beta(t)}{\Sigma} + \frac{1}{\Sigma} \right) \right\} \right. \\ &\quad \left. - \frac{1}{\Delta} \left\{ \frac{1}{2\omega_2} \left( \frac{\alpha(t)}{\Delta} - \frac{\beta(t)}{\Sigma} - \frac{1}{\Delta} \right) - \frac{it}{\Sigma} \right\} \right] \\ f_7 &= \lambda^3 f_1 \left[ \frac{1}{\Delta^2} \left\{ \frac{1}{2\omega_1} (1 - e^{2i\omega_1 t}) + \frac{\beta(t)}{\Sigma} \left( i\Delta t - \frac{\Delta}{\Sigma} + 1 \right) + \frac{\Delta}{\Sigma^2} - \frac{1}{\Sigma} \right\} \right. \\ &\quad \left. + \frac{1}{\Sigma^2} \left\{ \frac{2}{\Sigma} (1 - \beta(t)) + it(1 + \beta(t)) \right\} \right. \\ &\quad \left. - \frac{1}{\Sigma} \left\{ \frac{1}{2\omega_2} \left( \frac{\beta(t)}{\Sigma} - \frac{\alpha(t)}{\Delta} - \frac{1}{\Sigma} \right) + \frac{it}{\Delta} \right\} \right. \\ &\quad \left. - \frac{1}{\Delta} \left\{ \frac{1}{2\Sigma\omega_1} (1 - e^{2i\omega_1 t}) + \frac{1}{2\omega_2} \left( \frac{\beta(t)}{\Sigma} - \frac{\alpha(t)}{\Delta} + \frac{1}{\Delta} \right) \right\} \right] \quad (9) \end{aligned}$$

and

$$\begin{aligned}
h_1 &= e^{-i\omega_2 t} \\
h_2 &= -\frac{\lambda}{\Delta} h_1 \{1 - \alpha^*(t)\} \\
h_3 &= \frac{\lambda}{\Sigma} h_1 \{1 - \beta(t)\} \\
h_4 &= \lambda^2 h_1 \left[ \frac{1}{\Delta^2} \{ \alpha^*(t) + i\Delta t - 1 \} - \frac{1}{\Sigma^2} \{ \beta(t) - i\Sigma t - 1 \} \right] \\
h_5 &= \lambda^2 h_1 \left[ \frac{1}{\Sigma} \left\{ \frac{\alpha^*(t)}{\Delta} + \frac{e^{2i\omega_2 t}}{2\omega_2} - \frac{1}{2\omega_2} \right\} \right. \\
&\quad \left. - \frac{1}{\Delta} \left\{ \frac{\beta(t)}{\Sigma} - \frac{e^{2i\omega_2 t}}{2\omega_2} + \frac{1}{2\omega_2} \right\} \right] \\
h_6 &= \lambda^3 h_1 \left[ \frac{1}{\Delta^2} \left\{ \frac{2}{\Delta} (1 - \alpha^*(t)) - it(1 + \alpha^*(t)) \right\} \right. \\
&\quad + \frac{1}{\Sigma^2} \left\{ \frac{1}{2\omega_2} (e^{2i\omega_2 t} - 1) + \frac{\alpha^*(t)}{\Delta} \left( \frac{\Sigma}{\Delta} + i\Sigma t + 1 \right) - \frac{\Sigma}{\Delta^2} - \frac{1}{\Delta} \right\} \\
&\quad - \frac{1}{\Sigma} \left\{ \frac{1}{2\Delta\omega_2} (1 - e^{2i\omega_2 t}) - \frac{1}{2\omega_1} \left( \frac{\alpha^*(t)}{\Delta} + \frac{\beta(t)}{\Sigma} - \frac{1}{\Sigma} \right) \right\} \\
&\quad \left. - \frac{1}{\Delta} \left\{ \frac{1}{2\omega_1} \left( \frac{\alpha^*(t)}{\Delta} + \frac{\beta(t)}{\Sigma} - \frac{1}{\Delta} \right) + \frac{it}{\Sigma} \right\} \right] \\
h_7 &= \lambda^3 h_1 \left[ \frac{1}{\Delta^2} \left\{ \frac{1}{2\omega_2} (1 - e^{2i\omega_2 t}) + \frac{\beta(t)}{\Sigma} \left( \frac{\Delta}{\Sigma} + 1 - i\Delta t \right) - \frac{\Delta}{\Sigma^2} - \frac{1}{\Sigma} \right\} \right. \\
&\quad + \frac{1}{\Sigma^2} \left\{ \frac{2}{\Sigma} (1 - \beta(t)) + it(1 + \beta(t)) \right\} \\
&\quad - \frac{1}{\Sigma} \left\{ \frac{1}{2\omega_1} \left( \frac{\beta(t)}{\Sigma} + \frac{\alpha^*(t)}{\Delta} - \frac{1}{\Sigma} \right) - \frac{it}{\Delta} \right\} \\
&\quad \left. - \frac{1}{\Delta} \left\{ \frac{1}{2\Sigma\omega_2} (e^{2i\omega_2 t} - 1) + \frac{1}{2\omega_1} \left( \frac{1}{\Delta} - \frac{\beta(t)}{\Sigma} - \frac{\alpha^*(t)}{\Delta} \right) \right\} \right] \tag{10}
\end{aligned}$$

where  $\alpha(t) = \exp(i\Delta t)$ ,  $\beta(t) = \exp(i\Sigma t)$ ,  $\Delta = \omega_1 - \omega_2$  and  $\Sigma = \omega_1 + \omega_2$ . The validity of the solution is given by the limit  $\lambda t < 1$ . Therefore the dynamics of various observables are explored throughout the study with  $\lambda \ll 1$ . This is worth noting that the trial solutions (7) and (8) satisfy the equal time commutation relations  $[a(t), a^\dagger(t)] = 1$  and  $[b(t), b^\dagger(t)] = 1$ .

#### 4. Particle number dynamics

The initial states of both the oscillator modes are considered coherent. If  $|\alpha\rangle$  and  $|\beta\rangle$  are the eigenkets of the field operators  $a$  and  $b$  respectively, the eigenvalue equations are  $a|\alpha\rangle = \alpha|\alpha\rangle$  and  $b|\beta\rangle = \beta|\beta\rangle$ . The initial composite coherent state is given as

$$|\psi(0)\rangle = |\alpha\rangle \otimes |\beta\rangle \tag{11}$$

which gives rise to the following eigenvalue equations for composite system at  $t = 0$

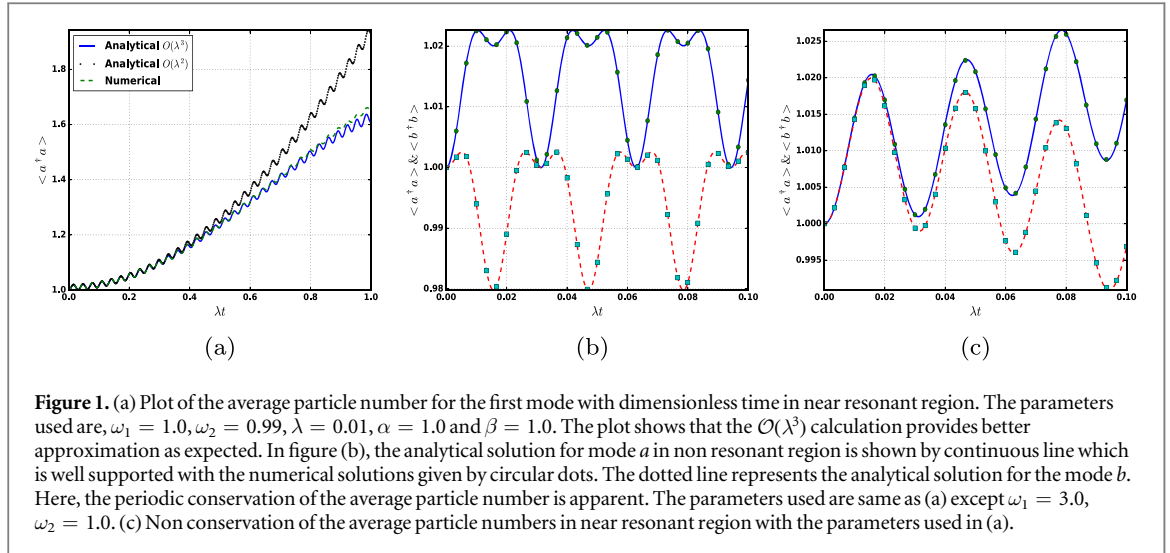
$$a|\psi(0)\rangle = \alpha|\alpha\rangle \otimes |\beta\rangle \tag{12}$$

$$b|\psi(0)\rangle = \beta|\alpha\rangle \otimes |\beta\rangle \tag{13}$$

The quantities  $|\alpha|^2$  and  $|\beta|^2$  are the average number of atoms for mode state  $|\alpha\rangle$  and  $|\beta\rangle$  respectively. The more general expression for  $\alpha$  and  $\beta$  can be given by

$$\alpha = |\alpha| e^{i\theta} \text{ and } \beta = |\beta| e^{i\phi} \tag{14}$$

where  $\theta$  and  $\phi$  are the phases of  $\alpha$  and  $\beta$  respectively.



We first consider the mean particle number for the first oscillator mode. The average particle number for the first one is given by

$$\begin{aligned}
 \langle a^\dagger a \rangle &= |\alpha|^2 + |f_2|^2 |\beta|^2 + |f_3|^2 [|\beta|^2 + 1] \\
 &+ [(f_1 f_2^* + f_1 f_6^* + f_3 f_5^* + f_2^* f_4) \alpha \beta^* \\
 &+ (f_1 f_3^* + f_1 f_7^* + f_2 f_5^* + f_3^* f_4) \alpha \beta \\
 &+ f_1 f_4^* |\alpha|^2 + f_1 f_5^* \alpha^2 + f_2 f_3^* \beta^2 + c.c.]
 \end{aligned}
 \tag{15}$$

The  $\mathcal{O}(\lambda^2)$  calculation for the above mean particle number is

$$\begin{aligned}
 \langle a^\dagger a \rangle &= |\alpha|^2 + |f_2|^2 |\beta|^2 + |f_3|^2 [|\beta|^2 + 1] \\
 &+ [f_1 f_2^* \alpha \beta^* + f_1 f_3^* \alpha \beta + f_1 f_4^* |\alpha|^2 \\
 &+ f_1 f_5^* \alpha^2 + f_2 f_3^* \beta^2 + c.c.]
 \end{aligned}
 \tag{16}$$

The similar expression for mode  $b$  is as follows

$$\begin{aligned}
 \langle b^\dagger b \rangle &= |\beta|^2 + |h_2|^2 |\alpha|^2 + |h_3|^2 [|\alpha|^2 + 1] \\
 &+ [h_1 h_2^* \alpha^* \beta + h_1 h_3^* \alpha \beta + h_1 h_4^* |\beta|^2 \\
 &+ h_1 h_5^* \beta^2 + h_2 h_3^* \alpha^2 + c.c.]
 \end{aligned}
 \tag{17}$$

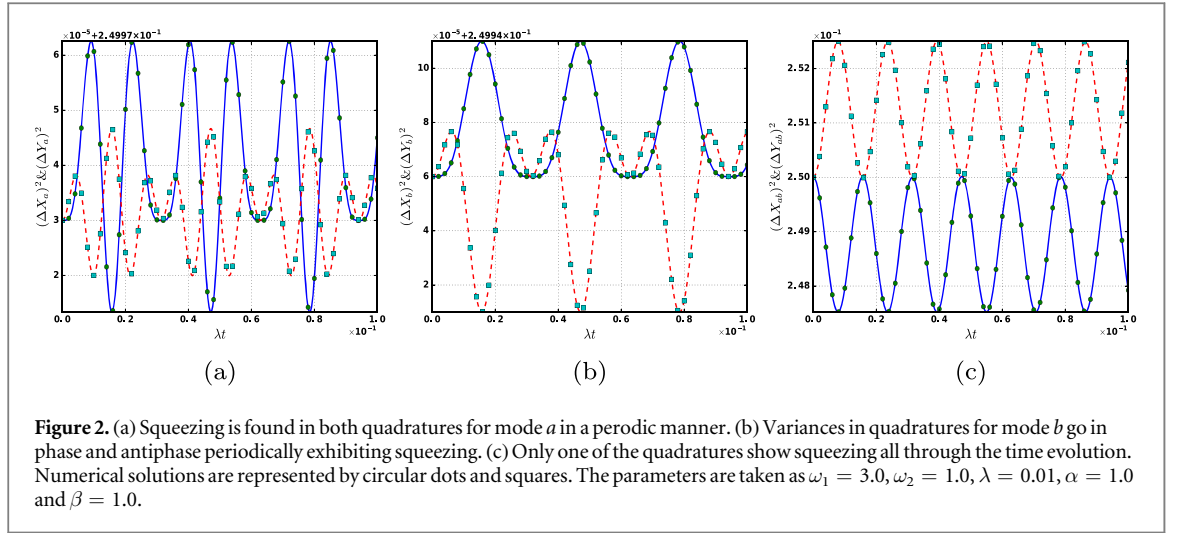
The solution (15) is a better approximation as expected. This is evident from figure 1(a). But both the  $\mathcal{O}(\lambda^3)$  and the  $\mathcal{O}(\lambda^2)$  solutions agree well with the numerical simulation in the region of dimensionless time  $\lambda t < 0.1$ . The simulations are done by the quantum optical toolbox QuTip 3.1.0 [43]. To investigate the unitary time evolution of closed system (4), the toolbox function qutip.essolve is used. Here the exponential-series technique is considered for the time evolution of the initial state of the system. Figure 1(c) shows that the average particle number is not conserved throughout the time evolution.

### 5. Nonclassicalities

As a matter of fact, all the states of light are nonclassical or quantum states. Among all, the single photon state is the most nonclassical one. For coherent states, the Glauber -Sudarshan  $P$  function is a  $\delta$  function, and for all other pure nonclassical states,  $P$  function becomes negative for some regions in phase space [44]. Among different nonclassical states, this study considers quadrature squeezing, sub-Poissonian quantum statistics with antibunching and quantum entanglement. Various experimentally realizable criteria for different types of nonclassicality are used in the present study as described below.

#### 5.1. Quadrature squeezing

The non vanishing commutator  $[a, a^\dagger] = 1$  is responsible for the vacuum fluctuation for the electromagnetic field. The fluctuation over one quadrature can be lower than the vacuum fluctuation at the cost of the other one. This phenomenon is known as squeezing. The quadrature defined for squeezing is as follows



$$X_a(t) = \frac{1}{2}[a(t) + a^\dagger(t)] \quad (18)$$

$$Y_a(t) = \frac{1}{2i}[a(t) - a^\dagger(t)] \quad (19)$$

Squeezing occurs if fluctuation in any of the quadrature in equations (18) and (19) assumes the value lower than the vacuum fluctuation, i.e.,  $(\Delta X_a)^2 < \frac{1}{4}$  or  $(\Delta Y_a)^2 < \frac{1}{4}$ . Using (7), (18) and (19), the simplified formula for the quadrature fluctuations in the first mode is given by

$$\begin{bmatrix} (\Delta X_a)^2 \\ (\Delta Y_a)^2 \end{bmatrix} = \frac{1}{4}[|f_2|^2 + |f_3|^2 + \{f_1 f_4^* \pm (f_1 f_5 + f_2 f_3) + c.c.\}] \quad (20)$$

where the upper '+' sign is for  $(\Delta X_a)^2$  and lower '-' is for  $(\Delta Y_a)^2$ . The complex conjugate terms are abbreviated by c.c.

Similarly for the second mode

$$\begin{bmatrix} (\Delta X_b)^2 \\ (\Delta Y_b)^2 \end{bmatrix} = \frac{1}{4}[|h_2|^2 + |h_3|^2 + \{h_1 h_4^* \pm (h_1 h_5 + h_2 h_3) + c.c.\}] \quad (21)$$

The quadrature for the coupled mode squeezing are written as

$$X_{ab}(t) = \frac{1}{2\sqrt{2}}[a(t) + a^\dagger(t) + b(t) + b^\dagger(t)] \quad (22)$$

$$Y_{ab}(t) = \frac{1}{2\sqrt{2}i}[a(t) - a^\dagger(t) + b(t) - b^\dagger(t)] \quad (23)$$

The coupled mode squeezing is given by the following expression

$$\begin{bmatrix} (\Delta X_{ab})^2 \\ (\Delta Y_{ab})^2 \end{bmatrix} = \frac{1}{8}[2(1 + |f_2|^2 + |f_3|^2) + f_1 f_4^* + f_1 h_2^* + f_2 h_1^* \pm h_1 h_4^* + f_1 f_5 + f_2 f_3 + 2f_1 h_3 + h_1 h_5 + h_2 h_3]. \quad (24)$$

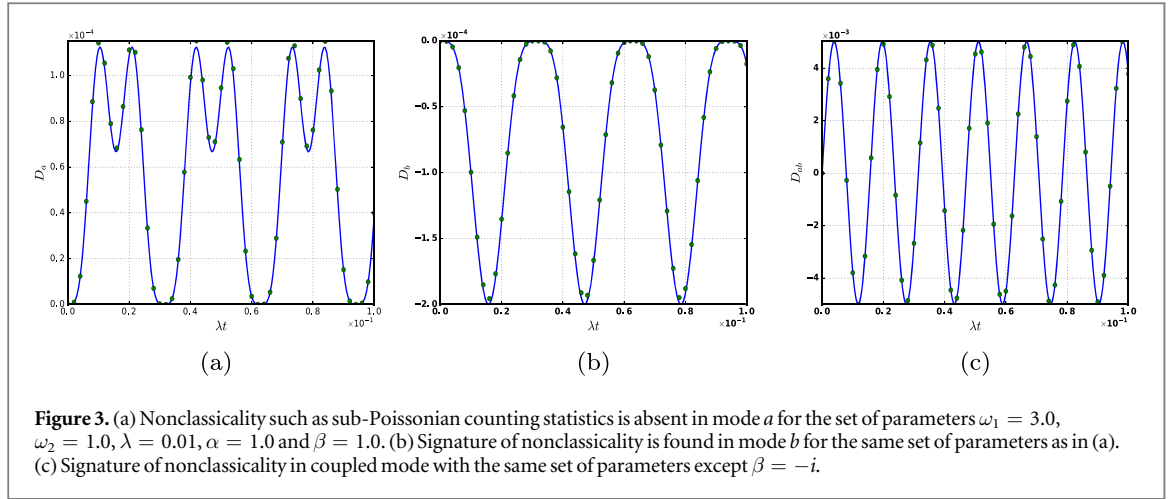
The Time evolution of quadrature variances are shown in figure 2.

## 5.2. Quantum statistics

The second order correlation function for the first mode corresponding to no time delay is given by

$$g^{(2)}(0) = \frac{\langle a^{\dagger 2}(t) a^2(t) \rangle}{\langle a^\dagger(t) a(t) \rangle \langle a^\dagger(t) a(t) \rangle}. \quad (25)$$

The condition  $0 < g^{(2)}(0) < 1$  implies the sub-Poissonian distribution of oscillator-mode number and is associated with the nonclassical effect, quantum antibunching [25, 45, 46]. As is well-known, the distribution



becomes super-Poissonian if  $g^{(2)}(0) = 1$  and particles are in coherent state if  $g^{(2)}(0) = 0$ . The equation (25) is rewritten as

$$g^{(2)}(0) = 1 + \frac{\langle(\Delta N_a)^2\rangle - \langle N_a\rangle}{\langle N_a\rangle^2}. \tag{26}$$

Writing the numerator that appeared in R.H.S of equation (26) as  $D_a$ , the condition of the mode  $a$  exhibiting sub-Poissonian distribution which may accompany antibunching, becomes

$$D_a = \langle(\Delta N_a)^2\rangle - \langle N_a\rangle < 0 \tag{27}$$

Similarly, the condition for  $b$  mode is as follows,

$$D_b = \langle(\Delta N_b)^2\rangle - \langle N_b\rangle < 0 \tag{28}$$

From equations (7), (8) and (27), the expression for  $D_a$  is,

$$D_a = (|f_2|^2 + |f_3|^2)|\alpha|^2 + \{f_1^* f_4 |\alpha|^2 + (f_1 f_5^* + f_1^2 f_2^* f_3^*)\alpha^2 + c.c.\} \tag{29}$$

and for the mode  $b$ ,

$$D_b = (|h_2|^2 + |h_3|^2)|\beta|^2 + \{h_1^* h_4 |\beta|^2 + (h_1 h_5^* + h_1^2 h_2^* h_3^*)\beta^2 + c.c.\} \tag{30}$$

The coupled mode antibunching is given by

$$D_{ab} = |f_3|^2(|\beta|^2 + 1) - |h_3|^2 + \{f_1 f_3^* (h_1^* h_2 \alpha^2 + h_1^* h_3 |\alpha|^2) + f_2 f_3^* \beta^2 + c.c.\} \tag{31}$$

Figure 3 exhibits the presence of antibunching in some modes. The nonclassical effect is present in  $b$  mode, whereas it is absent in  $a$  mode for the same set of parameters. In figure 3(c), the coupled mode shows antibunching for different phase of the eigenvalue  $\beta$ .

### 5.3. Entanglement

Intermodal entanglement is investigated in this study by the criteria conceived by Hillary and Zubairy [47–49]. The first sufficient criterion Hillary-Zubairy criterion 1 (HZ-1) is given as

$$\langle a^\dagger a b^\dagger b \rangle - |\langle a b^\dagger \rangle|^2 < 0 \tag{32}$$

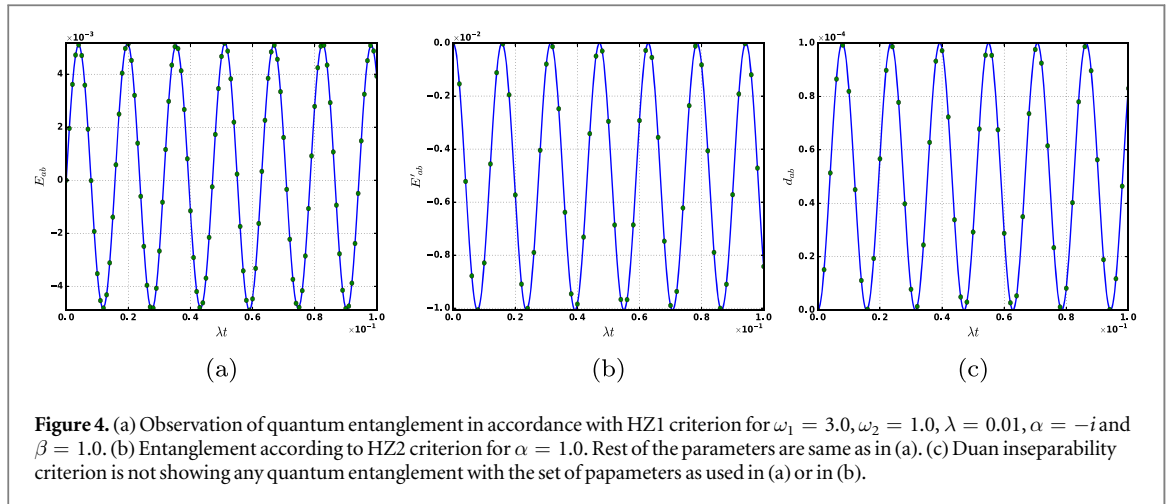
whereas the second sufficient criterion HZ-2 is

$$\langle a^\dagger a \rangle \langle b^\dagger b \rangle - |\langle a b \rangle|^2 < 0. \tag{33}$$

Using equations (7), (8), (32) and (33), we have the expression for HZ1 criterion

$$E_{ab} = |h_3|^2(3|\alpha|^2 + 1) + 3|f_3|^2 + \{f_1 f_3^* + h_2 h_3^* \alpha^2 + f_2 f_3^* \beta^2 + c.c.\} \tag{34}$$





and for HZ2 criterion

$$E'_{ab} = |f_3|^2 |\beta|^2 - |h_3|^2 (|\alpha|^2 + 1) - \{h_1 h_3^* (f_1^* f_2 \beta^2 + f_1^* f_3 |\beta|^2) + h_2 h_3^* \alpha^2 + c.c.\}. \quad (35)$$

The Duan inseparability criterion [40] is another criterion from which entanglement between two modes is investigated. This criterion is given by

$$\langle (\Delta u)^2 \rangle + \langle (\Delta v)^2 \rangle - 2 < 0. \quad (36)$$

Using (7), (8) and (36) we have the following expression of the Duan's inseparability criterion,

$$d_{ab} = 2(|f_2|^2 + |f_3|^2) + \{f_1 f_4^* + f_1 h_2^* + f_2 h_1^* + h_1 h_4^* + c.c.\}. \quad (37)$$

The equations (34), (35) and (37) are plotted in figure 4. The exhibits entanglement which is apparent from the plots 4(a) and (b). This is noteworthy that Duan's criterion (equations (36) and (37)) can assume negative values for different phases of  $\alpha$  or  $\beta$ .

## 6. Conclusion

We have derived the approximate analytical solution for the system of two linear oscillators bound with position-position coupling. The analytical solutions are then used to investigate the particle number dynamics and various nonclassicalities. We have chosen a region away from the resonance where the nonclassicalities namely squeezing, sub-Poissonian quantum statistics and quantum entanglement are found in individual modes as well as in coupled mode. The mean particle number shows a periodic kind of particle number conservation. The present findings would have potential applications in different fields of the likes of quantum information or optical simulations. This study can be extended to investigate higher order nonclassicalities such as higher order quantum entanglement.

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