2022

Mathematics

[Honours]

(B.Sc. Fifth Semester End Examination-2022)

PAPER-MTMH C502

Full Marks: 60 ***Time: 03 Hrs***

The figures in the right hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

Illustrate the answers wherever necessary

[Use separate answer script for each group]

[Partial Differential Equation & Metric Space - II]

Group-A

(Partial Differential Equation)

1. Answer any six questions $6 \times 2 = 12$

(a) Obtain the singular integral of the PDE: $z = px + qy + pq$, where p and q have their usual meanings.

(b) Define the 'domain of dependence' of the one-dimensional wave equation.

(c) Find the characteristic(s) of the equation

$$y^2 u_{xx} - x^2 u_{yy} = 0$$

(d) Using the method of separation of variables, solve the equation $u_{xx} - 2u_x - u_y = 0$.

(e) Discuss the nature of the partial differential equation
$(x^2 - 1)u_{xx} + 2yu_{xy} - u_{yy} = 0$.

(f) Write down the geometric interpretation of the equation

$$Pp + Qq = R$$

where p and q have their usual meanings and P, Q, R are the functions of x, y, z.

(g) Find the family of surfaces orthogonal to the family of surfaces given by the differential equation

$$(y+z)p + (z+x)q = x + y$$

where p and q have their usual meanings.

(h) Reduce the wave equation $u_{tt} = c^2 u_{xx}$ to canonical form.

(i) Obtain the partial differential equation of the family of spheres of radius r, having center in the xy-plane.

2. Answer any two questions of the following. $2 \times 5 = 10$

(a) Find complete and singular solutions of the PDE $(p^2 + q^2)y = qz$.

(b) Find the complete solution of the following PDE:

$$(x^2D^2 - xyDD' - 2y^2D'^2 + xD - 2yD')u = x^2y.$$

(c) Find the solution of the one-dimensional diffusion equation satisfying the following BCs:

(i) T is bounded as $t \to \infty$

(ii) $\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$, for all t

(iii)) $\left.\frac{\partial T}{\partial x}\right|_{x=a} = 0$, for all t

(iv) $T(x,0) = x(a-x), 0 < x < a$.

3. Answer any two question of the following. $2 \times 10 = 20$

(a) (i) Reduce the following equation to a canonical form

$$(1-k^2)u_{xx} + u_{yy} = 0, k > 1.$$

Hence solve it to find the general solution.

(ii) Find the integral surface of $x^2p + y^2q + z^2 = 0$ which passes through the hyperboloid $xy = x + y, z = 1$.

(b). (i) Prove that the total energy of a string, which is fixed at the points $x = 0, x = l$ and executing small transverse vibration, is given by

$$\frac{1}{2}T\int_0^l \left[\left(\frac{\partial y}{\partial x}\right)^2 + \frac{1}{c^2}\left(\frac{\partial y}{\partial t}\right)^2\right]dx$$

where $c^2 = \frac{T}{\rho}$, ρ is the uniform linear density and T is the tension. Show, also that if $y = f(x - ct), 0 \le x \le l$, then the energy of the wave is equally divided between the potential and kinetic energy. 5

(ii) Solve the vibrating string problem described by the PDE

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < l, t > 0$$

$$\text{BCs: } u(0,t) = u(l,t) = 0, \ t > 0$$

$$\text{ICs: } u(x,0) = 0, u_t(x,0) = sin^3\left(\frac{\pi x}{l}\right).$$

(c) (i) Consider the second order linear PDE in two variables x and y: $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$

where the coefficients A, B, C, D, E, F and G are the functions of x and y. Then show that for a suitable general transformation given by $\xi = \xi(x,y), \eta = \eta(x,y)$ of the variables x and y, the above PDE can be transform into a canonical form such that the nature of the PDE does not altered.

(ii) Find the characteristics of the equation $pq = z$ and determine the integral surfaces which passes through the straight line $x = 1, y = z$.

Group-B

(Metric Space - II)

1. **Answer any four questions** $4 \times 2 = 8$

(a) Let $f: (X, d_1) \to (Y, d_2)$ be continuous on X. Then show that any V is open in (Y, d_2) implies $f^{-1}(V)$ is open in (X, d_1).

(b) Show that$f: (\mathbb{R}, d_u) \to (\mathbb{R}, d_u)$ given by $f(x) = x^2, \forall x \in \mathbb{R}$, is not uniform continuous over $\mathbb{R}$., d_u be the usual metric.

(c) Define homeomorphism in a metric space. (X, d)

(d) Give the definition of open cover and define compact metric space.

(e) Is $(0,1)$ is compact in $\mathbb{R}$?

(f) Let $f: ((0,1], d_u) \to (\mathbb{R}, d_u)$ defined by $f(x) = \frac{1}{x}$. Examine whether f is uniform or not.

2. Answer any two questions of the following $2 \times 5 = 10$

(a) Let $f: (X, d) \to (\mathbb{R}, d_u)$ defined by $f(x) = d(x, A)$. where A is given subset of X. Then prove that f is continuous on X.

(b) Prove that a compact subset of a metric space (X, d) is closed and bounded.

(c) Prove that a function $f: (X, d_1) \to (Y, d_2)$ is uniform continuous if and only if f preserve parallel sequence.