Executive Summary

on

Modelling of Deteriorating/Defective Items in Crisp and Fuzzy Environment

under

UGC MINOR RESEARCH PROJECT

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Contents

Chapter-1

Introduction

1.1 Organisation of the Project

In this project, some realistic inventory models are formulated and solved in fuzzy environment. The proposed project has been divided into two parts on the basis of deteriorating and non-deteriorating inventory items in fuzzy environment. Part-I contains two inventory models and Part-II contains one inventory model. Model-3.

Part-I: Inventory Models of Deteriorating items in Fuzzy Environment

Chapter-2

Model-2.1: Multi-item fuzzy inventory model for deteriorating items in multi-outlet under single management

Multi-item inventory model with stock dependent demand is developed in fuzzy environment. Items are deteriorated in constant rate and are sold from different outlets in the city under single management. Due to the impreciseness of different parameters, objectives as well as the constraints are imprecise in nature. As optimization of fuzzy objectives as well as fuzzy constraints are not well defined, the model is formulated as a multi-objective chance constrained programming problem where optimistic/pessimistic return of the objectives with some degree of possibility/necessity are optimized and constraints are satisfied with some degree of necessity. The model is solved via Multi-Objective Genetic Algorithm (MOGA) when crisp equivalent of the problem is available. In other cases fuzzy simulation process is proposed to check the constraints as well as to determine the optimistic/pessimistic return of the objectives. The model is illustrated with some numerical examples.

Model-2.2: An optimal replenishment of fuzzy inventory model for time dependent deteriorating item with fuzzy planning horizon

This paper deals with an inventory model for single deteriorating item during its seasonal time where lifetime of an item has an upper limit. Deterioration rate increases with time and depends on the duration of lifetime left. Demand of the item is price dependent and unit cost of item is time dependent. Unit cost is a decreasing function at the beginning of the season and an increasing function at the end of the season and is constant during the remaining part of the season. So, the inventory model is formulated to maximize the average proceeds out of the system from the imprecise planning horizon. As the optimization of fuzzy objective is not well defined, optimistic/pessimistic return of the objective function (using possibility/necessity measure of the fuzzy event) is optimized. A fuzzy simulation process is proposed to evaluate this optimistic/pessimistic return. A genetic algorithm (GA) is developed based on entropy theory where region of the search space gradually decreases to a small neighbourhood of the optima. This is named as region reducing genetic algorithm (RRGA) and is used to solve this model when planning horizon is crisp. As simulation based region reducing genetic algorithm, called fuzzy simulation based region reducing genetic algorithm (FSRRGA) is developed to solve the fuzzy objective value. The model is illustrated with some numerical examples and some sensitivity analyses have been performed.

Part-II: Inventory Models of Non-Deteriorating items in Fuzzy Environment

Chapter-3

Model-3.1: Fuzzy economic production lot-size model under imperfect production process with cloudy fuzzy demand rate

The aim of the article is to develop classical economic production lot-size (EPL) model of an item produced in imperfect production process with fixed set up cost and without shortages in fuzzy environment where demand rate of an item is cloudy fuzzy number and production rate is demand dependent. In general, fuzziness of any parameter remains fixed over time but in practice, fuzziness of parameter begins to reduce as time progress because of gathering experience and knowledge. The model is solved in crisp, general fuzzy and cloudy fuzzy environment using Yager's index method and De and Beg's ranking index method and comparison are made for all cases. Here, the average cost function is minimized using dominance based Particle Swarm Optimization (DBPSO) algorithm to find decision for the decision maker (DM). The model is illustrated with some numerical examples and some sensitivity analyses have been done to justify the notion.

Part-I

Inventory Models of Deteriorating items in Fuzzy Environment

Chapter-2

Model-2.1: Multi-item fuzzy inventory model for deteriorating items in multi-outlet under single management

2.1.1 Introduction

One of the most important aspects of a successful retail outlet controlled by company/Businessman operating across various locations and channels are the key factors for inventory management system. Optimizing inventory can have huge effects to the overall profitability and progress of retail outlets of all sizes. But operating across different channels or locations has very specific challenges to run smooth inventory management. The biggest one is the time and space continuum. This can reduce the time, effort and stress involved with properly managed multiple retail stores and open up greater opportunities for sales and profits. In the past much of the quantitative work in this field has focused on tactical problems, particularly on site evaluation. In the last two decades extensive research work have been done on inventory control problems with retail outlets (Kamakura (1996), Mendes and Themido (2004), Naseraldin (2008), Li (2016), Smith (2017 etc.). Stanley andSewall (1978) investigated that supermarket plays a vital role for implications for marketing management. Strategic issues have largely been handled in an informal way. Robinson (1990) examines the optimal ordering policies for multi-period multi-locations inventory models with transshipments. Soysal and Krishnamurthy (2015) investigated how adoption of a retailer's factory outlet channel impacts customers' spending in the retailer's traditional retail store channel. Recently, Ngwee (2017) developed model with existence of outlet stores may enable firms to improve quality in their regular channels/ locations.

In many companies, convenience goods and products are offered to the consumers through the company controlled retail outlets. Examples of these products include packaged products, fast foods, fruits, vegetables etc where the respective outlets are situated in an important place like supermarkets, municipality markets etc. In these important places, it is almost impossible to have big show-rooms/shops due the scarcity of space and high rent. They run the outlets with a limited storage space and limited investment. Though inventory models with space and investment constraints have been published by Maiti and Maiti (2006, 2007), Chou et.al. (2009), Garaiet. al. (2016), El-Wakeel and Al-yazidi (2016) etc.), a very few have considered multi-item inventory models with different outlets under a single management.

Most of the inventory determines optimal policies for single item, assuming that inventory policy for single item does not influence the cost of inventory as well as profit of the system. Multi-item inventory first introduced by Federgruenet. al. (1984) who found out that coordinated replenishments for multiple items can significantly reduce total inventory costs because placing orders for multiple items in one replenishment order would reduce set up costs. Recently, researchers have started to realize the complicated natures of the multi-item inventory systems. The demands for multiple inventory items might be correlated, affecting the optimal order policies for a two-item inventory system (Liu andYuan (2000), Das et. at. (2000), Bera and Maiti (2012), Maiti and Maiti(2008), Yadav et. al. (2016), Garaiet. at. (2016), Bera and Jana(2017), . When one product is out of stock, the demand might be satisfied with other available products, requiring an inventory model for substitutable products (Yadavalli et. al (2006)). The existing literature, multi-item inventory model with multi-retail outlets under a single management has been discussed in fuzzy environment where inventory parameters are fuzzy. **None has considered multi-item inventory models with different outlets under a single management where objectives with some degree of possibility/necessity are optimized and constraints are satisfied with some degree of necessity**

It has been recognized that one's ability to make precise statement concerning different parameters of an inventory model with increasing complexities of the environment are not defined. As a result, different inventory parameters, especially purchase cost of an item fluctuates throughout the year. So, purchase cost is fuzzy in nature as well as selling price.

Normally, in inventory control systems, resource constraints are assumed deterministic. In real life, when different inventory parameters are imprecise then constraints also become imprecise. For example, at the beginning of a business, normally it is started with a fixed capital. But in course of business, to take some advantages like bulk transport, sudden increase of demand, price discount etc, decision of acquiring more items force the investor to augment the previously fixed capital by some amount in some situations. This augmented amount is clearly fuzzy in nature in the sense of degree of uncertainty (Dubois &Prade(1997)) and hence the total invested capital become imprecise in nature. When purchase costs and investment capital are fuzzy then the resource constraint becomes fuzzy in nature. As a fuzzy constraint represents a fuzzy event, it should be satisfied with some predefined necessity (Dubois &Prade (1983)), according to company's requirement. Like the chance constraint programming approach, proposed by Mohon (2000) in which minimum probability level for satisfying each of the constraint in stochastic environment, possibilistic constraints also may be defined as in (Zadeh (1978), Dubois &Prade(1983), Liu B. &Iwamura (1998a, 1998b)). When purchase costs are fuzzy, objective function (i.e., average profit) becomes fuzzy in nature. Since optimization of a fuzzy objective is not well defined , one can optimize the optimistic/pessimistic returns of the objectives with some degree of possibility/necessity according to requirement as proposed by Liu and Iwamura(1998a,1998b), Maiti and Maiti(2006,2007), Maiti et al. (2014), Garai et. al. (2016) etc.

In the present competitive market, the inventory/stock is decoratively displayed through electronic media to attract the customers and thus to boost the sale. Levin et al. (1972), Schary and Becker (1972), Wolfe (1968) and others established the impact of product availability for simulating demand. Mandal and Phaujder (1989), Datta and Pal (1988) and others considered linear form of stock-dependent demand, i.e., $D = c + dq$, where D, q represent demand and stock level respectively, c, d are two constants, so chosen to best fit the demand function, where as Urban(1992), Giri et al.(1996), Mandal and Maiti (2000), Maiti and Maiti (2006,2007) and others took the demand of the form $D = dq^{\beta}$, where β is a constant. So, extensive research work in inventory control problems have been reported in fuzzy environment (Yang (2014), Kumar & Kumar (2016a, 2016b), Shukla et. al. (2017), Tripathi et. al. (2018) etc.).

In real world problems, deterioration is also a natural phenomenon. There are some physical goods which deteriorate with the progress of time during their normal storage. In this area, a lot of research papers have been published by several researchers vizMandal and Phaujdar (1989), Gupta and Agarwal(2000), Chang(2004), Chang et al.(2003) , Balkhi (1998, 2004), Maragatham and Lakshmidevi (2014), Sharmila and Uthayakumar (2015), Muniappan et.al.(2015), Chang (2004), Saha and Chakraborti (2012), Kumar and Kumar (2016a, 2016b) , Pal et. al (2017) and others.

The main contributions of this paper are:

- Generally, inventory parameters may be considered precisely but due to practical situation inventory parameters like the purchase cost, investment amount, storage space are considered as fuzzy which are defuzzified using possibility/ necessity measures for a given level of optimistic/pessimistic sense.

- Though a considerable number of research papers have been published with single shop/selling point, much attention has not been paid for the situation where more than one shops are run under a single management. So, the model is formulated with multi-item multi-outlets for deteriorating items under single management.
- Average profits from different outlets gives different objectives. So the problem becomes multi-objective optimization problem.
- Due to fuzziness of the different parameters, the model is formulated as a multi-objective chance constrained programming problem where optimistic/pessimistic return of the objectives with some degree of possibilty/necessity are optimized and constraints are satisfied with some degree of necessity.
- The models are solved by Multi-Objective Genetic Algorithm (MOGA) and Fuzzy-Simulation based Multi-Objective Genetic Algorithm (FSMOGA) and results are compared.
- Finally, the model is illustrated with some numerical examples and results are verifies through sensitivity analyses.

2.1.2 Possibility/Necessity in fuzzy environment

Any fuzzy number $\tilde{a}$ of $\Re$ (where $\Re$ represents set of real numbers) with membership function $\mu_{\tilde{a}} : \Re \to [0,1]$ is called a fuzzy number. Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers with membership functions $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$ respectively. Then according to Zadeh (1978), Dubois and Prade (1983) and Liu and Iwamura(1998):

$$pos(\tilde{a} * \tilde{b}) = \sup\{\min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)), x, y \in \Re,\ x * y\} \qquad (1)$$

where abbreviation pos represents possibility and * is any one of the relations $<,\ >,\ =,\ \le,\ \ge$. Analogously, if $\tilde{b}$ is a crisp number, say, b, then

$$pos(\tilde{a} * b) = \sup\{\mu_{\tilde{a}}(x), \quad x \in \Re,\ x * b\} \qquad (2)$$

The necessity measure of an event $\tilde{a} * \tilde{b}$ is a dual of the possibility measure. The grade of an event is the grade of impossibility of the opposite event and is defined as:

$$nes(\tilde{a} * \tilde{b}) = 1 - pos(\overline{\tilde{a} * \tilde{b}}) \qquad (3)$$

where the abbreviation nes represents the necessity measure and $\overline{\tilde{a} * \tilde{b}}$ represents the complement of the event $\tilde{a} * \tilde{b}$.

If $\tilde{a}, \tilde{b} \in \Re$ *and* $\tilde{c} = f(\tilde{a}, \tilde{b})$ where $f : \Re \times \Re \to \Re$ is binary operation then, the extension principle by Zadeh(1978), the membership function $\mu_{\tilde{c}}$ *of* $\tilde{c}$ is given by

$$\mu_{\tilde{c}}(z) = \sup\{\min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)),\ x,\ y \in \Re \text{ and } z = f(x, y), \forall z \in \Re\} \quad (4)$$

2.1.3 Triangular Fuzzy Number (TFN): A TFN $\tilde{a} = (a_1, a_2, a_3)$ (cf. Fig-1) has three parameters a_1, a_2, a_3 where $a_1 < a_2 < a_3$ and is characterized by the membership function $\mu_{\tilde{a}}(x)$, is given by

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, a_1 \le x \le a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, a_2 \le x \le a_3 \\ 0, \quad \text{otherwise} \end{cases} \quad (5)$$

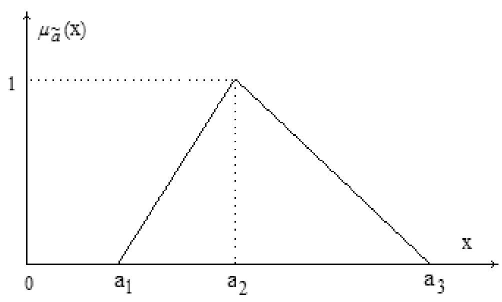

Fig-1: Membership function of a triangular fuzzy number

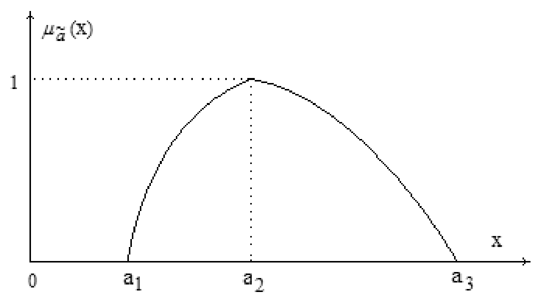

Fig-2: Membership function of a parabolic fuzzy number

2.1.4 Parabolic Fuzzy Number (PFN): A PFN $\tilde{a} = (a_1, a_2, a_3)$ (cf. Fig-2) has three parameters a_1, a_2, a_3 where $a_1 < a_2 < a_3$ and is characterized by the membership function $\mu_{\tilde{a}}(x)$, is given by

$$\mu_{\tilde{a}}(x)=\begin{cases}1-\left(\frac{x-a_1}{a_2-a_1}\right)^2, a_1\le x\le a_2\\ 1-\left(\frac{a_3-x}{a_3-a_2}\right)^2, a_2\le x\le a_3\\ 0, \quad otherwise\end{cases} \tag{6}$$

2.1.5 α cut of a fuzzy number: α cut of a fuzzynumber $\tilde{A}\ in\ \Re$ with membership function $\mu_{\tilde{A}}$ is denoted by A_α is defined as the crisp set

$A_\alpha=\{x:\mu_{\tilde{A}}(x)\ge\alpha,\ x\in\Re\}\ where\ \alpha\in[0,1]$ A_α is a non-empty bounded closed interval contained in $\Re$ and it can be denoted by $A_\alpha=[A_L(\alpha),A_R(\alpha)]$

Lemma 1: If $\tilde{a}=(a_1,a_2,a_3)$ and $\tilde{b}=(b_1,b_2,b_3)$ are TFNs with $0<a_1\ and\ 0<b_1$ then

$$nes(\tilde{a}>\tilde{b})\ge\alpha\ iff\ \frac{b_3-a_1}{a_2-a_1+b_3-b_2}\le1-\alpha$$

Proof: We have $nes(\tilde{a}>\tilde{b})\ge\alpha\Rightarrow\{1-pos(\tilde{a}\le\tilde{b})\}\ge\alpha\Rightarrow pos(\tilde{a}\le\tilde{b})\le1-\alpha$

So, from Fig-3, it is clear that $pos(\tilde{a}*\tilde{b})=\delta=\frac{b_3-a_1}{a_2-a_1+b_3-b_2}$ and hence the result follows.

Lemma 2: If $\tilde{a}=(a_1,a_2,a_3)$ be TFN with $0<a_1$ and b is a crisp number then

$$nes(\tilde{a}>b)\ge\alpha\ iff\ \frac{b-a_1}{a_2-a_1}\le1-\alpha$$

Proof: Proof follows from Lemma1.

Lemma 3: If $\tilde{a}=(a_1,a_2,a_3)$ be TFN with $0<a_1$ and b is a crisp number then

$$pos(\tilde{a}>b)\ge\alpha\ iff\ \frac{a_3-b}{a_3-a_2}\ge\alpha$$

Proof: Proof follows from formula (1) and Fig-4.

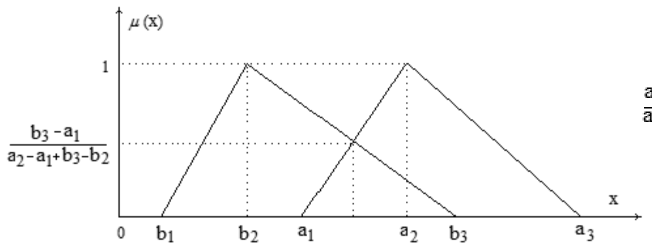

Fig-3: Comparison of two triangular fuzzy number

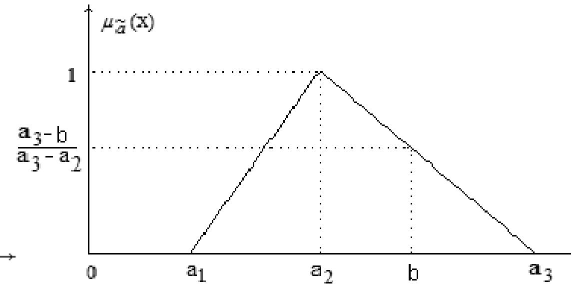

Fig-4: Comparison of a TFN with a crisp number

2.1.6 Multi-objective optimization using possibility/necessity measure

A general multi-objective mathematical programming should have the following form:

$$\text{Max} \quad f_j(x,\xi),\ j=1,2,......,m$$

$$\text{Subject to } g_i(x,\xi)\le 0,\ i=1,2,.....,n \qquad (7)$$

where x is a decision vector, ξ is a vector of crisp parameters, $f_j(x,\xi)$ are return functions, $g_i(x,\xi)$ are constraint functions, i = 1, 2, ...n. In the above problem when ξ is a fuzzy vector $\tilde{\xi}$ (i.e., a vector of fuzzy numbers), then return functions and constraint functions $g_i(x,\tilde{\xi})$ are imprecise in nature and can be represented by two fuzzy numbers whose membership functions involve the decision variable x as a parameter and can be obtained when membership functions of the fuzzy numbers in $\tilde{\xi}$ are known (since f_j and g_i are functions of decision vector x and the fuzzy numbers in $\tilde{\xi}$). In that case the statements maximize $f_j(x,\tilde{\xi})$ as well as $g_i(x,\tilde{\xi})\le 0$ are not well defined. Since $g_i(x,\tilde{\xi})$ represents a fuzzy number whose membership function involves decision vector x and for a particular value of x, one can measure the necessity of $g_i(x,\tilde{\xi})\le 0$ using formula (3), so a value x_0 of the decision vector x is said to be feasible if necessity measure of the event $\{\tilde{\xi}: g_i(x_0,\tilde{\xi})\le 0\}$ exceeds some predefined level α_i in pessimistic sense, i.e., if $nes\{g_i(x_0,\tilde{\xi})\le 0\}\ge \alpha_i$ which is also written as $nes\{\tilde{\xi}: g_i(x_0,\tilde{\xi})\le 0\}\ge \alpha_i$. If analytical form of membership function of

$g_i(x,\tilde{\xi})$ is available then one can transform this constraint to an equivalent crisp constraint (cf. Lemma1 of §2). Otherwise to check this necessity constraint one can follow simulation process as proposed by Maiti and Maiti(2006).

Again since maximize $f_j(x,\tilde{\xi})$ are not well defined one can find maximum value of $z_j$such that $f_j(x,\tilde{\xi})\geq z_j$. But $f_j(x,\tilde{\xi})\geq z_j$ are also not well defined and so one can measure their possibility/necessity in optimistic/pessimistic sense and if this possibility/necessity measure exceeds some predefined level β_j , i.e., if $pos/nes\{f(x,\tilde{\xi})\geq z\}\geq\beta$(which are also written as $pos/nes\{\tilde{\xi}:f(x,\tilde{\xi})\geq z\}\geq\beta$) then z_j taken as optimistic/pessimistic return of the fuzzy objective $f_j(x,\tilde{\xi})$ with degree of optimism/pessimism β_j . Since our aim is to maximize the objective functions it is worthwhile to maximize the optimistic/pessimistic returns $z_j$and so one can find x for which z_j are maximum. When analytical form of membership functions of $f_j(x,\tilde{\xi})$ are available one can transform $pos/nes\{\tilde{\xi}:f(x,\tilde{\xi})\geq z\}\geq\beta$ to equivalent crisp constraints, otherwise value of ξ are randomly generated from β_j cut set of fuzzy vector ξ and x is found from search space to maximize z_j as described in the next section (see Algorithm2 and Algorithm3 in §3.1).

So, when ξ is a fuzzy vector $\tilde{\xi}$ then one can convert the above problem (7) to the following chance constrained programming problems in optimistic and pessimistic sense respectively.

$$
\begin{aligned}
&\max && z_j,\ \ j=1,2,....,m \\
&\textit{subject to} && pos\{\tilde{\xi}: f_j(x,\tilde{\xi})\geq z_j\}\geq\beta_j,\ j=1,2,....,m \\
& && pos\{\tilde{\xi}: g_i(x,\tilde{\xi})\leq 0\}\geq\alpha_i,\ i=1,2,.....,n
\end{aligned}
\tag{8}
$$

$$\begin{aligned} &\max \quad z_j,\ j=1,2,....,m \\ &subject\,to\ \ nes\{\tilde{\xi}: f_j(x,\tilde{\xi}) \ge z_j\} \ge \beta_j,\ j=1,2,....,m \\ &\qquad\qquad pos\{\tilde{\xi}: g_i(x,\tilde{\xi}) \le 0\} \ge \alpha_i, i=1,2,.....,n \end{aligned} \tag{9}$$

where α_i, i = 1, 2, ...n, and β_j, j=1,2,….,m are predetermined confidence levels for fuzzy constraints and fuzzy objectives, respectively, pos/nes{.} denotes the possibility/necessity of the event in {.}. So a point x is feasible if and only if the necessity measure of the set $\{\tilde{\xi}: g_i(x_0,\tilde{\xi}) \le 0\}$ is at least α_i, i = 1, 2, ...,n. For each fixed feasible solution x, the objective value $z_j$should be the maximum that the objective function $f_j(x,\tilde{\xi})$ achieves with at least possibility/necessity β_j, j = 1, 2, ...,m.

2.1.7 Fuzzy simulation: The basic technique of chance constrained programming in a fuzzy environment is to convert the necessity constraints to their respective deterministic equivalents according to predetermined confidence level. However, the procedure is usually very hard and only successful for some special cases (cf. Lemma1). Maiti and Maiti (2007) propose fuzzy simulation process to check feasibility of a solution x of the problems (8) and (9). The algorithm is presented below.

Algorithm1: Algorithm to check $nes\{g_i(x,\tilde{\xi}) \le 0\} \ge \alpha_i$, i = 1, 2, ...,n, for a particular value of decision vector x, for problem (8) and (9).

We know that $nes\{g_i(x_0,\tilde{\xi}) \le 0\} \ge \alpha_i \Rightarrow pos\{g_i(x,\tilde{\xi}) > 0\} \le 1-\alpha_i,\ i=1,2,......,n$.Using these criteria required algorithm is developed as below:

1. Set i=1.

2. Generate ξ_0, uniformly from the $1-\alpha_i$ cut set of fuzzy vectors $\tilde{\xi}$.

3. If $g_i(x,\xi_0) > 0$ go to step 7.

4. Repeat steps 2 to 3, N times.

5. Set i = i + 1, if $i \leq n$ go to step 2.

6. Return feasible.

7. Return infeasible.

8. End algorithm.

Again as stated earlier if analytical form of membership function of $f_j(x,\tilde{\xi})$ is available then only one can determine value of z_j s in problem (8) and (9). However, in this case also, the procedure is usually very hard and only successful for some special cases (cf.Lemma2, Lemma3). To deal with the difficulties in evaluation of z_j s, following two simulation Algorithms are proposed for problems (8) and (9) respectively.

Algorithm2: Algorithm to determine z_j , j = 1, 2, ...,m., for problem (8).

1. Do for j = 1, 2, ...,m.

2. Set $z_j = -\infty$ i.e., a large negative number.

3. Generate ξ_0 uniformly from β_j cut set of fuzzy vector $\tilde{\xi}$.

4. If $z_j < f_j(x,\xi_0)$ *then* $z_j = f_j(x,\xi_0)$.

5. Repeat steps 3 and 4, N times, where N is a sufficiently large positive integer.

6. End Do

7. Return z_j, j = 1, 2, ...,m.

8. End algorithm.

Algorithm3: Algorithm to determine z_j , j = 1, 2, ...,m for problem (9).

We know that $nes\{\tilde{\xi} : f_j(x ,\tilde{\xi}) \geq z_j\} \geq \beta_j \Rightarrow pos\{\tilde{\xi} : f_j(x,\tilde{\xi}) < z_j\} < 1-\beta_j$. Now roughly find a point ξ_0 from fuzzy vector $\tilde{\xi}$, which approximately minimizes f_j. Let this value be z_0 *and* ε be a positive number. Set $z_j = z_0 - \varepsilon$ and if $pos\{\tilde{\xi} : f_j(x,\tilde{\xi}) < z_j\} < 1-\beta_j$ then increase z_j with ε. Again check $pos\{\tilde{\xi} : f_j(x,\tilde{\xi}) < z_j\} < 1-\beta_j$ and it continues until

$pos\{\tilde{\xi} : f_j(x,\tilde{\xi}) < z_j\} \geq 1-\beta_j$. At this stage decrease value of ε and again try to improve z_j. When ε becomes sufficiently small then we stop and final value of z_j is taken. Using this criterion, required algorithm is developed as below:

1. Do for j = 1, 2, ...,m.

2. Initialize z_0 *and* ε.

3. Set $z_j = z_0 - \varepsilon,\ F_j = z_0 - \varepsilon, F_0 = z_0 - \varepsilon$

4. Generate ξ_0 uniformly from the $1-\beta$ cut set of fuzzy vector $\tilde{\xi}$.

5. If $f_j(x,\tilde{\xi}_0) < z_j$

6. then go to step 12.

7. End If

8. Repeat step-4 to step-7 N times.

9. Set $F_j = z_j$.

10. Set $z_j = z_j + \varepsilon$.

11. Go to step-4.

12. If($z_j = F_j$) // In this case optimum value of $z_j < z_0 - \varepsilon$.

13. Set $z_j = z_j - \varepsilon,\ F_j = F_j - \varepsilon, F_0 = F_0 - \varepsilon$.

14. Go to step-4

15. End If

16. If ($\varepsilon < tol$)

17. go to step-22

18. End If

19. $\varepsilon = \varepsilon / N$

20. $z_j = F_j + \varepsilon$

21. Go to step-4.

22. End Do

23. Output F_j, j = 1, 2, ...,m.

It is not possible to find an optimum solution of problem (8) or (9) using any traditional gradient based optimization technique or using any soft computing algorithm (MOGA, Multi Objective Simulated Annealing (MOSA), etc.) until the necessity constraints are converted to equivalent crisp constraints and analytical expressions of z_j are available. In almost all real life problems it is not possible to convert the necessity constraints to their crisp equivalents and it is very hard to get analytical expressions for z_js. In that case with the help of above algorithms any soft computing algorithm (MOGA, MOSA, etc.) can be used to solve the above problem (8) or (9). In this paper MOGA is used for this purpose and since the above fuzzy simulation process is used to check the constraints in these situations as well as z_j are determined, the corresponding MOGA is called FSMOGA. In the next section a MOGA is discussed to solve (8) and (9) with the help of above algorithms. This algorithm is named as FSMOGA.

2.1.8 Fuzzy simulation based multi-objective genetic algorithm (FSMOGA)

Genetic Algorithms are exhaustive search algorithms based on the mechanics of natural selection and genesis (crossover, mutation etc.) and have been developed by Holland, his colleagues and students at the University of Michigan (c.f. Goldberg(1989), Michalewicz (1992) etc.). Because of its generality and other advantages over conventional optimization methods it has been successfully applied to different decision making problems. There are several approaches using genetic algorithms to deal with the multi-objective optimization problems. The better known ones include the plain aggregation approach, the population-based non-pareto approach, the pareto-based approach and Niche induction approach by Deb (2001, 2002). Proposed multi-objective genetic algorithm has been developed following Deb (2002) with the help of fuzzy simulation process to check the problem constraints and has the following two important components.

(a) **Division of a population of solutions into subsets having non-dominated solutions**: Consider a problem having M objectives and take a population P of feasible solutions of the problem of size N. We like to partition P into subsets F_1, F_2, ….,F_k, such that every subset contains non-dominated solutions, but every solution of F_i is not dominated by any solution of F_{i+1}, for i = 1, 2, ..k −1. To do this for each solution, x, of P, calculate the following two entities.

(i) Number of solutions of P which dominate x, let it be η_x .

(ii) Set of solutions of P that is dominated by x. Let it be S_x.

The above two steps require O (MN^2) computations. Clearly F_1 contains every solution x having η_x = 0. Now for each solution x $\in$ F_1, visit every member y of S_x and decrease η_y by 1. In doing so if for any member y, η_y = 0, then y $\in$ F_2. In this way F_2 is constructed. The above process is continued to every member of F_2 and thus F_3 is obtained. This process is continued until all subsets are identified.

For each solution x in the second or higher level of non-dominated subsets, η_x can be at most N − 1. So each solution x will be visited at most N − 1 times before η_x becomes zero. At this point, the solution is assigned a subset and will never be visited again. Since there is at most N − 1 such solutions, the total complexity is $O(N^2)$. So overall complexity of this component is $O(MN^2)$.

(b) **Determine distance of a solution from other solutions of a subset:**

To determine distance of a solution from other solutions of a sub set following steps are followed:

(i) First sort the subset according to each objective function values in ascending order of magnitude.

(ii) For each objective function, the boundary solutions are assigned an infinite distance value (a large value).

(iii) All other intermediate solutions are assigned a distance value for the objective, equal to the absolute normalized difference in the objective values of two adjacent solutions.

(iv) This calculation is continued with other objective functions.

(v) The overall distance of a solution from others is calculated as the sum of individual distance values corresponding to each objective. Since M independent sorting of at most N solutions (In case the subset contains all the solutions of the population) are involved, the above algorithm has O(MNlogN) computational complexity.

Using the above two operations proposed multi-objective genetic algorithm takes the following form:

1. Set probability of crossover p_c and probability of mutation p_m.

2. Set iteration counter T = 1.

3. Generate initial population set of solution P(T) of size N.

4. Select solution from P (T) for crossover and mutation.

5. Made crossover and mutation on selected solution and get the child set C (T).

6. Set P_1 = P (T)UC(T) // Here U stands for union operation.

7. Divide P_1 into disjoint subsets having non-dominated solutions. Let these sets be F_1, F_2, ..,F_k.

8. Select maximum integer n such that order of $P_2(= F_1UF_2U ... UF_n) \leq N$.

9. If $O(P_2) < N$ sort solutions of F_{n+1} in descending order of their distance from other solutions of

the subset. Then select first N –$O(P_2)$ solutions from F_{n+1} and add with P_2, where $O(P_2)$ represents order of P_2.

10. Set T = T + 1 and P(T) = P_2.

11. If termination condition does not hold go to step-4.

12. Output: P(T)

13. End algorithm.

MOGAs that use non-dominated sorting and sharing are mainly criticized for their

- O(MN3) computational complexity
- non-elitism approach
- the need for specifying a sharing parameter to maintain diversity of solutions in the population.

In the above algorithm, these drawbacks are overcome. Since in the above algorithm computational complexity of step-7 is O(MN2), step-9 is O(MNlogN) and other steps are ≤ O(N), so overall time complexity of the algorithm is $O(MN^2)$. Here selection of new population after crossover and mutation on old population is done by creating a mating pool by combining the parent and offspring population and among them, best N solutions are taken as solutions of new population. By this way, elitism is introduced in the algorithm. When some solutions from a non-dominated set F_j (i.e., a subset of F_j) are selected for new population, those are accepted whose distance compared to others (which are not selected) are much i.e., isolated solutions are accepted. In this way taking some isolated solutions in the new population, diversity among the solutions is introduced in the algorithm. Different procedures of the above MOGA are discussed in the following section.

2.1.9 Procedures of the proposed MOGA

(a) Representation: A 'K dimensional real vector' $X=(x_1, x_2, ...x_K)$ is used to represent a solution, where $x_1, x_2, ... ,x_K$ represent different decision variables of the problem such that constraints of the problem are satisfied.

(b) Initialization: N such solutions X_1, X_2, X_3, ...,X_N are randomly generated by random number generator from the search space such that each X_i satisfies the constraints of the problem. This solution set is taken as initial population P(1). Also set $p_c = 0.3$, $p_m = 0.2$, T=1.

(c) Crossover:

(i) **Selection for crossover**: For each solution of P(T) generate a random number r from the range [0, 1]. If $r < p_c$ then the solution is taken for crossover.

(ii) **Crossover process**: Crossover taken place on the selected solutions. For each pair of coupled solutions Y_1, Y_2 a random number c is generated from the [0, 1] and offsprings Y_{11} and Y_{21} are calculated by $Y_{11} = cY_1 + (1 - c)Y_2$, $Y_{21} = cY_2 + (1 - c)Y_1$.

(d) Mutation:

(i) Selection for mutation: For each solution of P(T) generate a random number r from the range [0, 1]. If $r < p_m$ then the solution is taken for mutation.

(ii) Mutation process: To mutate a solution $X = (x_1, x_2, x_3, \ldots, x_K)$ select a random integer r in the range [1, K]. Then replace x_r by randomly generated value within the boundary of r^{th} component of X.

(e) Division of P(T) into disjoint subsets having non-dominated solutions: Following the discussions of the previous section the following algorithm is developed for this purpose

For every $x \in P(T)$ do

Set $S_x = \Phi$, where Φ represents null set

$\eta_x = 0$

For every $y \in P(T)$ do

If x dominates y then

$S_x = S_x \cup \{y\}$

Else if y dominates x then

$\eta_x = \eta_x + 1$

End if

End For

If η_x = 0 then

$F_1 = F_1 U\{x\}$

End If

End For

Set i=1

While $F_i \neq \Phi$ do

$F_{i+1} = \Phi$

For every x $\in$ F_i do

For every y $\in S_x$ do

$\eta_y = \eta_y - 1$

If η_y = 0 then

$F_{i+1} = F_{i+1} U\{y\}$

End If

End For

End For

i=i+1

End While

Output: $F_1, F_2, .., F_{i-1}$.

(f) Determine distance of a solution of subset F from other solutions: Following algorithm is used for this purpose

Set n=number of solutions in F

For every x $\in$ F do

$x_{distance} = 0$

End For

For every objective m do

Sort F, in ascending order of magnitude of m^{th} objective.

F[1] = F[n] = M, where M is a big quantity.

For i=2 to n-1 do

$F[i]_{distance} = F[i]_{distance} + (F[i+1].objm - F[i-1].objm)/(f^{max}_m - f^{min}_m)$

End For

End For

In the algorithm F[i] represents i^{th} solution of F, F[i].objm represent m^{th} objective value of F[i]. f^{max}_m and f^{min}_m represent the maximum and minimum values of m^{th} objective function.

2.1.10 Assumptions and notations for the proposed models

The following notations and assumptions are used in developing the models.

Notations

This model is developed for i^{th} (i=1,) outlet and j^{th} item throughout the paper.

N	number of deteriorating items.
M	number of outlets.
W_i	storage area of i^{th} outlet.
λ	deterioration rate.
I_{NV}	maximum investment amount.
T_{ij}	cycle length.
Q_{ij}	order quantity.
Q_{0ij}	stock level above which stock has no effect on demand.
$q_{ij}(t)$	inventory level at time t.

D_{ij} demand rate per unit time.

$a_{ij}, b_{ij} (>0)$ parameters of demand.

A_{ij} storage area per unit.

T_{1ij} time when inventory level reaches Q_{0ij}.

c_{pij} purchase cost per unit.

c_{sij} selling price per unit.

c_{0ij} ordering cost per cycle.

c_{hij} holding cost per unit.

Z_{ij} average profit.

F_i average profit from ith outlet.

P_i / N_i optimistic/pessimistic return of the average profit F_i with degree of optimisim/pessimism β_i

α_1, α_2 confidence levels for investment and space constraints respectively.

Assumptions

(i) The model is developed for M outlets, N deteriorating items and n_i items are sold from this outlet. So, $\sum_{i=1}^{M} n_i = N$

(ii) Lead time is zero.

(iii) Shortages are not allowed.

(iv) Time horizon of the inventory system is infinite.

(v) The demand D_{ij} is linearly depend upon the stock level of the item and is of the form

$$D_{ij} = \begin{cases} a_{ij} + b_{ij}Q_{0ij} , Q_{0ij} < q_{ij} \le Q_{ij} \\ a_{ij} + b_{ij}q_{ij}, \quad q_{ij} \le Q_{0ij} \end{cases}$$

(vi) Selling price c_{sij} is the mark-up of purchase cost i.e. $c_{sij} = m\, c_{pij}$.

(vii) Ordering cost c_{0ij} is linearly depends on order quantity and is of the form

$c_{0ij} = c_{0ij1} + c_{0ij2}Q_{ij}$

(viii) The holding cost c_{hij} is multiple of purchase cost i.e. $c_{hij} = h_{ij}\, c_{pij}$

2.1.11 Model development and analysis

In the development of the model, it is assumed that the business man possesses M outlets and N items are sold from these outlets. For j[th] item in i[th] outlet a cycle starts with an inventory level Q_{ij} . Demand is stock dependent and when inventory level of the item reaches zero an order Q_{ij} for next cycle is made.

Formulation for the j[th] item in i[th] outlet

Depending upon the order quantity Q_{ij} two cases may arise (i) Case-I: $Q_{ij} > Q_{0ij}$ (ii) Case-II : $Q_{ij} \le Q_{0ij}$

2.1.11.1 Case-I ($Q_{ij} > Q_{0ij}$):

The instantaneous state $q_{ij}(t)$ is given by the following differential equation

$$\frac{dq_{ij}(t)}{dt} = \begin{cases} -(a_{ij} + b_{ij}Q_{0ij}) - \lambda q_{ij}(t) \quad , Q_{0ij} < q_{ij}(t) \le Q_{ij} \\ -(a_{ij} + b_{ij}q_{ij}(t)) - \lambda q_{ij}(t), Q_{0ij} \ge q_{ij}(t) \ge 0 \end{cases} \tag{10}$$

with the boundary conditions $q_{ij}(0) = Q_{ij}, q_{ij}(T_{1ij}) = Q_{0ij}, q_{ij}(T_{ij}) = 0$.

Solving (10) we get $T_{1ij} = \frac{1}{\lambda}\log\left|\frac{a_{ij} + b_{ij}Q_{0ij} + \lambda Q_{ij}}{a_{ij} + b_{ij}Q_{0ij} + \lambda Q_{0ij}}\right|$ (11)

$$T_{ij} = T_{1ij} + \frac{1}{b_{ij} + \lambda}\log\left|\frac{a_{ij} + (b_{ij} + \lambda)Q_{0ij}}{a_{ij}}\right| \tag{12}$$

$$q_{ij}(t)=\begin{cases}\frac{1}{\lambda}\left[(a_{ij}+b_{ij}Q_{0ij}+\lambda Q_{ij})e^{-\lambda t}-(a_{ij}+b_{ij}Q_{0ij})\right],\ 0<t\le T_{1ij}\\ \frac{1}{b_{ij}+\lambda}\left[\left\{a_{ij}+(b_{ij}+\lambda)Q_{0ij}\right\}e^{-(b_{ij}+\lambda)(t-T_{1ij})}-a_{ij}\right],T_{1ij}\le t\le T_{ij}\end{cases}\tag{13}$$

Sales revenue in $[0,T_{ij}]$, is $S_{pij}=c_{sij}S_{ij}$ *where* S_{ij} is given by

$$S_{ij}=\int_{0}^{T_{1ij}}(a_{ij}+b_{ij}\,Q_{0ij})\,dt\ +\int_{T_{1ij}}^{T_{ij}}(a_{ij}+b_{ij}\,q_{ij})\,dt$$

$$=(a_{ij}+b_{ij}\,Q_{0ij})T_{1ij}+\frac{a_{ij}\lambda}{(b_{ij}+\lambda)^{2}}\log\left|\frac{a_{ij}+(b_{ij}+\lambda)Q_{0ij}}{a_{ij}}\right|+\frac{b_{ij}Q_{0ij}}{b_{ij}+\lambda}\tag{14}$$

Holding cost in $[0,T_{ij}]$ is $c_{hij}H_{ij}$ *where* H_{ij} is given by

$$H_{ij}=\int_{Q_{ij}}^{Q_{0ij}}\frac{-q_{ij}}{a_{ij}+b_{ij}Q_{0ij}+\lambda q_{ij}}dq_{ij}\ +\int_{Q_{0ij}}^{0}\frac{-q_{ij}}{a_{ij}+(b_{ij}+\lambda)q_{ij}}dq_{ij}$$

$$\begin{aligned}&=-\frac{(a_{ij}+b_{ij}\,Q_{0ij})}{\lambda^{2}}\log\left|\frac{a_{ij}+b_{ij}Q_{0ij}+\lambda Q_{ij}}{a_{ij}+(b_{ij}+\lambda)Q_{0ij}}\right|+\frac{1}{\lambda}(Q_{ij}-Q_{0ij})+\\&\quad\frac{Q_{0ij}}{b_{ij}+\lambda}-\frac{a_{ij}}{(b_{ij}+\lambda)^{2}}\log\left|\frac{a_{ij}+(b_{ij}+\lambda)Q_{0ij}}{a_{ij}}\right|\end{aligned}\tag{15}$$

2.1.11.2 Case-II ($Q_{ij}\le Q_{0ij}$):

The instantaneous state $q_{ij}(t)$ is given by the following differential equation

$$\frac{dq_{ij}(t)}{dt}=-(a_{ij}+b_{ij}q_{ij}(t))-\lambda q_{ij}(t)\tag{16}$$

with the boundary conditions $q_{ij}(0)=Q_{ij},\ \ q_{ij}(T_{ij})=0$.

Solving (16) we get $T_{ij}=\dfrac{1}{b_{ij}+\lambda}\log\left|\dfrac{a_{ij}+(b_{ij}+\lambda)Q_{ij}}{a_{ij}}\right|$ (17)

$$q_{ij}(t)=\frac{1}{b_{ij}+\lambda}\left[\left\{a_{ij}+(b_{ij}+\lambda)Q_{ij}\right\}e^{-(b_{ij}+\lambda)t}-a_{ij}\right]\tag{18}$$

Sales revenue in $[0,T_{ij}]$, is $S_{pij}=c_{sij}S_{ij}$ *where* S_{ij} is given by

$$S_{ij} = \int_0^{T_{ij}} (a_{ij} + b_{ij}\, q_{ij})\, dt$$

$$= \frac{a_{ij}\lambda}{(b_{ij}+\lambda)^2}\log\left|\frac{a_{ij}+(b_{ij}+\lambda)Q_{ij}}{a_{ij}}\right| + \frac{b_{ij}Q_{ij}}{b_{ij}+\lambda} \tag{19}$$

Holding cost in $[0, T_{ij}]$ is $c_{hij}H_{ij}$ *where* H_{ij} is given by

$$H_{ij} = \int_{Q_{ij}}^{0} \frac{-q_{ij}}{a_{ij} + (b_{ij}+\lambda)q_{ij}}\, dq_{ij}$$

$$= \frac{Q_{ij}}{b_{ij}+\lambda} - \frac{a_{ij}}{(b_{ij}+\lambda)^2}\log\left|\frac{a_{ij}+(b_{ij}+\lambda)Q_{ij}}{a_{ij}}\right| \tag{20}$$

Combining both the cases average profit from j[th] item in i[th] outlet Z_{ij} is given by

$$\begin{aligned} Z_{ij} &= \left[c_{sij}S_{ij} - c_{pij}Q_{ij} - c_{hij}H_{ij} - (c_{0ij1} + c_{0ij2}Q_{ij}) \right] / T_{ij} \\ &= \left[\{mS_{ij} - Q_{ij} - h_{ij}H_{ij}\}c_{pij} - (c_{0ij1} + c_{0ij2}Q_{ij}) \right] / T_{ij} \end{aligned} \tag{21}$$

Average profit F_i from i[th] outlet is given by $F_i = \sum_{j=1}^{n_i} Z_{ij}$ (22)

2.1.12 Crisp model in mathematical form

From the above discussion the problem reduces to the following multi-objective constrained optimization problem as:

Model 1: Maximize F_i ,i=1,2,…….,M (23)

$$\text{Subject to} \sum_{i=1}^{M}\sum_{j=1}^{n_i} Q_{ij}c_{pij} \le I_{NV}$$

$$\sum_{j=1}^{n_i} Q_{ij}A_{ij} \le W_i \;, i = 1,2,....,M$$

2.1.13 Fuzzy models in mathematical form

In the real world, purchase cost (c_{pij}), investment amount (I_{NV}), warehouse space (W_i) are normally imprecise, i.e., vaguely defined in some situations. So we take c_{pij} , I_{NV} , W_i are fuzzy numbers, i.e., as $\tilde{c}_{pij}, \tilde{I}_{NV}, \tilde{W}_i$ respectively. Then, due to this assumption, F_i become fuzzy number $\tilde{F}_i$ and constraints in (23) also become imprecise in nature. Then as discussed in §3, the statement Maximize $\tilde{F}_i$ and constraints in (23) are not well defined. In this case following §3, one can maximize an optimistic (pessimistic) return value $P_i\,(N_i)$ for the objective function $\tilde{F}_i$ with some degree of optimism (pessimism) β_i and the fuzzy constraints are satisfied with degree of pessimism α_1 *and* α_2 for investment and space constraints respectively. So in this case the problem reduces to the following multi-objective chance constrained programming problems in optimistic and pessimistic senses respectively.

Model 2: Maximize P_i ,i=1,2,…….,M (24)

$$pos\{\tilde{F}_i \geq P_i\} \geq \beta_i$$

$$\text{Subject to } nes\{\sum_{i=1}^{M}\sum_{j=1}^{n_i} Q_{ij}\tilde{c}_{pij} \leq \tilde{I}_{NV}\} \geq \alpha_1$$

$$nes\{\sum_{j=1}^{n_i} Q_{ij}A_{ij} \leq \tilde{W}_i\} \geq \alpha_2 \quad , i=1,2,...,M$$

Model 3: Maximize N_i ,i=1,2,…….,M (25)

$$nes\{\tilde{F}_i \geq N_i\} \geq \beta_i$$

$$\text{Subject to } nes\{\sum_{i=1}^{M}\sum_{j=1}^{n_i} Q_{ij}\tilde{c}_{pij} \leq \tilde{I}_{NV}\} \geq \alpha_1$$

$$nes\{\sum_{j=1}^{n_i} Q_{ij}A_{ij} \leq \tilde{W}_i\} \geq \alpha_2 \quad , i=1,2,...,M$$

If purchase cost $\tilde{c}_{pij}$, investment amount $\tilde{I}_{NV}$ and outlet capacity $\tilde{W}_i$ are TFNs with components $(c_{pij1}, c_{pij2}, c_{pij3})$, $(I_{NV1}, I_{NV2}, I_{NV3})$ and (W_{i1}, W_{i2}, W_{i3}) respectively then according to formula (4) $\tilde{Z}_{ij}$ becomes TFN with components $(Z_{ij1}, Z_{ij2}, Z_{ij3})$ where

$$Z_{ijk} = [\{mS_{ij} - Q_{ij} - h_{ij}H_{ij}\}c_{pijk} - (c_{0ij1} + c_{0ij2}Q_{ij})]/T_{ij},\ k = 1,2,3$$

Then $\tilde{F}_i$ becomes TFN with components (F_{i1}, F_{i2}, F_{i3}) where $F_{ik} = \sum_{j=1}^{n_i} Z_{ijk}$, $k = 1,2,3$. Also the quantity $\sum_{i=1}^{M}\sum_{j=1}^{n_i} Q_{ij}\,\tilde{c}_{pij}$ becomes TFN (R_1, R_2, R_3) *where* $R_k = \sum_{i=1}^{M}\sum_{j=1}^{n_i} Q_{ij}\tilde{c}_{pijk}$, $k = 1,2,3.$ Then using the lemmas of §3, the problems (24) and (25) reduce to following crisp multi-objective constrained optimization problems (26) and (27) respectively.

$$\text{Maximize} \quad P_i \ , i=1,2,...,M \tag{26}$$

$$\text{Subject to} \quad \frac{F_{i3}-P_i}{F_{i3}-F_{i2}} \geq \beta_i$$

$$\frac{R_3 - I_{NV1}}{I_{NV2}-I_{NV1}+R_3-R_2} \leq 1-\alpha_1$$

$$\frac{\sum_{j=1}^{n_i} Q_{ij}A_{ij} - W_{i1}}{W_{i2}-W_{i1}} \leq 1-\alpha_2 \quad , i=1,2,....,M \text{ and}$$

$$\text{Maximize} \quad N_i \ , i=1,2,...,M \tag{27}$$

$$\text{Subject to} \quad \frac{N_i - F_{i1}}{F_{i2}-F_{i1}} \geq \beta_i$$

$$\frac{R_3 - I_{NV1}}{I_{NV2}-I_{NV1}+R_3-R_2} \leq 1-\alpha_1$$

$$\frac{\sum_{j=1}^{n_i} Q_{ij}A_{ij} - W_{i1}}{W_{i2}-W_{i1}} \leq 1-\alpha_2 \quad , i=1,2,....,M$$

So when fuzzy parameters are TFN type then problems (24) and (25) can be transformed to equivalent crisp problems and can be solved via MOGA. But if the parameters are of PFN type then sum of two PFNs is not a PFN so in that case problems (24) and (25) can not be transformed to equivalent crisp problems. In that case problems can be solved using FSMOGA with the help of simulation algorithms in §2.1.7

2.1.14. Numerical Illustration

The models are illustrated for two outlets (M=2) and five items (N=5). First three items are sold from first outlet and last two from second one. Common parametric values used in different examples to illustrate the models are presented in Table-1 below. Other common parametric values are λ = 0.02, m = 1.5.

Table-1. Common input data for different examples

Outlet(i)	Item(j)	a_{ij}	b_{ij}	h_{ij}	c_{0ij1}	c_{0ij2}	A_{ij}	Q_{0ij}
1	1	5	2.5	0.15	50	0.5	0.5	10
	2	10	2.2	0.15	50	0.5	0.45	10
	3	12	2.0	0.15	50	0.5	0.55	10
2	1	14	1.8	0.15	50	0.5	0.35	10
	2	8	2.1	0.15	50	0.5	0.45	10

Crisp model: For illustration of the crisp model (23) following example (Example 1) is used.

Example1: Along with the common parametric values other assumed parametric values are c_{p11} = 9.5, c_{p12} = 10.5, c_{p13} = 8.5, c_{p21} = 9, c_{p22} = 8, W_1 = 60, W_2 = 35, I_{NV} = \$1550.

For this example results are obtained via MOGA and few pareto optimal solutions are presented in Table-2 below. It is observed that pareto optimality of a solution does not imply total profit (F_1+F_2) from the system is maximum, which agrees with reality.

Table-2: Results of Example-1 (Model 1) via MOGA

Q_{11}	Q_{12}	Q_{13}	Q_{21}	Q_{22}	F_1	F_2	F_1+F_2
36.21	37.84	29.64	30.80	34.33	140.77	72.47	213.23
36.84	34.40	28.05	35.06	35.22	137.16	76.49	213.64
36.71	33.08	35.93	28.74	35.89	143.44	71.06	214.50
34.11	31.08	36.82	32.75	36.10	140.60	75.26	215.86
36.77	31.26	32.48	35.73	34.02	139.30	76.19	215.48

Fuzzy model: For illustration of the fuzzy models (24) and (25) following examples (Example2 and Example3) are used.

Example 2: Here it is assumed that $\tilde{c}_{pij} = (c_{pij1}, c_{pij2}, c_{pij3})$,$\tilde{I}_{NV} = (I_{NV1}, I_{NV2}, I_{NV3})$, $\tilde{W}_i =$ (W_{i1}, W_{i2}, W_{i3}) as TFNs with $c_{p111} = 9,\ c_{p112} = 9.5,\ c_{p113} = 10,\ c_{p121} = 9,\ c_{p122} = 10.5,$ $c_{p123} = 11,\ c_{p131} = 8,\ c_{p132} = 8.5,\ c_{p133} = 9.5,\ c_{p211} = 8.5,\ c_{p212} = 9,\ c_{p213} = 10.5,\ c_{p221} = 7.5,$ $c_{p222} = 8,\ c_{p223} = 9,\ I_{NV1} = \$1500,\ I_{NV2} = \$1550,\ I_{NV3} = \$1600,\ W_{11} = 50,\ W_{12} = 60,$ $W_{13} = 65,\ W_{21} = 30,\ W_{22} = 35,\ W_{23} = 40,\ \alpha_1 = 0.5, \alpha_2 = 0.5$.

In this case problem (24), (25) can be transformed into corresponding crisp problem (26), (27) respectively. Problem (26) is solved for $\beta_1 = 0.9, \beta_2 = 0.9$ and (27) is solved for $\beta_1 = 0.1, \beta_2 = 0.1$ via MOGA and results are presented in Table-3 and Table-4 respectively. In this case problem (24), (25) are also directly solved via FSMOGA and results are presented in Table-3 and Table-4 respectively. It is observed that results obtained by both the techniques are almost same.

Table-3: Results of Example 2 in model 2

Method	Q_{11}	Q_{12}	Q_{13}	Q_{21}	Q_{22}	P_1	P_2
MOGA	27.84	32.27	29.97	36.96	31.98	132.47	78.08
	30.60	34.38	32.82	31.48	29.01	139.85	71.51
	34.14	33.50	31.21	34.08	25.26	140.59	69.10
	31.68	37.63	32.27	29.42	26.34	142.25	66.28
FSMOGA	35.91	26.89	32.35	35.51	29.06	135.50	74.66
	36.03	28.96	32.78	33.54	28.15	138.73	72.34
	30.60	34.38	32.82	31.48	29.01	139.85	71.51
	31.18	36.85	32.39	29.78	27.54	141.51	68.18

Table-4: Results of Example 2 in model 3

Method	Q_{11}	Q_{12}	Q_{13}	Q_{21}	Q_{22}	N_1	N_2
MOGA	33.08	30.78	35.21	30.31	30.06	135.93	67.65
	31.77	31.32	33.67	35.08	27.17	134.31	68.61
	31.85	33.00	31.71	34.42	27.95	134.19	69.07
	28.46	31.94	31.35	35.12	32.77	129.59	73.92
FSMOGA	29.64	29.14	35.44	33.23	32.36	131.22	72.27
	30.88	30.61	32.53	36.59	28.78	131.86	71.31
	32.01	29.31	35.67	30.45	32.35	133.76	69.71
	31.67	30.30	35.21	31.37	30.95	134.29	69.52

Example 3: Here it is assumed that $\tilde{c}_{pij} = (c_{pij1}, c_{pij2}, c_{pij3})$ as PFNs with $c_{p111} = 9$,

$c_{p112} = 9.5,\ c_{p113} = 10,\ c_{p121} = 9,\ c_{p122} = 10.5,\ c_{p123} = 11,\ c_{p131} = 8,\ c_{p132} = 8.5,\ c_{p133} = 9.5,$
$c_{p211} = 8.5,\ c_{p212} = 9,\ c_{p213} = 10.5,\ c_{p221} = 7.5,\ c_{p222} = 8,\ c_{p223} = 9,\ \alpha_1 = 0.5, \alpha_2 = 0.5$. All other parameters are same as Example 2.

In this case problem (24), (25) cannot be reduced to equivalent crisp problem and so are solved via FSMOGA only and results are presented in Table-5 and Table-6 respectively.

Table-5: Results of Example 3 in model 2

Method	Q_{11}	Q_{12}	Q_{13}	Q_{21}	Q_{22}	P_1	P_2
FSMOGA	31.23	32.65	31.74	31.31	29.10	142.68	77.21
	27.76	28.86	32.06	35.42	32.51	135.43	83.33
	26.51	28.03	32.44	36.39	33.11	133.08	84.31
	26.26	27.93	32.42	36.66	33.42	132.61	84.67

Table-6: Results of Example 3 in model 3

Method	Q_{11}	Q_{12}	Q_{13}	Q_{21}	Q_{22}	N_1	N_2
FSMOGA	31.82	38.14	30.24	26.75	27.57	130.24	58.28
	31.27	32.43	30.81	29.35	32.55	126.63	66.38
	28.82	33.67	30.37	30.71	32.46	124.77	67.84
	26.61	32.23	30.03	32.67	34.87	120.54	71.23

2.1.15 Conclusion

For the first time deteriorating multi-item inventory model is developed with multi-outlet facilities under a single management. The model is formulated as a multi-objective chance constrained programming problem in fuzzy environment. An approach is proposed where instead of objective functions optimistic/pessimistic returns of the objective functions are optimized. Also a simulation approach is proposed to determine these optimistic/pessimistic returns in fuzzy environment. So, from the economical point of view, the proposed model will be useful to the business houses in the present context as it gives better inventory control system. Further extension of this model can be done considering some realistic situation like quantity discount policy. Also, the present MOGA can be applied to other inventory models with variable demand, fixed time horizon etc. along with quantity discount formulated in stochastic and fuzzy-stochastic environments.

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Model-2.2: An optimal replenishment of fuzzy inventory model for time dependent d deteriorating item with fuzzy planning horizon

2.2.1 Introduction

In general, planning horizon of many seasonal items fluctuate to some extend. As for example, in India, winter starts with November and ends with February. But its duration is not always fixed. A little variability can be easily noticed over the years. Thus, planning horizon of seasonal products such as fruits, potato, onion, cabbage, cauliflower, food grains, etc. is a fuzzy variable instead of a fixed deterministic constant. For the seasonable item, it is normally observed that price of the item decreases with time at the beginning of the production season due to availability in the market and reaches to a minimum value. Price of the item remains constant at this minimum value during the major part of the season due to sufficient availability of the item in the market and towards the end of the season due to scarcity, cost again increases gradually and reaches its off season value. This price remains stable during the remaining part of the year. A considerable number of research works have been done for seasonal products by several researchers Zhou *et al*. (2004), Chen and Chang (2007), Panda *et al*. (2008), Banerjee and Sharma (2010A, 2010B), Skouri and Konstantaras (2013),Tayal*et. al*. (2015),Krommyda*et. al*.(2017) etc. Recently, Mohanty*et. al* (2018) developed an trade credit inventory modeling of deteriorating items over random planning horizon due to fluaction of season.

In most of these research works, it is assumed that price of the item decreases with time or demand increases with time. But the above mentioned real life phenomenon of a seasonal product is overlooked by the researchers. Another shortcoming of these research work is the assumption that the duration of the season of such products as crisp value. Although, the duration of the season for an item is finite but it varies from year to year due to environmental changes. So, it is worthwhile to assume this duration as a fuzzy parameter. Occurrence of fuzzy seasonal time leads to optimization problem with fuzzy objective function. In the last two decades extensive research work has been done on inventory control problems in fuzzy environment (Lee *et al*. (1991), Lam and Wong (1996), Roy and Maiti (2000), Mondal and Maiti (2002), Kao and Hsu (2002), Bera*et al*. (2012), Bera and Maiti (2012), Maiti*et al*. (2014), De and Sana (2015), Garai*et. al*. (2016), Bera and Jana (2017), De and Mahata (2017) etc. These problems considered different inventory parameters as fuzzy numbers which render fuzzy objective function. As optimization in fuzzy environment is not well defined some of these researcher transform the fuzzy parameters as equivalent crisp number or crisp interval and then the objective function is transformed to an equivalent crisp number/interval (Maiti and Maiti (2007), Bera*et al*.(2012)). Some of the researchers (Mondal and Maiti, (2002)) set the fuzzy objective as fuzzy goal whose membership function as a linear/non-linear fuzzy number and try to optimize this membership function using Bellman Zadeh's principle (Bellman and Zadeh, 1970). Maiti and Maiti (2006) propose a technique where instead of objective function pessimistic return of the fuzzy objective is optimized. They use necessity measure on fuzzy event to determine this pessimistic return and propose fuzzy simulation process to find this return function. Maiti (2008, 2011) proposes a technique where possibility/necessity measure of objective function (fuzzy profit) on fuzzy goal is optimized to find optimal decision. Recently, Manna *et.al*.(2016), Garai*et. al*. (2016) and others developed inventory models using possibility and necessity constraints for a given level of optimistic/pessimistic sense. All these studies transform the fuzzy objective of the problem to an equivalent crisp objective and solution of the reduced problem is taken as approximate solution of the fuzzy problem. But there exist always some error in such approximation. In present day competitive market, an erroneous inventory decision may invite a huge loss in business. So modelling of present day inventory control problems should be very realistic and a methodology is required which can deal with fuzzy objective function directly without reducing it to crisp form.

Most of the seasonal products have finite lifetime and are deteriorating in nature (Mahataand Goswami, (2010)). Rate of deterioration increases with time and actually depends on the length of lifetime left. Rate of deterioration becomes 100% when age of product covers the lifetime. In the literature, there are several investigations for deteriorating items such as Jaber*et al*. (2009); Yadav*et al.* (2011); Sana (2011), Skouri and Konstantaras (2013),Chaudhury*et. al*. (2015), Tayal*et. al*. (2015, Dutta and Kumar (2015), Karmakar and Chaudhury(2014), Kumar and Rajput (2015), , Mohanty*et. al* (2018), Rastogi et. al. (2018) and others. Most of the inventory articles are developed with constant deterioration. But the deterioration mentioned earlier, deterioration increases with time as stress of units on others causes damage. According to the author's best knowledge, very few articles have been published incorporating time varying deterioration (Sarkar (2011)). However, Janssen *et. al*. (2016) presented a review article on deteriorating items including this publication from 2012 to 2015.

Use of soft computing techniques for inventory control problems is a well-established phenomenon. Several authors use Genetic Algorithm (GA) in different forms to find marketing decisions for their problems. Pal *et al*. (2009) uses GA to solve an EPQ model with price discounted promotional demand in an imprecise planning horizon. Roy *et al*. (2009) used a GA with varying population size to solve a production inventory model with stock dependent demand incorporating learning and inflationary effect in a random planning horizon. Bera and Maiti (2012) used GA to solve multi-item inventory model incorporating discount. Maiti*et al*. (2009) used GA to solve inventory model with stochastic lead time and price dependent demand incorporating advance payment. Mondal*et al*. (2002) uses a dominance based GA to solve a production-recycling model with variable demand, demand-dependent fuzzy return rate. Combining the features of GA and PSO a hybrid algorithm PSGA is used by Guchhait*et al*. (2014) to solve an inventory model of a deteriorating item with price and credit linked fuzzy demand. All these soft computing techniques are not capable to deal with fuzzy objective directly.

From the above discussion it is clear that there are some lacunas in fuzzy inventory models of deteriorating items, especially for seasonal products. In this research work an attempt has been made to reduce these lacunas. The aim of this research work is fourfold:

The aim of this research work is fourfold:

- Firstly to model price of a seasonal product as a function $f_1(t)$ of time which decreases monotonically for a duration H_1 at the beginning of the season and reaches a minimum value $f_1(H_1)$. The price remains at this value $f_1(H_1)$ during a period H_2. Then it again follows an increasing function $f_2(t)$ and after a period H_3 it reaches the off season value, i.e., $f_1(0)=f_2(H_1+H_2+H_3)$.
- Secondly to model the season length $(H_1+H_2+H_3)$ as imprecise parameter.
- Thirdly for such a realistic inventory model, rate of deterioration as increasing function of time which actually depends on the lifetime of the item.
- At length to introduce an approach which can deal with fuzzy optimization problem, without reducing the objective function to any deterministic form.

Here, inventory model for a deteriorating seasonal product is developed whose demand depends upon the unit cost of the product. Unit cost of the product is time dependent. During the beginning of the period as availability of the item gradually increases, unit cost decreases monotonically with time and reaches a constant value when availability of the item becomes stable. Unit cost remains constant until the items availability again decreases towards the end of the season. Then as availability decreases, unit cost gradually increases and reaches its value as it was at the beginning of the season and then the season ends. Here exponential increasing and decreasing rate of unit cost function is considered. Rate of deterioration $\theta$of the item increases with time and is of the form $\theta=[1/(1+R-t)]$, where R is the lifetime of the product, t is the time passed after the arrival of the units in the inventory.

Clearly as $t \to R, \theta \to 1$, i.e., when t=R, all units in the inventory will be spoiled. It is assumed that time horizon of the season is fuzzy in nature. In fact three parts in which unit cost function can be divided are considered as fuzzy number. The model is formulated to maximize the total proceeds out of the system which is fuzzy in nature. As the optimization of fuzzy objective is not well defined, optimistic/pessimistic return of the objective function (using possibility/necessity measure of the fuzzy event) is optimized. A fuzzy simulation process is proposed to evaluate this optimistic/pessimistic return. A genetic algorithm (GA) is developed based on entropy theory where region of the search space gradually decreases to a small neighbourhood of the optima. This is named as region reducing genetic algorithm (RRGA) and simulation based region reducing genetic algorithm, called fuzzy simulation based region reducing genetic algorithm (FSRRGA) is developed to solve the fuzzy objective value. The models are illustrated with some numerical examples and some sensitivity analyses have been presented.

2.2.2 Definitions and Preliminaries

2.2.2.1 Possibility/Necessity in fuzzy environment

Any fuzzy number $\tilde{a}$ of $\Re$ (where $\Re$ represents set of real numbers) with membership function $\mu_{\tilde{a}} : \Re \to [0,1]$ is called a fuzzy number. Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers with membership functions $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$ respectively. Then according to Zadeh(1978), Dubois and Prade (1983) and Liu and Iwamura(1998a, 1998b):

$$pos(\tilde{a} * \tilde{b}) = \sup\{\min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)), x, y \in \Re,\ x * y\} \quad (1)$$

where abbreviation pos represents possibility and * is any one of the relations $<, >, =, \leq, \geq$. Analogously, if $\tilde{b}$ is a crisp number, say, b, then

$$pos(\tilde{a} * b) = \sup\{\mu_{\tilde{a}}(x),\quad x \in \Re,\ x * b\} \quad (2)$$

The necessity measure of an event $\tilde{a} * \tilde{b}$ is a dual of the possibility measure. The grade of an event is the grade of impossibility of the opposite event and is defined as:

$$nes(\tilde{a} * \tilde{b}) = 1 - pos(\overline{\tilde{a} * \tilde{b}}) \quad (3)$$

where the abbreviation nes represents the necessity measure and $\overline{\tilde{a} * \tilde{b}}$ represents the complement of the event $\tilde{a} * \tilde{b}$.

If $\tilde{a},\tilde{b} \in \Re$ *and* $\tilde{c} = f(\tilde{a},\tilde{b})$ where $f : \Re \times \Re \rightarrow \Re$ is binary operation then, the extension principle by Zadeh(1978), the membership function $\mu_{\tilde{c}}$ *of* $\tilde{c}$ is given by

$$\mu_{\tilde{c}}(z) = \sup\{\min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)),\ x,\ y \in \Re \text{ and } z = f(x, y), \forall z \in \Re\} \quad (4)$$

2.2.2.2 Triangular Fuzzy Number (TFN): A TFN $\tilde{a} = (a_1, a_2, a_3)$ (cf. Fig-1) has three parameters a_1, a_2, a_3 where $a_1 < a_2 < a_3$ and is characterized by the membership function $\mu_{\tilde{a}}(x)$, is given by

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, a_1 \le x \le a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, a_2 \le x \le a_3 \\ 0, \quad \text{otherwise} \end{cases}$$

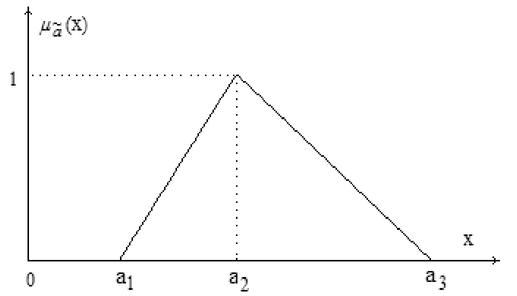


Fig-1: Membership function of a triangular fuzzy number

2.2.3. Optimization of fuzzy objective using possibility/necessity measure

A general single-objective unconstrained mathematical programming problem is of the following form:

$$\text{Max} \quad f(x, \xi)$$
$$\text{subject to } x \in X \quad (5)$$

where x is a decision vector, ξ is a vector of crisp parameters, $f(x, \xi)$ is the return function, X is the search space. In the above problem when ξ is a fuzzy vector $\tilde{\xi}$, then return function f

(x, $\tilde{\xi}$) becomes imprecise in nature. In that case the statement maximize $f(x, \tilde{\xi})$ is not well defined. In that situation one can maximize the optimistic (pessimistic) return zcorresponding to the objective function using possibility (necessity) measure of the fuzzy event $\{\tilde{\xi} \mid f(x, \tilde{\xi}) \geq z\}$ as suggested by Liu and Iwamura (1998a, 1998b), Maiti and Maiti (2006). So when ξ is a fuzzy vector one can convert the above problem (5) to the following equivalent possibility/necessity constrained programming problem (analogous to the chance constrained programming problem).

$$\begin{array}{ll} \max & z \\ \textit{subject to} & pos / nes\{\tilde{\xi} : f(x, \tilde{\xi}) \geq z\} \geq \beta \\ & x \in X \end{array} \qquad (6)$$

where $\beta$is the predetermined confidence level for fuzzy objective, pos{.} nes{.}denotes the possibility (necessity) of the event in {.}. Here the objective value z should be the maximum that the objective function $f(x, \tilde{\xi})$ achieves with at least possibility (necessity) β, in optimistic (pessimistic) sense.

2.2.4 Fuzzy simulation

The basic technique to deal problem (6) is to convert the possibility/necessity constraint to its deterministic equivalent. However, the procedure is usually very hard and successful in some particular cases (Maiti and Maiti, 2006). Liu and Iwamura (1998a,1998b) proposed fuzzy simulation process to determine optimum value of z for the problem (6) under possibility measure of the event $\{\tilde{\xi} \mid f(x, \tilde{\xi}) \geq z\}$. Following Liu and Iwamura (1998b) two algorithms are developed to determine z in (6) and are presented below.

Algorithm 1 Algorithm to determine z, for problem (6) under possibility measure of the event

$\{\tilde{\xi} \mid f(x, \tilde{\xi}) \geq z\}$

1. Set $z = -\infty$.

2. Generate ξ_0 uniformly from the β cut set of fuzzy vector $\tilde{\xi}$.
3. If $z < f(x, \xi_0)$ then set $z = f(x, \xi_0)$.

4. Repeat Steps 2 and 3, N times, where N is a sufficiently large positive integer.

5. Return z.

6 End algorithm.

We know that $nes\{\tilde{\xi} \mid f(x, \tilde{\xi}) \geq z\} \geq \beta \Rightarrow pos\{\tilde{\xi} \mid f(x, \tilde{\xi}) < z\} < 1-\beta$. Now roughly find a point ξ_0 from fuzzy vector ξ, which approximately minimizes f. Let this value be z_0 and ε be a positive number. Set $z = z_0 - \varepsilon$ and if $pos\{\tilde{\xi} \mid f(x, \tilde{\xi}) < z\} < 1-\beta$ then increase z with ε. Again check $pos\{\tilde{\xi} \mid f(x, \tilde{\xi}) < z\} < 1-\beta$ and it continues until $pos\{\tilde{\xi} \mid f(x, \tilde{\xi}) < z\} \geq 1-\beta$. At this stage decrease value of ε and again tries to improve z.

When ε becomes sufficiently small then we stop and final value of z is taken as value of z. Using this criterion, Algorithm 2 is developed.

Algorithm 2 Algorithm to determine z, for problem (6) under necessity measure of the event

$\{\tilde{\xi} \mid f(x, \tilde{\xi}) \geq z\}$

1. Set $z = z_0 - \varepsilon$, F $= z_0 - \varepsilon$, $F_0 = z_0 - \varepsilon$.

2 .Generate ε_0 uniformly from the $1-\beta$ cut set of fuzzy vector $\tilde{\xi}$.

3 .If $f(x, \tilde{\xi}_0) < z$.

4 .then go to Step 10.

5. End If

6. Repeat Step 2 to Step 5 N times

7. Set $F = z$.

8. Set $z = z + \varepsilon$.

9. Go to Step 2.

10. If($z = F$) //In this case optimum value of $z < z_0 - \varepsilon$

11. Set $z = z_0 - \varepsilon$, $F = F - \varepsilon$, $F_0 = F_0 - \varepsilon$.

12. Go to Step 2

13. End If

14. If (ε <*tol*)

15. go to Step 20

16. End If

17. $\varepsilon = \varepsilon/N$

18. $z = F + \varepsilon$

19. Go to Step 2.

20. Output F.

2.2.5 Fuzzy simulation-based region reducing genetic algorithm

GAs are exhaustive search algorithms based on the mechanics of natural selection and genesis (crossover, mutation, etc.) and have been developed by Holland, his colleagues and students at the University of Michigan (Goldberg (1989)). Because of its generality and other advantages over conventional optimization methods it has been successfully applied to different decision making problems (Zegordi*et al*.(2010), Simon *et al*.(2011), Das et al., (2012), Maiti *et. al.* (2014) and others). Generally a GA starts with a single population (Goldberg (1989), Michalewicz (1992)), randomly generated in the search space. Consequently they are easily trapped into local optima of the objective function. This difficulty is mainly due to the premature loss of diversity of the population during the search. To overcome this difficulty, Bessaou and Siarry (2001) propose a GA where initially more than one population of solutions are generated. Genetic operations are done on every population a finite number of times to find a promising zone of optimum solution. Finally a population of solutions is generated in this zone and genetic operations are performed on this population a finite number of times to get a final solution. Again the convergence towards the global optima of a GA, operating with a constant probability of crossover p_c, is ensured if the probability of mutation $p_m(k)$ follows a given decreasing law, in function of the generation number k (Davis and Principe, 1991). Following Bessaou and Siarry (2001) a GA is developed using them entropy generated from information theory, where promising zone is gradually reduces to a small neighbourhood of the optimal solution. In the algorithm any possibility constraint on objective function is checked via fuzzy simulation technique. This algorithm is named as FSRRGA and is used to solve our models. The algorithm is given below:

Algorithm 3 FSRRGA algorithm

1. Initialize probability of crossover $p_c$and probability of mutation p_m.

2. Set iteration counter T = 0.

3. Generate M sub-populations of solutions, each of order N (i.e., each sub-population contains N solutions), from search space of optimization problem under consideration, such that the diversity among the solutions of each population is maintained. Diversity is maintained using the entropy originating from information theory [cf., § 5.1-(b)]. Solutions for each of the population are generated randomly from the search space in such a way that the constraints of the problem are satisfied. Possibility constraints are checked using the algorithms of Section

2.2.4. Let P_1, P_2, … $P_M$be these populations.

4. Evaluate fitness of each solution of every population.

5. Repeat

A. Do for each sub-populations P_i.

a. Select N solutions from $P_i$for mating pool using Roulette-wheel selection process (Michalewicz, 1992) (These N solutions may not be distinct.

Solution with higher fitness value may be selected more than once). Let this set be P_i^1.

b. Select solutions from P_i^1, for crossover and mutation depending on $p_c$and p_m respectively.

c. Make crossover on selected solutions for crossover.

d. Make mutation on selected solutions for mutation.

e. Evaluate fitness of the child solutions.

f. Replace the parent solutions with the child solutions.

g. Replace $P_i$with P_i^1

B. End Do

C. Reduce probability of mutation p_m.

6. Until number of generations < Maxgen1, where Maxgen1 represents the maximum number of generations to be made on initial populations.

7. Select optimum solutions from each sub-populations and S* be the best among these solutions.

8. Select a neighbourhood V(T) of S*

9. Repeat

a. Generate a population of solutions of size N in V(T). Let it be P.

b. Evaluate fitness of each solutions.

c. Initialize probability of mutation p_m.

d. Repeat

(i) Select N solutions from P for mating pool using Roulette-wheel selection process. Let this set be P^1.

(ii) Select solutions from P^1 for crossover and mutation depending on $p_c$and p_m respectively.

(iii) Make crossover on selected solutions for crossover.

(iv) Make mutation on selected solutions for mutation.

(v) Evaluate fitness of the child solutions.

(vi) Replace the parent solutions with the child solutions.

(vii) Replace P with P^1.

(viii) Reduce probability of mutation p_m.

e. Until number of generations < Maxgen2, where Maxgen2 represents the maximum number of generations to be made on this population.

f. Update S* by the best solution found.

g. Reduce the neighbourhood V(T).

h. Increment T by 1.

10. Until T< Maxgen3, where Maxgen3 represents the maximum number of times for which the search space to be reduced.

11. Output S^*.

2.2.6 FSRRGA procedures for the proposed model

a. ***Representation***: A '*K*-dimensional real vector' $X_{li}= (x_{li1}, x_{li2}, ...x_{liK})$ is used to represent i^{th} solution in l^{th} population, where x_{li1}, x_{li2}, ... x_{liK}represent different decision variables of the problem such that constraints of the problem are satisfied.

b. ***Initialization***: At this step *M* sub-populations, each of size *N* are randomly generated in the search space in such a way that diversity among the solutions of each of the populations is maintained and the constraints of the problem are satisfied. Possibility constraints are checked using the algorithms of Section 4.1. Let X_{l1}, X_{l2}, ...X_{lN}, are the solutions of l^{th} population P_l, l = 1, 2, ...*M*. Diversity can be maintained using the entropy originating from information theory. Entropy of j^{th} variable for the l^{th}population $P_l$can be obtained by the formula:

$$E_j(P_l) = \sum_{i=1}^{N} \sum_{k=i+1}^{N} -p_{ik} \log(p_{ik})$$

where p_{ik} represents the probability that the value of j^{th} variable of i^{th} solution (x_{lij}) is different from the one of the j^{th} variable of the k^{th} solution (x_{lkj}) and is determined by the formula:

$$p_{ik} = 1 - \frac{|X_{lij} - X_{lkj}|}{U_j - L_j}$$

Where [L_j, U_j] is the variation domain of the j^{th} variable. The average entropy $E(P_l)$ of the l^{th} subpopulation $P_l$is taken as the average of the entropies of the different variables fo the population, i.e.,

$$E(P_l) = \frac{1}{k} \sum_{j=1}^{K} E_j(P_l)$$

It is clear that if $P_l$is made-up of same solutions, then $E(P_l)$ vanishes and more varied the solutions, higher the value of $E(P_l)$ and the better is its quality. So to maintain diversity, every time a new solution is randomly generated for $P_l$from the search space, the entropy between this one and the previously generated individuals for $P_l$is calculated. If this value is higher than a fixed threshold E_0, fixed from the beginning, the current chromosome is accepted. This process is repeated until *N* solutions are generated. Following the same procedure all the sub-populations P_l,

$l = 1, 2, ..M$ are generated. This solution sets are taken as initial sub-populations.

c. ***Fitness value:***Value of the objective function due to the solution X_{ij}(j^{th} solution in i^{th} population), is taken as fitness of X_{ij}. Let it be $f(X_{ij})$. Objective function is calculated using Algorithm 2 of Section 4.1.

d. ***Selection process for mating pool****:* The following steps are followed for this purpose:

1. For each population P_i, find total fitness of the population $F_i = \sum_{j=1}^{N} f(X_{ij})$
2. Calculate the probability of selection $pr_{ij}$of each solution $X_{ij}$by the formula $pr_{ij}=f(X_{ij})/F_i$.
3. Calculate the cumulative probability $qr_{ij}$for each solution $X_{ij}$by the formula $qr_{ij} = \sum_{k=0}^{j} pr_{ik}$
4. Generate a random number 'r' from the range [0, 1].
5. If $r < qr_{i1}$ then select X_{i1} otherwise select X_{ij}($2 \le j \le N$) where $qr_{ij-1} \le r < qr_{ij}$.

6. Repeat Step 4 and 5 *N* times to select N solutions for mating pool. Clearly one solution may be selected more than once.

7. Selected solution set is denoted by P_i^1 in the proposed FSRRGA algorithm.

e. ***Crossover:***

1. **Selection for crossover**: For each solution of P_i^1 generate a random number r from the range [0, 1]. If $r < p_c$ then the solution is taken for crossover, where p_c is the probability of crossover.

2. **Crossover process**: Crossover taken place on the selected solutions. For each pair of coupled solutions Y_1, Y_2 a random number c is generated from the range [0, 1] and Y_1, Y_2 are replaced by their offspring's Y_{11} and Y_{21} respectively where $Y_{11} = cY_1 + (1 - c)Y_2$, $Y_{21} = cY_2 + (1 - c)Y_1$.

f. ***Mutation:***

1. **Selection for mutation**: For each solution of P_i^1 generate a random number r from the range [0, 1]. If $r < p_m$ then the solution is taken for mutation, where p_m is the probability of mutation.

2. **Mutation process**: To mutate a solution $X_{li} = (x_{li1}, x_{li2}, \ldots x_{lik})$ select a random integer r in the range [1, *k*]. Then replace x_{ijr} by randomly generated value within the boundary of r^{th} component of X_{ij}.

g. ***Reduction process of* p_m:** Let $p_m(0)$ is the initial value of p_m. $p_m(T)$ is calculated by the formula $p_m(T) = p_m(0)\exp(-T/\alpha)$, where α is calculated so that the final value of p_m is small enough (10^{-3} in our case). So, $\alpha = Maxgen1/\log\left[\frac{p_m(0)}{10^{-3}}\right]$ for the population P_i, i=1,2,….M and $\alpha = Maxgen2/\log\left[\frac{p_m(0)}{10^{-3}}\right]$ for the population P(T) in the promising zone.

h. ***Reduction process of neighbourhood:*** V(0) is the initial neighbourhood of S*. V(T) is calculated by the formula $V(T) = V(0)\exp(-T/\alpha)$, where α is calculated so that the final neighbourhood is small enough(10^{-2} in our case). So $\alpha = Maxgen3/\log\left[\frac{V(0)}{10^{-2}}\right]$

2.2.7. Assumptions and notations for the proposed model

The following notations and assumptions are used in developing the model.

Notations

c_h holding cost per unit/unit time.

H time horizon.

$p(t)$ purchase cost per unit.

$s(t)$ selling price per unit .

$\theta(t)$ deterioration rate

C0 ordering cost.

$Q(T_i)$ order quantity at $t=T_i$.

$q(t)$ inventory level at time t.

Z total profit from the planning horizon H.

$D(t)$ Demand per unit time.

n_1, n_2, n_3 number of replenishment made during $(0, H_1)$, (H_1, H_1+H_2), $(H_1+H_2, H_1+H_2+H_3)$ respectively.

m_1, m_2, m_3 mark up of purchasing cost during $(0, H_1)$, (H_1, H_1+H_2), $(H_1+H_2, H_1+H_2+H_3)$ respectively.

R maximum lifetime of the product.

t_1 first cycle length over the time interval $(0, H_1)$.

$t_1^{/}$ initial cycle length over the time interval $(H_1+H_2, H_1+H_2+H_3)$.

T_i Total time elapses upto and including i^th cycle ($i=1,2,\ldots\ldots.. n_1+n_2+n_3$)

Assumptions

(i) Inventory system involves only one item.

(ii) Time horizon (H) is finite and $H=H_1+H_2+H_3$.

(iii) Shortages are not allowed.

(iv) Unit cost, i.e., purchase price p(t) is a function of t and is of the form

$$p(t)=\begin{cases} be^{-ct} & for\ 0\le t\le H_1 \\ be^{-cH_1} & for\ H_1\le t\le H_1+H_2 \\ Ae^{\frac{cH_1(t-H_1-H_2)}{H_3}} & for\ H_1+H_2\le t\le H_1+H_2+H_3 \end{cases} \tag{7}$$

where A=be^{-cH_1}

(v) Selling price $s(t)$ is mark-up m of $p(t)$ and m takes the values m_1, m_2 and m_3 during $(0,H_1)$, (H_1, H_1+H_2) and $(H_1+H_2, H_1+H_2+H_3)$ i.e. $s(t)=m[=m_1, m_2,m_3]\ p(t)$.

(vi) Demand is a function of selling price $s(t)$ and is of the form $D(t)=\frac{D_0}{[s(t)]^\gamma}=\frac{D_1}{[p(t)]^\gamma}\ where\ D_1=\frac{D_0}{m^\gamma}$, $D_0>0$

(vii) The lead time is zero.

(viii) Deterioration rate $\theta(t)$ is a function of time where $\theta(t)=\frac{1}{1+R+T_{j-1}-t}$ where R is the maximum lifetime of the product. This form of deterioration comes from the fact that as $(t-T_{j-1})\rightarrow R$, $\theta(t)\rightarrow 1$ i.e. rate of deterioration tends to 100%.

(ix) T_i is the total time that elapses up to and including the i-th cycle ($i=1,2,\ldots,n_1+n_2+n_3$) where $n_1+n_2+n_3$ denotes the total number of replenishment to be made during the interval $(0, H_1+H_2+H_3)$ and $T_0=0$.

(x) n_1 is the number of replenishment to be made during $(0,H_1)$ at $t=T_0, T_1,\ldots, T_{n_1-1}$ So, there are n_1 cycles in this duration. As purchase cost decreases during this session, so demand increases. Hence, successive cycle length must decrease. Here, α is the rate of reduction of successive cycle length and t_1 is the first cycle length. So, i-th cycle length $t_i=t_1-(i-1)\alpha$.

$$T_i=\sum_{j=1}^{i}t_j=it_1-\alpha\frac{i(i-1)}{2},\ i=1,2,\ldots,n_1. Clearly, T_{n_1}=H_1$$

$$Thus,\ n_1t_1-\alpha\frac{n_1(n_1-1)}{2}=H_1$$
$$\Rightarrow\alpha=\frac{2(n_1t_1-H_1)}{n_1(n_1-1)} \qquad (8)$$

Here, t_1 is decision variable.

(xi) n_2 be the number of replenishment to be made during (H_1, H_1+H_2). Since purchase cost is constant, demand is also constant during this interval. So, all the sub-cycle length in this interval is assumed as constant. Replenishment are done at

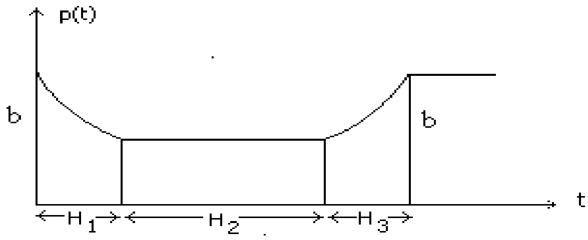
$$t=T_{n_1},T_{n_1+1},\ldots,T_{n_1+n_2-1}\ where\ T_{n_1+j}=T_{n_1}+(j-1)\frac{H_2}{n_2},\ j=1,2,\ldots,n_2$$

(xii) n_3 is the number of replenishment to be made during $(H_1+H_2, H_1+H_2+H_3)$. During this interval, purchase cost increases, as a result demand decreases. So, the duration of placing of order gradually increases. Here, β be the rate of increase of cycle length. Let $t_1^{/}$ be the initial cycle length. Then i-th cycle length $t_i^{/}=t_1^{/}+(i-1)\beta$. $Thus,\ t^{/}_{n_3}=t_1^{/}+(n_3-1)\beta.$ Orders are made at $t=T_{n_1+n_2},T_{n_1+n_2+1},\ldots,T_{n_1+n_2+n_3-1}$ Where

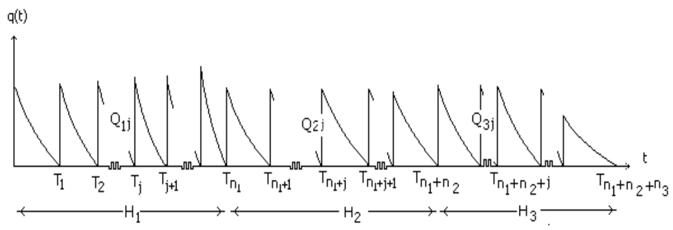
$$T_{n_1+n_2+i}=T_{n_1+n_2}+\sum_{j=1}^{i}t^{/}_{j}=H_1+H_2+it_1^{/}+\beta\frac{i(i-1)}{2}$$
$$Clearly,\ T_{n_1+n_2+n_3}=H_1+H_2+H_3$$

$$H_1 + H_2 + n_3 t'_1 + n_3(n_3-1)\beta/2 = H_1 + H_2 + H_3$$
$$\Rightarrow \beta = \frac{2(H_3 - n_3 t'_1)}{n_3(n_3-1)} \tag{9}$$

A wavy bar (~) is used with this symbol to represent corresponding fuzzy numbers when required.



Pictorial representation of p(t)
Fig-2



Inventory situation of the model
Fig-3

2.2.8 Model development and analysis

In the development of the model, it is assumed that at the beginning of every *j*-th cycle $[T_{j-1}, T_j]$, an amount $Q1_j$ units of item is ordered. As lead time negligible, replenishment of an item occurs as soon as order is made. Item is sold during the cycle and inventory level reaches zero at time $t=T_j$. Then order for next cycle is made. Here, selling price is a markup of initial purchase cost for each cycle. The inventory situation and the purchase cost are shown in Fig-2 and Fig-3.

2.2.8.1Formulation of the model in crisp environment

This part is formulated in three phases.

2.2.8.1.1 Formulation for first phase (i.e., $0 \le t \le H_1$**):** Duration of *j*-th ($1 \le j \le n_1$) cycle is $[T_{j-1}, T_j]$ where $T_{j-1} = jt_1 - \alpha j(j-1)/2$ at the beginning of the cycle inventory level is $Q1_j$. So, the governing differential equation of the model in the presence of deterioration of the item during $T_{j-1} \le t \le T_j$ is given by

$$\frac{dq(t)}{dt} + \theta(t)\, q = -D_j \tag{10}$$

where $D_j = \frac{D_1}{\left(m_1 b e^{-cT_{j-1}}\right)^{\gamma}}$ and $\theta(t) = \frac{1}{1+R+T_{j-1}-t}$

Solving the above differential equation using the initial condition at t=T_j, q(t)=0, we get

$$q(t) = (1+R+T_{j-1}-t)D_j \log\left(\frac{1+R+T_{j-1}-t}{1+R+T_{j-1}-T_j}\right) \tag{11}$$

$$\text{When } t = T_{j-1},\ Q1_j = q(T_{j-1}) = (1+R)D_j \log\left(\frac{1+R}{1+R+T_{j-1}-T_j}\right) \tag{12}$$

So, the holding cost for jth ($1 \le j \le n_1$) cycle, $H1_j$ is given by

$$H1_j = c_h \int_{T_{j-1}}^{T_j} q(t)\, dt$$

$$= c_h D_j \left[\frac{1}{4}\left\{\left(1+R+T_{j-1}-T_j\right)^2 - \left(1+R\right)^2\right\} + \frac{(1+R)^2}{2} \log\left(\frac{1+R}{1+R+T_{j-1}-T_j}\right)\right]$$

Thus, the total holding cost during (0, H_1), HOC1, is given by $HOC1 = \sum_{j=1}^{n_1} H1_j$ (13)

Total purchase cost during (0, H_1), PC1, is given by

$$PC1 = \sum_{j=1}^{n_1}\left[Q1_j\, p(T_{j-1})\right]$$

$$= \sum_{j=1}^{n_1}\left[(1+R)D_j \log\left(\frac{1+R}{1+R+T_{j-1}-T_j}\right) p(T_{j-1})\right] \tag{14}$$

where $p(T_{j-1}) = b e^{-cT_{j-1}}$

Total ordering cost during *(0 ,H_1),* OC1, is given by $OC1 = \sum_{j=1}^{n_1}\left[c_{o1} + c_{o2} Q1_j\right]$ (15)

where $Q1_j$ is given by (12)

Selling price for j-th ($1 \le j \le n_1$) cycle $SP1_j$, is given by $SP1_j = m_1 p(T_{j-1}) \int_{T_{j-1}}^{T_j} D_j \, dt$

$$= m_1 p(T_{j-1}) D_j (T_j - T_{j-1})$$

Total selling price during *(0, H₁),* SP1, is given by $SP1 = \sum_{j=1}^{n_1} SP1_j$ (16)

2.2.8.1.2 Formulation of second phase (i.e., $H_1 \le t \le H_1 + H_2$): In the second phase, the purchase price of an item remains constant. So, the demand of customer is taken as constant. During of j-th

($n_1 \le j \le n_1 + n_2$) cycle is *$[T_{j-1}, T_j]$*. The governing differential equation of the model of deteriorating item during $T_{j-1} \le t \le T_j$ is given by

$$\frac{dq(t)}{dt} + \theta(t)\, q = -D_j \tag{17}$$

where $D_j = \dfrac{D_1}{\left(m_2 b e^{-cH_1}\right)^{\gamma}}$ and $\theta(t) = \dfrac{1}{1 + R + T_{j-1} - t}$

Solving the above differential equation using the initial condition $t=T_j$, $q(t)=0$, we get

$$q(t) = (1 + R + T_{j-1} - t) D_j \log\left(\frac{1 + R + T_{j-1} - t}{1 + R + T_{j-1} - T_j}\right) \tag{18}$$

When $t = T_{j-1}$, $Q2_j = q(T_{j-1}) = (1+R) D_j \log\left(\dfrac{1+R}{1 + R + T_{j-1} - T_j}\right)$ (19)

So, the holding cost for j-th ($n_1 \le j \le n_1 + n_2$) cycle, $H2_j$, is given by

$$H2_j = c_h \int_{T_{j-1}}^{T_j} q(t)\, dt$$

$$= c_h D_j \left[\frac{1}{4}\left\{ \left(1 + R + T_{j-1} - T_j\right)^2 - \left(1+R\right)^2 \right\} + \frac{(1+R)^2}{2} \log\left(\frac{1+R}{1 + R + T_{j-1} - T_j} \right) \right]$$

Thus, the total holding cost during (H_1, H_1+H_2), $HOC2$, is given by $HOC2 = \sum_{j=n_1+1}^{n_1+n_2} H2_j$ (20)

Total purchase cost during (H_1, H_1+H_2), $PC2$, is given by

$$\begin{aligned} PC2 &= \sum_{j=n_1+1}^{n_1+n_2} \left[Q2_j\, p(T_{j-1}) \right] \\ &= \sum_{j=n_1+1}^{n_1+n_2} \left[(1+R) D_j \log\left(\frac{1+R}{1+R+T_{j-1}-T_j} \right) p(T_{j-1}) \right] \end{aligned} \quad (21)$$

where $p(T_{j-1}) = be^{-cH_1}$

Total ordering cost during (H_1, H_1+H_2), $OC2$, is given by $OC2 = \sum_{j=n_1+1}^{n_1+n_2} \left[c_{o1} + c_{o2} Q2_j \right]$ (22)

where $Q2_j$ is given by (19)

Selling price for j-th ($n_1 \le j \le n_1 + n_2$) cycle $SP2_j$, is given by $SP2_j = m_2 p(T_{j-1}) \int_{T_{j-1}}^{T_j} D_j \, dt$

$$= m_2 p(T_{j-1}) D_j (T_j - T_{j-1})$$

Total selling price during (H_1, H_1+H_2), $SP2$, is given by $SP2 = \sum_{j=n_1+1}^{n_1+n_2} SP2_j$ (23)

2.2.8.1.3. Formulation of third phase (i.e., $H_1 + H_2 \le t \le H_1 + H_2 + H_3$): In the second phase, duration of j-th $(n_1 + n_2 \le j \le n_1 + n_2 + n_3)$ cycle is $[T_{j-1}, T_j]$ where $T_j = H_1 + H_2 + (j - n_1 - n_2) t_1^{/} + (j - n_1 - n_2)(j - n_1 - n_2 - 1)\beta/2$ and at the beginning of cycle inventory level is $Q3j$. So, instantaneous state q(t) of deteriorating item during $T_{j-1} \le t \le T_j$ is given by

$$\frac{dq(t)}{dt} + \theta(t) q = -D_j \quad (24)$$

where $D_j = \dfrac{D_1}{\left(m_3 A e^{\frac{cH_1}{H_3}\left(T_{j-1} - H_1 - H_2\right)} \right)^{\gamma}}$, $\theta(t) = \dfrac{1}{1 + R + T_{j-1} - t}$ and $A = be^{-H_1}$

Solving the above differential equation using the initial condition t=T_j, q(t)=0, we get

$$q(t) = (1+R+T_{j-1}-t)D_j \log\left(\frac{1+R+T_{j-1}-t}{1+R+T_{j-1}-T_j}\right) \tag{25}$$

When $t = T_{j-1}$, $Q3_j = q(T_{j-1}) = (1+R)D_j \log\left(\frac{1+R}{1+R+T_{j-1}-T_j}\right)$ (26)

So, the holding cost for j-th ($n_1 + n_2 \le j \le n_1 + n_2 + n_3$) cycle, $H3_j$, is given by

$$H3_j = c_h \int_{T_{j-1}}^{T_j} q(t)\,dt$$

$$= c_h D_j \left[\frac{1}{4}\left\{(1+R+T_{j-1}-T_j)^2 - (1+R)^2\right\} + \frac{(1+R)^2}{2}\log\left(\frac{1+R}{1+R+T_{j-1}-T_j}\right)\right]$$

Thus, the total holding cost during ($H_1 + H_2 \le t \le H_1 + H_2 + H_3$), HOC3, is given by

$$HOC3 = \sum_{j=n_1+1}^{n_1+n_2} H3_j \tag{27}$$

Total purchase cost during ($H_1 + H_2 \le t \le H_1 + H_2 + H_3$), PC3, is given by

$$\begin{aligned} PC3 &= \sum_{j=n_1+n_2+1}^{n_1+n_2+n_3} \left[Q3_j\, p(T_{j-1})\right] \\ &= \sum_{j=n_1+n_2+1}^{n_1+n_2+n_3} \left[(1+R)D_j \log\left(\frac{1+R}{1+R+T_{j-1}-T_j}\right) p(T_{j-1})\right] \end{aligned} \tag{28}$$

where $p(T_{j-1}) = Ae^{\frac{cH_1}{H_3}(T_{j-1}-H_1-H_2)}$, $A = be^{-H_1}$

Total ordering cost during ($H_1 + H_2 \le t \le H_1 + H_2 + H_3$), *OC3*, is given by

$$OC3 = \sum_{j=n_1+n_2+1}^{n_1+n_2+n_3} \left[c_{o1} + c_{o2} Q3_j\right] \tag{29}$$

where $Q3_j$ is given by (26)

Selling price for j-th ($n_1 + n_2 \le j \le n_1 + n_2 + n_3$) cycle ,*SP3ⱼ*is given by $SP3_j = m_3 p(T_{j-1}) \int_{T_{j-1}}^{T_j} D_j \, dt$

$$= m_3 p(T_{j-1}) D_j (T_j - T_{j-1})$$

Total selling price during ($H_1 + H_2 \le t \le H_1 + H_2 + H_3$), *SP3,* is given by $SP3 = \sum_{j=n_1+n_2+1}^{n_1+n_2+n_3} SP3_j$ (30)

Thus, total profit Z, for this model over the planning horizon ($H_1 + H_2 + H_3$), is given by

Z=(SP1+SP2+SP3)-(PC1+PC2+PC3)-(HOC1+HOC2+HOC3)-(OC1+OC2+OC3) (31)

2.2.9 Mathematical Model in Crisp Environment: According to the above discussion, as lifetime of the product is R, so, no cycle should exceed R which implies $t_1 \le R, H_2 / n_2 \le R, t'_{n_3} \le R.$ Therefore, the problem reduces to determine the decision variables $t_1, t'_1, m_1, m_2, m_3, n_1, n_2$ *and* n_3. The problem becomes

Maximize Z

Subjectto $t_1 \le R, H_2 / n_2 \le R, t'_{n_3} \le R.$ (32)

This constrained optimization problem is solved using proposed RRGA for crisp objective function.

2.2.10. Mathematical Model in Fuzzy Environment: As discussed in introduction section, in real life phase intervals H_1, H_2 and H_3 are imprecise in nature i.e $\widetilde{H}_1, \widetilde{H}_2$ and $\widetilde{H}_3$ respectively, then the profit function Z reduces to the fuzzy number $\widetilde{Z}$ whose membership function is a function of the decision variables t_1, t'_1, m_1, m_2, m_3, n_1, n_2 and n_3. Also the last cycle length t'_{n3} becomes imprecise $\widetilde{t}'_{n3}$. So, the problem reduces to fuzzy optimization problem

Maximize $\widetilde{Z}$ (33)

subject to $t_1 \le R, \widetilde{H}_2 / n_2 \le R, \widetilde{t}'_{n3} \le R$

If $\widetilde{H}_1, \widetilde{H}_2$ and $\widetilde{H}_3$ are considered as TFNs *(H_{11}, H_{12} ,H_{13}), (H_{21}, H_{22} ,H_{23})* and *(H_{31}, H_{32} ,H_{33})* respectively, then $\widetilde{Z}$ becomes a TFN (Z_1, Z_2, Z_3), where Z_i=value of Z for H_1=H_{1i},

H_2=H_{2i}, H_3=H_{3i}, i=1,2,3. In this case $\tilde{t}'_{n3}$ also becomes a TFN (t'_{n31}, t'_{n32}, t'_{n33}) .So it is an obvious assumption that fuzzy constraints should necessarily hold. The problem reduces to

$$\text{Maximize } \tilde{Z}$$

$$\text{subject to } \quad t_1 \le R, H_{23}/n_2 \le R, t'_{n33} \le R \tag{34}$$

Since optimization of fuzzy number is not well defined one can optimize the optimistic (pessimistic) return of the fuzzy objective $\tilde{Z}$ with some degree of possibility (necessity) $\alpha_1(\alpha_2)$ as described in §2.1. Accordingly, in optimistic sense the problem reduces to

$$\text{Maximize } z_1$$

$$\text{subject to } pos\{Z \ge z_1\} \ge \alpha_1$$

$$\text{and } t_1 \le R, H_{23}/n_2 \le R, t'_{n33} \le R \tag{35}$$

and in pessimistic sense the problem reduces to

$$\text{Maximize } z_1$$

$$\text{subject to } \begin{array}{l} nes\{Z \ge z_1\} \ge \alpha_2 \\ i.e., pos\{Z \le z_1\} < 1-\alpha_2 \end{array}$$

$$\text{and } t_1 \le R, H_{23}/n_2 \le R, t'_{n33} \le R \tag{36}$$

This constraint optimization problem is solved using proposed FSRRGA.

2.2.11. Numerical Experiments

2.2.11.1 Results obtained for crisp environment: To illustrate the model following hypothetical set of data is used. This data set is taken for items like rice, potato, wheat, onion, cabbage, cauliflower, etc, whose demand exists in the market throughout the year. When new crops come in the market, then its price gradually decreases during some weeks (say H_1) and reaches a lowest level. This minimum price prevails for few weeks (say H_2). Then again it gradually increases during few weeks (say H_3) and reaches its normal value. This normal price prevails remaining part of the year. For an item like potato, values of H_1, H_2 and H_3 are about 5 weeks, 15 weeks, 7 weeks in the state of West Bengal, India. Normal price of the item throughout the year is about \$3 for a 10 kg bag. Lowest price of it in the season is about \$2 for a 10 kg bag. Keeping this real situation, following data set is fixed to illustrate the model in crisp environment. In the data set 10 kg of the item is considered as one unit item, one week is considered as unit time and costs are represented in $.

b=10, c=0.2, H_1=5(weeks), H_2=(15 weeks), H_3=7(weeks), D_0=1500, γ=2.5, c_h=0.5, c_{01}=10, c_{02}=0.5, R=3.

For the above parametric values, results are obtained via RRGA and presented in Table-1.

Table-1

Results obtained for crisp model via RRGA

n_1	n_2	n_3	m_1	m_2	m_3	t_1	t'_1	Profit($)
3	13	4	2.432	2.380	2.577	2.051	1.408	280.981

For above parametric values, results are obtained for different values of γ and presented in Table-2. It is observed that as γincreases, profit decreases due to decrease of demand which agrees with reality. It is also found that as γ increases for same values of n_1, n_2 and n_3, t_1 increases but t'_1decreases. Moreover, m_1, m_2 and m_3 also decrease with increase ofγ. It happens because as γ increases demand decreases in each cycle and demand is minimum when purchase cost is maximum. According to assumption, purchase cost is maximum in first and last cycle of the whole planning horizon. As demand decreases length of first and last cycle increases as a result t_1 increases and t'_1 decreases. Again as demand decreases due to increase ofγ to keep the demand high markup of selling price m_1, m_2 and m_3 also decreases. From the table-2, it has been seen that the parameter γ is highly sensible. The observation is more practical and hence realistic one.

Table-2

Results obtained for crisp model due to different γ via RRGA

γ	n_1	n_2	n_3	m_1	m_2	m_3	t_1	t'_1	Profit($)
2.40	4	14	4	2.372	2.400	2.641	1.573	1.434	407.980
2.42	4	14	4	2.358	2.386	2.628	1.567	1.432	379.800
2.44	4	14	4	2.345	2.372	2.614	1.562	1.427	353.961
2.45	4	14	4	2.331	2.359	2.602	1.558	1.423	342.311
2.46	3	13	4	2.463	2.402	2.620	2.058	1.422	320.955
2.48	3	13	4	2.448	2.388	2.588	2.052	1.416	299.105
2.50	3	13	4	2.434	2.373	2.573	2.047	1.412	278.648
2.52	3	13	4	2.421	2.360	2.560	2.041	1.409	258.864

For the above parametric values, results are obtained for different values of R and presented in Table-3. It is observed that as R increases profit increases. It happens because increase of R, i.e., increase of lifetime of the product, decreases rate of deterioration which in turn increases profit.

Table-3

Results obtained for crisp model due to different R

R	n_1	n_2	n_3	m_1	m_2	m_3	t_1	t'_1	Profit($)
2.70	3	13	4	2.487	2.411	2.653	2.029	1.430	267.983
2.80	3	13	4	2.472	2.398	2.629	2.036	1.423	272.680
2.90	3	13	4	2.454	2.387	2.606	2.043	1.417	277.804
3.00	3	13	4	2.436	2.375	2.581	2.049	1.412	281.379
3.10	3	13	4	2.416	2.365	2.569	2.054	1.404	286.675
3.20	3	13	4	2.389	2.354	2.549	2.061	1.397	289.172
3.30	3	13	4	2.377	2.343	2.521	2.065	1.391	294.784
3.40	3	13	4	2.365	2.335	2.497	2.071	1.384	296.226

2.2.11.2 Results obtained for fuzzy environment: To illustrate the proposed inventory models, following input data are considered. In this case also hypothetical data set is used and source of this data has been discussed for crisp model. For crisp model it was considered that unit price of the item decreases during the period H_1=5 weeks, but in reality it is about 5 weeks which is fuzzy in nature. Due to this reason here H_1 is considered as TFN (4.75, 5, 5.2). Following the same argument other parameters are fixed and the data set are presented below. In the data set fuzzy numbers are considered as TFN types.

b=10, c=0.2, $\tilde{H}_1$=(4.75, 5, 5.2), $\tilde{H}_2$=(14.5,15, 15.4), $\tilde{H}_3$=(6.8, 7, 7.3), D_0=1500, γ=2.5, c_h=0.5, c_{01}=10, $\alpha_1 = 0.9, \alpha_2 = 0.1$, c_{02}=0.5, R=3.

For the above parametric values, results are obtained via FSRRGA in optimistic and pessimistic sense and presented in Table-4 and 5.

Table-4
Results obtained for fuzzy model via FSRRGA in optimistic sense

n_1	n_2	n_3	m_1	m_2	m_3	t_1	t'_1	Profit ($)
3	13	4	2.422	2.370	2.577	2.051	1.408	311.242

Table-5
Results obtained for fuzzy model via FSRRGA in pessimistic sense

n_1	n_2	n_3	m_1	m_2	m_3	t_1	t'_1	Profit ($)
4	13	4	2.430	2.380	2.587	2.156	1.439	245.644

From the Tables 6 and 7, it is observed that as the degree of acceptability (α_1) for optimistic sense increases, the profit decreases and the increase of degree of acceptability (α_2) for pessimistic sense brings down, the profit also decreases. All these observations agree with reality.

Table-6
Sensitivity analysis in optimistic sense

α_1	n_1	n_2	n_3	m_1	m_2	m_3	t_1	t'_1	Profit ($)
0.92	3	13	4	2.422	2.370	2.577	2.051	1.408	310.560
0.94	3	13	4	2.422	2.370	2.577	2.051	1.408	309.614
0.96	3	13	4	2.422	2.370	2.577	2.051	1.408	308.664
0.98	3	13	4	2.422	2.370	2.577	2.051	1.408	307.714
1.00	3	13	4	2.422	2.370	2.577	2.051	1.408	306.774

Table-7
Sensitivity analysis in pessimistic sense

α_2	n_1	n_2	n_3	m_1	m_2	m_3	t_1	t'_1	Profit ($)
0.12	4	13	4	2.430	2.380	2.587	2.156	1.439	244.804
0.14	4	13	4	2.430	2.380	2.587	2.156	1.439	243.974
0.16	4	13	4	2.430	2.380	2.587	2.156	1.439	243.144
0.18	4	13	4	2.430	2.380	2.587	2.156	1.439	242.304
0.20	4	13	4	2.430	2.380	2.587	2.156	1.439	241.494

For the above parametric values, results are obtained for different values of γ and presented in Table-8 . In this case also same trend of result is obtained as found in crisp model.

Table-8

Results obtained for fuzzy model due to different γ via FSRRGA

γ	n_1	n_2	n_3	m_1	m_2	m_3	t_1	t'_1	Profit ($)
2.40	4	14	4	2.307	2.405	2.653	1.569	1.421	446.593
2.42	4	14	4	2.294	2.391	2.637	1.571	1.417	416.973
2.44	4	14	4	2.281	2.378	2.620	1.576	1.415	388.757
2.46	3	13	4	2.458	2.406	2.607	2.040	1.413	359.851
2.48	3	13	4	2.444	2.394	2.591	2.044	1.409	334.977
2.50	3	13	4	2.431	2.380	2.577	2.050	1.407	311.285
2.52	3	13	4	2.416	2.368	2.564	2.058	1.405	288.713

For the above parametric values, results are obtained for different values of R and presented in Table-6. As expected in this case also same trend of result is obtained as in crisp model, i.e., profit increases with increase of R, which agrees in reality.

Table-9

Results obtained due to different R for fuzzy model via FSRRGA

R	n_1	n_2	n_3	m_1	m_2	m_3	t_1	t'_1	Z($)
2.90	3	13	4	2.494	2.392	2.652	2.046	1.411	306.583
3.00	3	13	4	2.431	2.380	2.577	2.050	1.407	311.285
3.10	3	13	4	2.412	2.369	2.555	2.055	1.402	315.735
3.20	3	13	4	2.394	2.359	2.537	2.059	1.394	319.966
3.30	3	13	4	2.379	2.350	2.520	2.062	1.390	323.993
3.40	3	13	4	2.364	2.340	2.502	2.066	1.387	327.826
3.50	3	13	4	2.351	2.376	2.486	2.071	1.379	330.440

2.2.12. Conclusion

Here, a real-life inventory model for deteriorating seasonal product is developed whose demand depends upon the unit cost of the product in fuzzy environment. Unit cost of product is time dependent. Lifetime of each item is finite and rate of deterioration depend on the age of the item. Unique contribution of the paper is fourfold:

- The model is developed for such items like food grains, pulses, potato, onion etc., whose stable demand exists in the market throughout the year but it fluctuates for a part of the year when they are produced in the field. Here modeling is done for such products during their season of grown. These items are normally stored in cold storage and when bought in the market items are fully deteriorated after a finite time R, which is considered here as lifetime of the product. For the best of author's knowledge none have considered this type of inventory model.
- Here for the first time unit cost of an item is modeled following real life situation, which gradually decreases with time during grown of the item in the field, then it retains the lowest value for a period and again gradually increases with time to normal price of the year. Though it is found for above mentioned items in every year, inventory practitioners overlooked this real life phenomenon.
- It is assumed that time horizon of the season is fuzzy in nature. For the first time season of an item is considered as a combination of three imprecise intervals. In fact three parts in which unit cost function can be divided are considered as fuzzy numbers, which agree with reality.
- As optimization of fuzzy objective is not well defined, optimistic/pessimistic return of the objective function (using possibility/necessity measure of the fuzzy event) is optimized. A fuzzy simulation based region reducing genetic algorithm is proposed to evaluate this fuzzy objective value.

At length, though the model is formulated in fuzzy environment, demand or lifetime/deterioration of the product is not considered as imprecise in nature, though it is appropriate for these types of products. In fact, consideration of fuzzy demand or deterioration the inventory model leads to fuzzy differential equation for formulation of the model. Using proposed solution approach one cannot consider imprecise demand which is the major limitation of the approach. So, further research work can be done incorporating fuzzy demand and or deterioration in the imprecise planning horizon. Though the model is presented in crisp environment and fuzzy, it can be formulated in stochastic, fuzzy-stochastic environment.

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Part-II

Inventory Models of Non-Deteriorating items in Fuzzy Environment

Chapter-3

Model-3.1: Fuzzy economic productionlot-size model under imperfect production process with cloudy fuzzy demand rate

3.1.1 Introduction

In the development of economic production lot-size model, usually researchers consider the demand rate as constant in nature. In the real world, it is observed that these quantities will have little changes from the exact values. Thus in practical situations, demand variable should be treated as fuzzy in nature. Recently fuzzy concept is introduced in the production/ inventory problems. At first, Zadeh (1965) introduced the fuzzy set theory. After that, it has been applied by Bellman and Zadeh (1970) in decision making problems. Numerous researches have been done in this area. Researchers like Kaufmann and Gupta(1992), Mandal and Maiti (2002), Maiti et.al (2014), Maiti and Maiti(2006,2007), Bera and Maiti (2012), Mahata and Goswami(2007, 2013), De and Sana(2015) etc. have investigated extensively over this subject. Kau and Hsu(2002) developed a lot-size reorder point inventory model with fuzzy demands. In this study, a cloudy fuzzy inventory model is developed depending upon the learning from past experience. In defuzzification methods, specially on ranking fuzzy numbers, after Yager (1981), some researchers like Ezzati et at. (2012), Deng (2014), Zhang et al. (2014) and others adopted the method for ranking of fuzzy numbers based on centre of gravity. Moreover, De and Beg (2016) and De and Mahata (2016) invented new defuzzification method for triangular dense fuzzy set and triangular cloudy fuzzy set respectively. In this model, fuzziness depends upon time. As the time progress, fuzziness become optimum at the optimum time. This idea is incorporated in cloudy fuzzy environment. ***Till now***, n***one has addressed this type of realistic production inventory model with cloudy fuzzy demand rate***.

In the classical economic production lot-size (EPL) model, the rate of production of single item or multiple items is assumed to be inflexible and predetermined. However, in reality, it is observed that the production is influenced by the demand. When the demand increases, consumption by the customer obviously more and to meet the additional requirement of the customer, the manufactures bound to increase their production. Converse is true for reverse situation. In this connection, several researchers developed EPL models for single/multiple items considering either uniform or variable production rate (depend on time, demand and/or on hand inventory level). Bhunia and Maiti (1997), Balkhi and Benkherouf (1998), Abad (2000), Mandal and Maiti (2000) etc. developed their inventory models considering either uniform or variable production rate. However, manufacturing flexibility has become more important factor in inventory management. Different types of flexibility in manufacturing system have been identified in the literature among which volume flexibility is the most important one. Volume flexibility of a manufacturing system is defined as its ability to be operated profitably at different output levels. Cheng (1989) first developed the demand dependent production unit cost in EPQ model; Khouja (1995) introduced volume flexibility and reliability consideration in EPQ model. Shah and shah (2014) developed EPQ model for time declining demand with imperfect production process under inflationary conditions and reliability.

Items are produced using conventional production process with a certain level of reliability. Higher reliability means that the products with acceptable quality are more consistently produced by the process reducing the cost of scraps, rework of substandard products, wasted materials, labor hours etc. A considerable number of research paper have been done on imperfect production by Rosenblatt and Lee(1986), Ben-Daya and Hariga(2000), Goyal et al. (2003), Maiti et al. (2006), Sana et al. (2007), Manna et al. (2014), Pal et al. (2014), etc. Recently, Manna et al. (2016) considered multi-item EPQ model with learning effect on imperfect production over fuzzy random planning horizon. Khara et al. (2017) developed an inventory model under development dependent imperfect production and reliability dependent demand.

Use of soft computing techniques for inventory control problems is a well established phenomenon. Several authors use Genetic Algorithm (GA) in different forms to find marketing decisions for their problems. Pal *et al*. (2009) uses GA to solve an EPQ model with price discounted promotional demand in an imprecise planning horizon. Bera and Maiti (2012) used GA to solve multi-item inventory model incorporating discount. Maiti*et al*.(2009) used GA to solve inventory model with stochastic lead time and price dependent demand incorporating advance payment.Mondal and Maiti (2002), Maiti(2006,2007), Maiti et.al (2014) many other researchers uses GA in inventory control problems. Also, Bhunia and Shaikh (2015) used PSOto solve two-warehouse inventory model for deteriorating item under permissible delay in payment. ***Here, dominance based particle swarm optimization has been developed to solve this fuzzy inventory model.***

Here, fuzzy inventory model under imperfect production process with cloudy fuzzy demand rate is developed where production rate is demand dependent. The model is solved in crisp , general fuzzy and cloudy fuzzy environment using Yager's index method and De and Beg's ranking index method for defuzzification and compare the results obtained in crisp, fuzzy and cloudy fuzzy environment. In this study, objective is to minimize average total cost to obtain the optimum order quantity and the cycle time using dominance based Particle Swarm Optimization (PSO) algorithmto find decision for the decision maker (DM). The model is illustrated with some numerical examples and some sensitivity analyses have been presented.

3.1.2 Definitions and Preliminaries

3.1.2.1Normalized General Triangular Fuzzy Number (NGTFN):

A NGTFN $\tilde{A}=(a_1,a_2,a_3)$ (cf. Fig-1) has three parameters a_1, a_2, a_3 where $a_1<a_2<a_3$ and is characterized by its continuous the membership function $\mu_{\tilde{A}}(x): X \rightarrow [0,1]$, where X is the set and x∈X, is defined by

$$\mu_{\tilde{a}}(x)=\begin{cases}\dfrac{x-a_1}{a_2-a_1}, a_1 \le x \le a_2 \\ \dfrac{a_3-x}{a_3-a_2}, a_2 \le x \le a_3 \\ 0, \quad otherwise\end{cases} \tag{1}$$

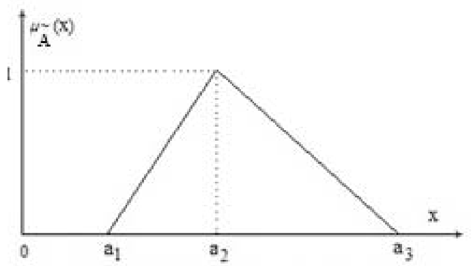

Fig-1: Membership function of a triangular fuzzy number

3.1.2.2 α -Cut of a fuzzy number:

A α cut of a fuzzy number $\tilde{A}$ in X is denoted by A_α and is defined as crisp set A_α= $\{x:\mu_{\tilde{A}}(x)\geq\alpha, x\in X\}$ *where* $\alpha\in[0,1]$. Here, A_α is a non-empty bounded closed interval contained in X and it can be denoted by $A_\alpha=[A_L(\alpha),A_R(\alpha)]$ where $A_L(\alpha)=a_1+(a_2-a_1)\alpha$ is alled left α -cut and $A_R(\alpha)=a_3-(a_3-a_2)\alpha$ is called the right α -cut of $\mu_{\tilde{A}}(x)$ respectively.(2)

3.1.2.3 Yager's Ranking Index:

If $A_L(\alpha)$ *and* $A_R(\alpha)$ be the left and right α cuts of a fuzzy number $\tilde{A}$ then the Yager's Ranking index is computed for defuzzification as

$$I(\tilde{A})=\frac{1}{2}\int_0^1[A_L(\alpha)+A_R(\alpha)]d\alpha=\frac{1}{4}(a_1+2a_2+a_3) \qquad (3)$$

Also, the degree of fuzziness (d_f) is defined by the formula $d_f=\frac{U_b-L_b}{m}$ where U_b and L_b are the upper and lower bounds of the fuzzy numbers respectively and m being their respective mode.

3.1.2.4 Cloudy Normalized Triangular Fuzzy Number (CNTFN) (De and Beg (2016)):

After infinite time, the normalized triangular fuzzy number $\tilde{A}=(a_1,a_2,a_3)$ becomes a crisp singleton then fuzzy number $\tilde{A}=(a_1,a_2,a_3)$ is called the cloudy fuzzy number. This means that both $a_{1,}\ a_3\rightarrow a_2$ *as* $t\rightarrow\infty$.

So, the cloudy fuzzy number takes the form $\tilde{A}=(a_2(1-\frac{\rho}{1+t}),\ a_2, a_2(1+\frac{\sigma}{1+t}))$ for $0<\rho,\sigma<1$ (4)

It is to be noted that $\lim_{t\to\infty}(1-\frac{\rho}{1+t})a_2=a_2$ *and* $\lim_{t\to\infty}(1+\frac{\sigma}{1+t})a_2=a_2$. So, $\tilde{A}\rightarrow\{a_2\}$

Its membership function becomes a continuous function of x and t , defined by

$$\mu(x,t)=\begin{cases}\dfrac{x-a_2(1-\frac{\rho}{1+t})}{\frac{a_2\rho}{1+t}}, & if\ \ a_2(1-\frac{\rho}{1+t})\le x\le a_2\\ \dfrac{a_2(1+\frac{\sigma}{1+t})-x}{\frac{a_2\sigma}{1+t}}, & if\ \ a_2\le x\le a_2(1+\frac{\sigma}{1+t})\\ 0, & otherwise\end{cases} \quad (5)$$

The graphical representation of CNTFN is appeared in the Fig-2. Let left and right α-cut of $\mu(x,t)$ from (5) denoted as $L(\alpha,t)$ *and* $R(\alpha,t)$ respectively. According to definition of α-cut defined in subsection 2.2, $L(\alpha,t)=a_2(1-\frac{\rho}{1+t}+\frac{\rho\alpha}{1+t})$ and $R(\alpha,t)=a_2(1+\frac{\sigma}{1+t}-\frac{\sigma\alpha}{1+t})$ (6)

3.1.2.5 De and Beg's Ranking Index on CNTFN:

Let left and right α-cut off $\mu(x,t)$ from (5) denoted as $L(\alpha,t)$ *and* $R(\alpha,t)$ respectively. Then the defuzzification formula under time extension of Yager's ranking index is given by

$$J(\tilde{A})=\frac{1}{2T}\int_{\alpha=0}^{1}\int_{t=0}^{T}\{L(\alpha,t)+R(\alpha,t)\}\,d\alpha\ dt \quad (7)$$

Note that α *and* t independent variables. Thus using (5), (6) becomes

$$J(\tilde{A})=\frac{a_2}{2T}\left[2T+\frac{\sigma-\rho}{2}\log(1+T)\right] \quad (8)$$

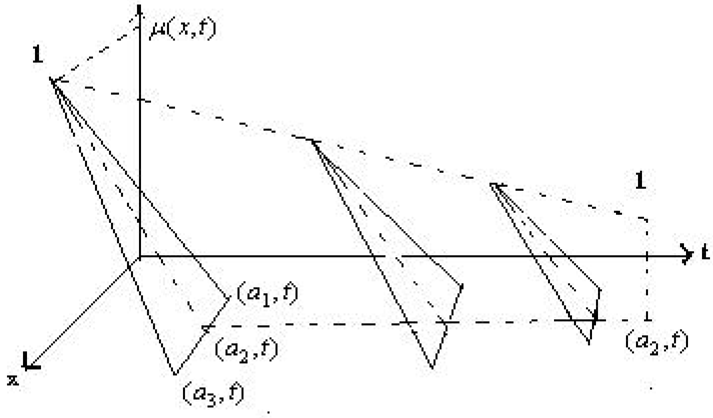

Fig2: Membership function of CNTFN

Obviously, $\lim_{T\to\infty}\frac{\log(1+T)}{T}=0$ (Using L'Hopital's rule) and therefore $J(\tilde{A})\to a_2 \ as \ T\to\infty$.

Note that $\frac{\log(1+T)}{T}$ is taken as cloud index(CI) (9)

In practices, T is measured in days/months.

3.1.2.6 Arithmetic Operations on Normalized General Triangular Fuzzy Number (NGTFN):

Let $\tilde{A}=(a_1,a_2,a_3) \ and \ \tilde{B}=(b_1,b_2,b_3)$ are two triangular fuzzy numbers, then for usual arithmetic operations $+, -, \times, \div$ respectively namely addition, subtraction, multiplication and division between $\tilde{A} \ and \ \tilde{B}$ are defined as follows:

(i) $\tilde{A}+\tilde{B}=(a_1+b_1,\ a_2+b_2,\ a_3+b_3)$

(ii) $\tilde{A}-\tilde{B}=(a_1-b_3,\ a_2-b_2,\ a_3-b_1)$

(iii) $\tilde{A}\times\tilde{B}=(a_1 b_1, a_2 b_2, a_3 b_3)$

(iv) $\frac{\tilde{A}}{\tilde{B}}=(\frac{a_1}{b_3},\frac{a_2}{b_2},\frac{a_3}{b_1}),\ b_1,b_2,b_3\neq 0$

(v) $k\tilde{A}=(ka_1,ka_2,ka_3) if\ k\geq 0$

and $k\tilde{A}=(ka_3,ka_2,ka_1) if\ k<0$

3.1.3Dominance based Particle Swarm Optimization technique (DBPSO)

During the last decade, nature inspired intelligence becomes increasingly popular through the development and utilization of intelligent paradigms in advance information systems design. Among the most popular nature inspired approaches, when task is to optimize with in complex decisions of data or information, PSO draws significant attention. Since its introduction a very large number of applications and new ideas have been realized in the context of PSO (Najafi*et al.*, 2009; Marinakis and Marinaki, 2010). A PSO normally starts with a set of solutions (called swarm) of the decision making problem under consideration. Individual solutions are called particles and food is analogous to optimal solution. In simple terms, the particles are flown through a multi-dimensional search space, where the position of each particle is adjusted according to its own experience and that of its neighbors. The particle *i*has a position vector *($X_i(t)$),* velocity vector *($V_i(t)$),* the position at which the best fitness $X_{pbesti}(t)$ encountered by the particle so far and the best position of all particles $X_{gbest}(t)$ in current generation t. In generation $(t+1)$, the position and velocity of the particle are changed to $X_i(t+1)$ and $V_i(t+1)$ using following rules:

$$V_i(t+1) = wV_i(t) + \mu_1 r_1 (X_{pbest\,i}(t) - X_i(t)) + \mu_2 r_2 (X_{gbest}(t) - X_i(t)) \quad (10)$$

$$X_i(t+1) = X_i(t) + V_i(t+1) \quad (11)$$

The parameters μ_1 and μ_2 are set to constant values, which are normally taken as 2, r_1 and r_2 are two random values uniformly distributed in [0,1], w (0<w<1) is inertia weight which controls the influence of previous velocity on the new velocity. Here $(X_{pbesti}(t))$ and $(X_{gbest}(t))$ are normally determined by comparison of objectives due to different solutions. So for optimization problem involving crisp objective the algorithm works well. If objective value due to solution $X_i$dominates objective value due to solution X_j, we say that X_i dominates X_j. Using this dominance property PSO can be used to optimize crisp optimization problem. This form of the algorithm is named as dominance based PSO (DBPSO) and the algorithm takes the following form. In the algorithm V_{max}represent maximum velocity of a particle, $B_{il}(t)$ and $B_{iu}(t)$ represent lower and upper boundary of the i-th variable respectively. check_constraint$(X_i(t))$ function check whether solution $X_i(t)$ satisfies the constraints of the problem or not. It returns 1 if the solution $X_i(t)$ satisfies the constraints of the problem otherwise it returns 0.

3.1.3.1 Proposed DBPSO algorithm

1. Initialize μ_1, μ_2, w, N and Maxgen.
2. Set iteration counter $t=0$ and randomly generate initial swarm $P(t)$ of N particles (solutions).
3. Determine objective value of each solution $X_i(t)$ and find $X_{gbest}(t)$ using dominance property.
4. Set initial velocity $V_i(t)$, $\forall X_i(t) \in P(t)$ and set $X_{pbesti}(t)=X_i(t)$, $\forall X_i(t) \in P(t)$.
5. While (t<Maxgen) do
6. For i=1:N do
7. $V_i(t+1) = wV_i(t) + \mu_1 r_1 (X_{pbesti}(t) - X_i(t)) + \mu_2 r_2 (X_{gbest}(t) - X_i(t))$
8. If $(V_i(t+1) > V_{max})$ then set $V_i(t+1)=V_{max}$.
9. If $(V_i(t+1) < -V_{max})$ then set $V_i(t+1)=-V_{max}$

10. $X_i(t+1)=X_i(t)+V_i(t+1)$
11. If $(X_i(t+1)>B_{iu}(t))$ then set $X_i(t+1)=B_{iu}(t)$.
12. If $(X_i(t+1)<B_{il}(t))$ then set $X_i(t+1)=B_{il}(t)$.
13. If check_constraint$(X_i(t+1))=0$
14. Set $X_i(t+1)=X_i(t)$, $V_i(t+1)=V_i(t)$
15. Else
16. If $X_i(t+1)$ dominates $X_{pbesti}(t)$ then set $X_{pbesti}(t+1)=X_i(t+1)$.
17. If $X_i(t+1)$ dominates $X_{gbest}(t)$ then set $X_{gbest}(t+1)=X_i(t+1)$.
18. End If.
19. End For.
20. Set $t=t+1$.
21. End While.
22. Output: $X_{gbest}(t)$.
23. End Algorithm

3.1.3.2 Implementation of DBPSO

(a) Representation of solutions: A n-dimensional real vector $X_i=(x_{i1}, x_{i2},\ldots\ldots,x_{in})$, is used to represent i-th solution, where $x_{i1,} x_{i2},\ldots\ldots, x_{in}$ represent n decision variables of the decision making problem under consideration.

(b) Initialization: N such solutions $X_i=(x_{i1}, x_{i2},\ldots\ldots,x_{in})$, $i=1,2,\ldots,N$, are randomly generated by random number generator within the boundaries for each variable $[B_{jl}, B_{ju}]$, $j=1,2,\ldots\ldots,n$. Initialize (P(0)) sub function is used for this purpose.

(c) Dominance property: For crisp maximization problem, a solution X_i dominates a solution X_j if objective value of X_i is greater than that of X_j.

(d) Implementation: With the above function and values the algorithm is implemented using C-programming language. Different parametric values of the algorithm used to solve the model are as follows (Engelbrech, 2005), $\mu_1 = 1.49618$, $\mu_2 = 1.49618$, $w = 0.7298$.

3.1.4 Notations and Assumptions

The following notations and assumptions are adopted to develop the proposed inventory model.

Notations

k	Production rate per cycle.
d	Demand rate per cycle (d<k).
r	Production process reliability.
q(t)	Instantaneous inventory level
Q	Maximum inventory level(decision variable)
T	Cycle length (decision variable).
t_1	Production period (decision variable)
c	Production cost per unit.
c_3	Setup cost per cycle.
h	Inventory carrying cost per unit quantity per unit time.
Z	Average total inventory cost.
Q^*	Optimum value of Q.
T^*	Optimum value of T.
Z^*	Optimum value of Z.
t_1^*	Optimum value of t_1.

Assumptions

(i) Replenishment occurs instantaneously on placing of order quantity so lead time is zero.

(ii) The inventory is developed for single item in an imperfect production process.

(iii) Shortages are not allowed.

(iv) The time horizon of the inventory system is infinite.

(vi) The production rate k is demand dependent and is of the form $k=a + b\,d$ (12)
wherea and b arepositive constants.

(vii) At the beginning of inventory system, ambiguity of demand rate is high because the decision maker (DM) has no any definite information how many people are accepting the product and how much will be demand rate. As the time progress, DM will begin to get more information about the expected demand over the process of inventory and learn whether it is below or over expected. It is generally observed that when new product comes into the market, people will take much more time (no matter what offers /discounts have been declared or what's the quality of product) to adopt/accept the item. Gradually, the uncertain region (cloud) getting thinner to DM's mind. In this respect, demand rate is assumed to be cloudy fuzzy (§**3.1.2.4**).

3.1.5 Model development and analysis

The process reliability r means that amongst the items produced in a production run, only *r* percent are acceptable that can be used to meet the customer's demand. Initially, the production process starts with zero inventories with production rate *k* and demand rate *d*. During the interval [0, t_1], inventory level gradually built up at a rate $rk - d$ and reaches at its maximum level Q at the end of production process. The inventory level gradually depleted during the period [t_1, T]due to customer's demand and ultimately becomes at zero at t=T. The graphical representation of this model is shown in Fig-2. The instantaneous state of *q(t)* describing the differential equations in the interval [$0,T$] of that item is given by

$$\frac{dq(t)}{dt} = rk - d, \quad 0 \le t \le t_1$$
$$= -d, \qquad t_1 \le t \le T \qquad where \; rk - d > 0 \tag{13}$$

with boundary condition $q(0)=0,\ q(t_1)=Q,\ q(T)=0$ 14)

The solution of the differential equation (13) using the boundary condition (14) is given by

$$q(t) = \begin{cases} (rk-d)t, \; 0 \le t \le t_1 \\ d(T-t) \;, \; t_1 \le t \le T \end{cases} \tag{15}$$

The length of each cycle is $T = \frac{Q}{rk-d} + \frac{Q}{d} = \frac{Qrk}{d(rk-d)}$ (16)

Total holding cost for each cycle is given by $hH_1(Q,r,k)$, where (17)

$$H_1(Q,r,k) = \int_0^T q(t)\,dt = \int_0^{t_1} (rk-d)t\,dt + \int_{t_1}^T d(T-t)dt = \frac{Q^2 rk}{2d(rk-d)}$$

Total production cost per cycle is $cP_c(Q,r,k)$, where

$$P_c(Q,r,k) = \int_0^{t_1} kdt = kt_1 = k\frac{Q}{rk-d} \; where \; Q = (rk-d)t_1 \tag{18}$$

Total cost=Production cost+ Set up cost + Holding cost

$$= cP_c(Q,r,k) + c_3 + hH_1(Q,r,k)$$

$$= \frac{ckQ}{rk-d} + c_3 + \frac{hQ^2 rk}{2d(rk-d)}$$

Therefore, the total average cost is $Z=\left[\frac{ckQ}{rk-d}+c_3+\frac{hQ^2rk}{2d(rk-d)}\right]/T$

$$=\frac{cd}{r}+\frac{c_3}{T}+\frac{hT(rk-d)d}{2rk}$$

$$=\frac{cd}{r}+\frac{c_3}{T}+\frac{hdT(ar+(br-1)d)}{2(a+bd)r}$$

Hence, our problem is given by Minimize $Z=\frac{cd}{r}+\frac{c_3}{T}+\frac{hdT(ar+(br-1)d)}{2(a+bd)r}$

$subject\ to\ \ d(T-t_1)=(rk-d)t_1\ \ i.e.\ rkt_1=dT,\ Q=d(T-t_1)$ (19)Now, the problem is reduced to minimize the average cost Z and to find the optimum value of Q and T for which $Z(Q, T)$ is minimum and the corresponding value of t_1. The average cost is minimized by DBPSO.

3.1.5.1 Fuzzy mathematical model

Initially, when production process starts, demand rate of an item is ambiguous. Naturally, demand rate is assumed to be general fuzzy over the cycle length. Then fuzzy demand rate $\tilde{d}$ as follows $\tilde{d}=(d_1,\ d_2,\ d_3)$ for NGTFN.

Therefore the problem (19) becomes fuzzy problem, is given by

$$Minimize\ \tilde{Z}=\frac{c\tilde{d}}{r}+\frac{c_3}{T}+\frac{h\tilde{d}T(ar+(br-1)\tilde{d})}{2(a+b\tilde{d})r}$$

subject to $r\tilde{k}t_1=\tilde{d}T,\ \tilde{Q}=\tilde{d}\ (T-t_1)$ (20)

Now, using (1), the membership function of the fuzzy objective, fuzzy order quantity and fuzzy production rate under NGTFN are given by

$$\mu_1(Z)=\begin{cases}\dfrac{Z-Z_1}{Z_2-Z_1}, & Z_1\le Z\le Z_2\\ \dfrac{Z_3-Z}{Z_3-Z_2}, & Z_2\le Z\le Z_3\\ 0, & otherwise\end{cases}\qquad where\begin{cases}Z_1=\dfrac{cd_1}{r}+\dfrac{c_3}{T}+\dfrac{hd_1T\{ar+(br-1)d_1\}}{2r(a+bd_3)}\\ Z_2=\dfrac{cd_2}{r}+\dfrac{c_3}{T}+\dfrac{hd_2T\{ar+(br-1)d_2\}}{2r(a+bd_2)}\\ Z_3=\dfrac{cd_3}{r}+\dfrac{c_3}{T}+\dfrac{hd_3T\{ar+(br-1)d_3\}}{2r(a+bd_1)}\end{cases}$$

(21)

$$\mu_2(Q)=\begin{cases}\dfrac{Q-Q_1}{Q_2-Q_1}, & Q_1\le Q\le Q_2\\ \dfrac{Q_3-Q}{Q_3-Q_2}, & Q_2\le Q\le Q_3\\ 0, & otherwise\end{cases}\qquad where\begin{cases}Q_1=d_1(T-t_1)\\ Q_2=d_2(T-t_1)\\ Q_3=d_3(T-t_1)\end{cases}\quad(22)$$

$$\mu_3(k)=\begin{cases}\dfrac{k-k_1}{k_2-k_1}, & k_1\le k\le k_2\\ \dfrac{k_3-k}{k_3-k_2}, & k_2\le k\le k_3\\ 0, & otherwise\end{cases}\qquad where\begin{cases}r\,k_1\,t_1=d_1\,T\\ r\,k_2\,t_1=d_2\,T\\ r\,k_3\,t_1=d_3\,T\end{cases}\qquad(23)$$

The index value of the fuzzy objective, fuzzy order quantity and fuzzy production rate are respectively obtained using (2) and (3) as

$$\begin{cases}I(\tilde{Z})=\dfrac{1}{4}(Z_1+2Z_2+Z_3)\\ \quad=\dfrac{c(d_1+2d_2+d_3)}{4r}+\dfrac{c_3}{T}+\dfrac{hT}{8r}\left[\dfrac{d_1\{ar+(br-1)d_1\}}{a+bd_3}+\dfrac{2d_2\{ar+(br-1)d_2\}}{a+bd_2}+\dfrac{d_3\{ar+(br-1)d_3\}}{a+bd_1}\right]\\ I(\tilde{Q})=\dfrac{1}{4}(Q_1+2Q_2+Q_3)=\dfrac{(T-t_1)}{4}(d_1+2d_2+d_3)\\ I(\tilde{k})=\dfrac{1}{4}(k_1+2k_2+k_3)=\dfrac{T}{4rt_1}(d_1+2d_2+d_3)\ [using\ (21),(22)\ and\ (23)]\end{cases}\qquad(24)$$

3.1.5.2 Particular cases

Subcase-3.1.5.2.1: If $d_1,d_2,d_3\to d$ then $I(\tilde{Z})\to\dfrac{cd}{r}+\dfrac{c_3}{T}+\dfrac{hdT(ar+(br-1)d)}{2(a+bd)r}$

$$I(\tilde{Q})\to d(T-t_1)$$

$$and\ I(\tilde{k})\to\frac{dT}{rt_1}$$

This is a classical EPQ model with process reliability ***r***.

Subcase-3.1.5.2.2 If $r\to 1, b\to 0$ then $I(\tilde{Z})\to cd+\dfrac{c_3}{T}+\dfrac{hdT}{2a}(a-d)$

$$I(\tilde{Q})\to d(T-t_1)$$

$$I(\tilde{k})\to\frac{dT}{t_1}$$

Also, this is classical EPQ model with production rate ***a***.

3.1.5.2Cloudy fuzzy mathematical model

Initially, when production process starts, demand rate of an item is ambiguous. As the time progress, hesitancy of demand rate tends to certain demand rate over the cycle length. Then fuzzy demand rate $\tilde{d}$ becomes cloudy fuzzy following the equation (4)

Now, using (5), the membership function of the fuzzy objective, fuzzy order quantity and fuzzy production rate under CNTFN are given by

$$\chi_1(Z,T)=\begin{cases}\dfrac{Z-Z_{11}}{Z_{12}-Z_{11}},\ Z_{11}\le Z\le Z_{12}\\ \dfrac{Z_{13}-Z}{Z_{13}-Z_{12}},\ Z_{12}\le Z\le Z_{13}\\ 0,\quad otherwise\end{cases}\quad where\ \begin{cases}Z_{11}=\dfrac{c(1-\frac{\rho}{1+T})d}{r}+\dfrac{c_3}{T}+\dfrac{hTd(1-\frac{\rho}{1+T})}{2r}\left[\dfrac{ar+(br-1)d(1-\frac{\rho}{1+T})}{a+bd(1+\frac{\sigma}{1+T})}\right]\\ Z_{12}=\dfrac{cd}{r}+\dfrac{c_3}{T}+\dfrac{hdT\{ar+(br-1)d\}}{2r(a+bd)}\\ Z_{13}=\dfrac{c(1+\frac{\sigma}{1+T})d}{r}+\dfrac{c_3}{T}+\dfrac{hTd(1-\frac{\rho}{1+T})}{2r}\left[\dfrac{ar+(br-1)d(1+\frac{\sigma}{1+T})}{a+bd(1-\frac{\rho}{1+T})}\right]\end{cases}\tag{25}$$

$$\chi_2(Q,T)=\begin{cases}\dfrac{Q-Q_{11}}{Q_{12}-Q_{11}},\ Q_{11}\le Q\le Q_{12}\\ \dfrac{Q_{13}-Q}{Q_{13}-Q_{12}},\ Q_{12}\le Q\le Q_{13}\\ 0,\quad otherwise\end{cases}\quad where\ \begin{cases}Q_{11}=d(1-\dfrac{\rho}{1+T})(T-t_1)\\ Q_{12}=d(T-t_1)\\ Q_{13}=d(1+\dfrac{\sigma}{1+T})(T-t_1)\end{cases}\tag{26}$$

$$\chi_3(k,T)=\begin{cases}\dfrac{k-k_{11}}{k_{12}-k_{11}},\ k_{11}\le k\le k_{12}\\ \dfrac{k_{13}-k}{k_{13}-k_{12}},\ k_{12}\le k\le k_{13}\\ 0,\quad otherwise\end{cases}\quad where\ \begin{cases}k_{11}=d(1-\dfrac{\rho}{1+T})\dfrac{T}{rt_1}\\ k_{12}=\dfrac{dT}{r\,t_1}\\ k_{13}=d(1+\dfrac{\sigma}{1+T})\dfrac{T}{rt_1}\end{cases}\tag{27}$$

Using (7) the index value of the fuzzy objective, fuzzy order quantity and fuzzy production rate are respectively are given by

$$J(\tilde{Z})=\frac{1}{4\tau}\int_{T=0}^{\tau}(Z_{11}+2Z_{12}+Z_{13})\,dT$$

$$=\frac{1}{4\tau}\int_0^{\tau}\left[\frac{cd}{r}(4+\frac{\sigma-\rho}{1+T})+\frac{4c_3}{T}\right]dT$$

$$+\frac{1}{4\tau}\int_0^{\tau}\frac{hdT}{2r}\left[(1-\frac{\rho}{1+T})\frac{ar+(br-1)d(1-\frac{\rho}{1+T})}{a+bd(1+\frac{\sigma}{1+T})}+2\frac{ar+(br-1)d}{a+bd}+(1+\frac{\sigma}{1+T})\frac{ar+(br-1)d(1+\frac{\sigma}{1+T})}{a+bd(1-\frac{\rho}{1+T})}\right]dT$$

[Using (25)]

$$=I_1+\frac{hd}{8\tau r}(I_2+I_3+I_4) \quad (28)$$

The expression of I_1, I_2, I_3 *and* I_4 are given in Appendix-1

$$J(\tilde{Q})=\frac{1}{\tau}\int_0^{\tau}\frac{1}{4}(Q_{11}+2Q_{12}+Q_{13})\,dT=\frac{d}{4\tau}\int_0^{\tau}\left(4+\frac{\sigma-\rho}{1+T}\right)(T-t_1)\,dT \quad \text{[Using (26)]}$$

$$=\frac{d}{4\tau}\left[2\tau^2-4\tau t_1+(\sigma-\rho)(\tau-(1+t_1)\ln|1+\tau|)\right] \quad (29)$$

$$J(\tilde{k})=\frac{1}{\tau}\int_0^{\tau}\frac{1}{4}(k_{11}+2k_{12}+k_{13})\,dT=\frac{1}{4\tau}\int_0^{\tau}\frac{dT}{rt_1}\left\{4+\frac{(\sigma-\rho)}{1+T}\right\}dT \quad \text{[Using (27)]}$$

$$=\frac{d}{4\tau r t_1}\left[2\tau^2+(\sigma-\rho)(\tau-\ln|1+\tau|)\right] (30)$$

3.1.6 Stability analysis and particular cases

(i) If $\rho,\sigma\to 0$ *then* $p\to q$ *and* $u\to v$ Also, $I_2\to\frac{p}{2u}\tau^2$, $I_4\to\frac{p}{2u}\tau^2$, $I_3=\frac{p}{u}\tau^2$

So, $J(\tilde{Z})\to\frac{cd}{r}+\frac{c_3}{\tau}\ln\left|\frac{\tau}{\varepsilon}\right|+\frac{hd}{4r\tau}\frac{p\tau^2}{u}$, $J(\tilde{Q})\to d(\frac{\tau}{2}-t_1)$, $J(\tilde{k})\to\frac{d}{2rt_1}\tau$

(ii) If $\rho,\sigma\to 0$ then the model reduces to (i). The above expressions deduced in (i) are in the form of classical EPQ model. Thus we choose ε in such a way that above expressions reduced to classical EPQ model.

Hence, $\frac{cd}{r}+\frac{c_3}{\tau}\ln\left|\frac{\tau}{\varepsilon}\right|+\frac{hd}{4r\tau}\frac{p\tau^2}{u}\cong\frac{cd}{r}+\frac{c_3}{T}+\frac{hdT}{2}\frac{p}{ur}$. Comparing we have

$\frac{1}{T}=\frac{1}{\tau}\ln\left|\frac{\tau}{\varepsilon}\right|$, $T=\frac{\tau}{2}$

From these, we get $\varepsilon=\frac{2T}{e^2}$. Also, if $\tau=2$ *then* $T=1$. *Hence,* $\varepsilon\to 2e^{-2}\ll 1$

Since $2<e\Rightarrow\frac{2T}{e^2}<\frac{T}{2}\Rightarrow\varepsilon<\frac{T}{2}$

3.1.7 Numerical Illustration

The following values of inventory parameters are used to calculate the minimum values of average cost function (Z^*) along with the optimum inventory level (Q^*), optimum production period (t_1^*) and optimum cycle length (T^*)

a=100, b=1.22, c_3=\$300 ,h=\$ 1.5 per unit, c= \$ 3 per unit, r=.8, d= 500 units for the crisp model; for fuzzy model demand rate <d_1, d_2, d_3>=<460, 500, 600> units keeping other inventory parameters are same as taken in crisp model and that for the cloudy fuzzy model, $\sigma = 0.16$, $\rho = 0.13, \varepsilon = 0.6$. Optimum results are obtained via dominance based particle swarm optimization and presented in Table-1.

It is noted that for computation of degree of fuzziness, apply formula $d_f = \frac{U_b - L_b}{m}$ where U_b, L_b respectively are the upper and lower bounds of fuzzy components and m is the Mode which is obtained using the formula Mode(m)=3$\times$Median-2$\times$Mean. For fuzzy demand rate <460, 500, 600>, Median=500, Mean=520, U_b=600, L_b=460, m=460

Table-1: Optimum values of EPL model by DBPSO

Model	t_1*(months)	T^*(months)	Q^* units	Z^*(\$)	$d_f = \frac{U_b - L_b}{m}$	CI= $\frac{\log(1+T)}{T}$
Crisp	1.5	1.704	102.00	2127.56	……………..	……………
Fuzzy	1.9	2.58	346.30	2164.49	0.304	……………
Cloudy Fuzzy	1.85	2.22	183.03	2115.33	……………..	0.227

From the above results, it has been observed that minimum cost is obtained in cloudy fuzzy model and the value of optimum cost Rs. 2115.33 after the completion 2.22 months. In cloudy fuzzy environment degree of fuzziness is less than the general triangular number as the hesitancy of fuzzy gradually decreases due to the taking experience over time.

3.1.7.1 Sensitivity Analysis of Cloudy Fuzzy Model

Table-2: Sensitivity analysis for cloudy fuzzy model

Parameters	% change	Average cost (z^*)	$\frac{(z^{/}-z)}{z}\times 100\%$
d	-15%	1833.44	-13.32
	-10%	1927.49	-8.88
	-5%	2021.45	-4.44
	5%	2209.13	+4.43
	10%	2302.87	+8.86
	15%	2396.55	+13.29
a	-15%	2099.51	-0.75
	-10%	2104.86	-0.49
	-5%	2110.13	-0.25
	5%	2120.45	+0.24
	10%	2125.51	+0.48
	15%	2130.48	+0.69
b	-15%	2006.40	-5.15
	-10%	2046.12	-3.27
	-5%	2082.28	-1.56
	5%	2145.66	+1.43
	10%	2173.58	+2.75
	15%	2199.39	+3.97
c_3	-15%	2108.56	-0.32
	-10%	2110.82	-0.21
	-5%	2113.07	-0.11
	5%	2122.09	+0.32
	10%	2128.87	+0.64
	15%	2135.63	+0.96
c	-15%	1833.27	-13.37
	-10%	1927.29	-8.90
	-5%	2021.31	-4.44
	5%	2209.35	+4.44
	10%	2303.37	+8.89
	15%	2397.38	+13.33
h	-15%	2100.38	-0.71
	-10%	2105.36	-0.47
	-5%	2110.35	-0.23
	5%	2120.31	+0.23
	10%	2125.28	+0.47
	15%	2130.28	+0.71

Using the above numerical illustration, the effect of under or over estimation of various parameters on average cost is studied. Here using $\Delta z = \frac{(z^{/}-z)}{z}\times 100\%$ as a measure of sensitivity where z is the true value and $z^{/}$ is the estimated value. The sensitivity analysis is shown by increasing or decreasing the parameters by 5%, 10% and 15% , taking one at a time and keeping the others as true values. The results are presented in Table-2.

It is seen form the Table-3 that the parameters *d* and *c*are highly sensitive. For the changes of demand at -15% , average inventory cost reduces to -13.32% and for 15%, the average inventory cost increases at +13.29% respectively. Also the same results observed for the changes of unit production cost. These phenomena agree with reality. But for the changes of $a,\ b,\ c_3,\ h$ from -15% to +15%, there are moderately variations on the average cost. This sensitivity table reveals that the observations done on inventory model are more realistic and more practicable.

3.1.7.2 Effect of changing cycle time

Comparing the results obtained in crisp, general fuzzy and cloudy fuzzy environment, it has been observed from the graphical illustration (**Fig-3**) that cloudy fuzzy model predicts the minimum cost 2068.57 ($) and the minimum cost is obtained at cycle time 4 months which is shown in **Fig-4**. In Fig-4, the curve shown U shape pattern under the cloudy fuzzy model. So the curve is convex. So, it is interesting to note that cloudy fuzzy model is more reliable.

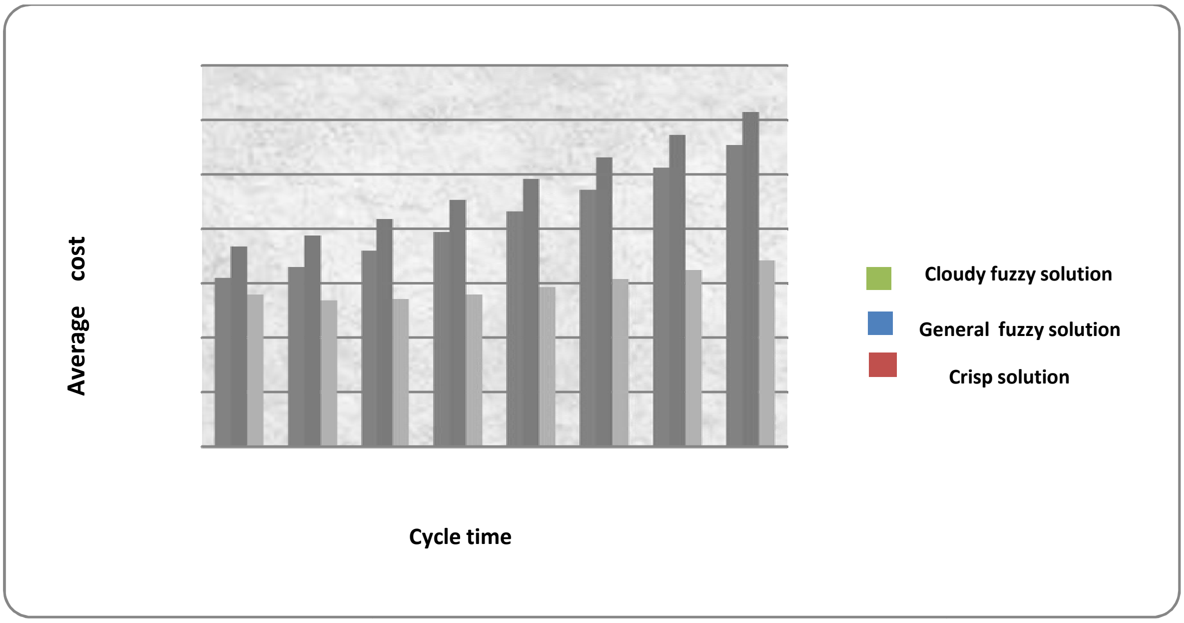

Fig-3:Average cost vs cycle time

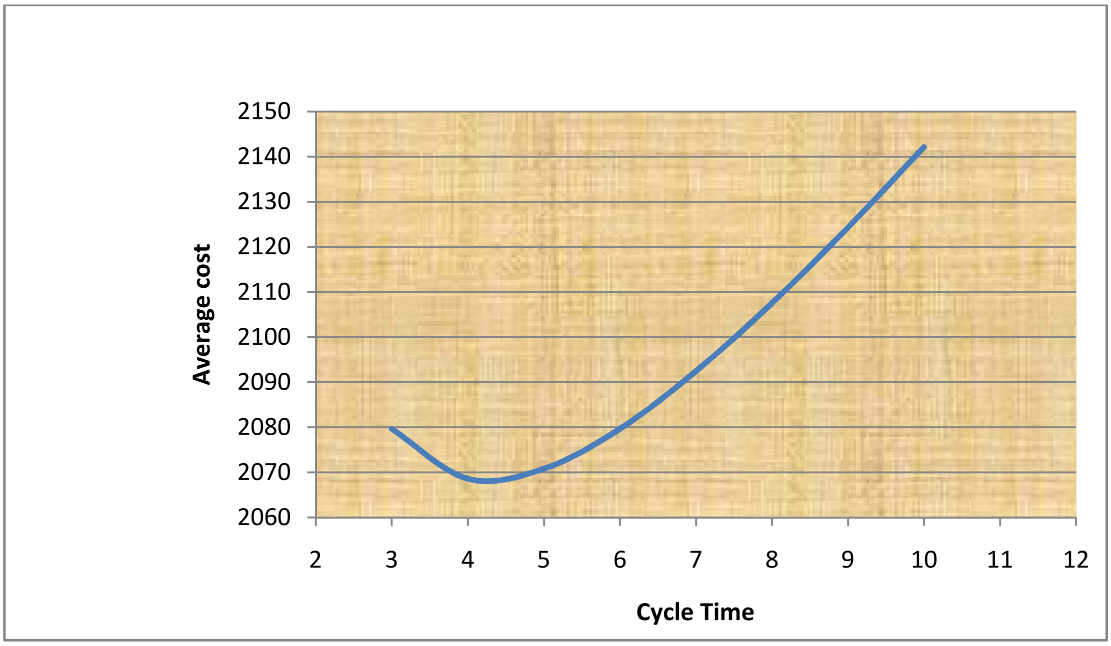

Fig-4: Average cost vs cycle time for cloudy fuzzy model

3.1.7.3 Effect of changing reliability

Relibility is the most important factor in manufacturing system as reliability defined to be capability of manufacturing units without breakdown of the system. It has been observed from the graphical illustration (**Fig-5)** that as the reliability increases, average cost gradually decreases as because increase of reliabilty resulted in increase of production rate. So, cost of finished good consistently decreases.

Also, the performance level as measured by reliability can significantly improved the manufacturing system. Since the present is minimization problem, so averege cost decreases with the increase of reliability.

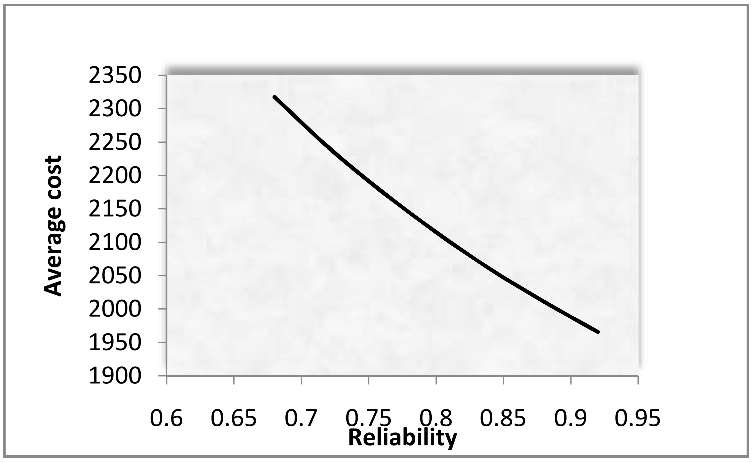

Fig-5:Average cost vs reliability for cloudy fuzzy model

3.1.7.4 Comparison of average cost under different cycle time

It has been observed that difference of average inventory cost of crisp model as well as general fuzzy model with respect to cloudy fuzzy model for different value of cycle time are shown in Table-4. From this Table-4, it is seen that cloudy fuzzy model giving the minimum average inventory cost at time 4 months which is the better choice of inventory practinioner as well as decision maker.

Table-4: Average cost under diferent model

Crisp model				Genral fuzzy model			Cloudy fuzzy model		
Cycle time T	t_1^*	Q^*	Z^*	t_1^*	Q^*	Z^*	t_1^*	Q^*	Z^*
3	2.64	179.58	2109.86	2.68	164.80	2167.25	1.35	74.68	2079.64
4	3.52	239.46	2129.58	3.62	195.70	2187.59	1.80	99.52	**2068.57**
5	4.41	299.29	2159.47	4.55	231.75	2217.92	2.21	149.04	2070.79
6	5.20	359.15	2194.36	5.51	252.35	2253.35	2.69	154.26	2079.68
7	6.11	419.01	2232.11	6.45	283.25	2291.45	3.13	189.16	2092.40
8	7.04	478.87	2271.65	7.37	323.42	2331.43	3.59	203.02	2107.53
9	7.92	538.73	2312.33	8.34	339.90	2372.59	4.04	228.91	2124.26
10	8.81	598.59	2353.95	9.20	394.49	2414.60	4.53	238.13	2142.13

3.1.8 Conclusion and future research

In this paper, fuzzy inventory model under imperfect production process with cloudy fuzzy demand rate is developed where production rate is demand dependent. The model is solved in crisp , general fuzzy and cloudy fuzzy environment using Yager's index method and De and Beg's ranking index method using new defuzzification method and the results obtained in crisp, fuzzy and cloudy fuzzy environment are compared. ***For the first time, this type of inventory model has been successfully solved by DBPSO in cloudy fuzzy environment.*** Further extension of this model can be done considering some realistic situation such as multi-item, quantity discount, price and reliability dependent, learning effect etc. Moreover, in future, this model can be formulated with random planning horizon, fuzzy planning horizon in stochastic, fuzzy stochastic environments.

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Appendix-1: The expression of I_1, I_2, I_3 *and* I_4 are given below.

$$I_1 = \frac{1}{4\tau}\int_0^{\tau}\left\{\frac{c\,d}{r}(4+\frac{\sigma-\rho}{1+T})+\frac{4c_3}{T}\right\}dT \quad = \frac{c\,d}{r}(1+\frac{\sigma-\rho}{4\tau}\ln|1+\tau|)+\frac{c_3}{\tau}\ln\left|\frac{\tau}{\varepsilon}\right|$$

$$I_2 = \int_0^{\tau} T(1-\frac{\rho}{1+T})\frac{\left\{a\,r+(b\,r-1)\,d(1-\frac{\rho}{1+T})\right\}}{a+b\,d\,(1+\frac{\sigma}{1+T})}dT$$

$$= \int_0^{\tau} T(1-\frac{\rho}{1+T})\frac{T(a\,r+(b\,r-1)d)+a\,r+(b\,r-1)\,d(1-\rho)}{T(a+b\,d)+a+b\,d(1+\sigma)}dT$$

$$= \int_0^{\tau} T(1-\frac{\rho}{1+T})\frac{pT+q}{T\,u+v}dT$$

[$p = a\,r+(b\,r-1)\,d$, $q = a\,r+(b\,r-1)\,d\,(1-\rho)$, $u = a+b\,d$, $v = a+b\,d\,(1+\sigma)$]

$$= p\int_0^{\tau}\frac{T^2}{T\,u+v}dT + q\int_0^{\tau}\frac{T}{T\,u+v}dT - \rho\,p\int_0^{\tau}\frac{T^2}{(T\,u+v)(1+T)}dT - \rho\,q\int_0^{\tau}\frac{T}{(T\,u+v)(1+T)}dT$$

$$= I_{21} + I_{22} - I_{23} - I_{24}$$

where $I_{21} = p\int_0^{\tau}\frac{T^2}{T\,u+v}dT = \frac{p}{u^3}\left[\frac{u^2\,\tau^2}{2} - u\,v\,\tau + v^2\ln\left|\frac{v+u\,\tau}{v}\right|\right]$

$$I_{22} = q\int_0^{\tau}\frac{T}{T\,u+v}dT = \frac{q}{u}\left[\tau - \frac{v}{u}\ln\left|\frac{v+\tau\,u}{v}\right|\right]$$

$$I_{23} = \rho\,p\int_0^{\tau}\frac{T^2}{(T\,u+v)(1+T)}dT = \frac{\rho p}{u-v}\left[\tau - \ln|1+\tau| - \frac{v\tau}{u} + \frac{v^2}{u^2}\ln\left|\frac{v+\tau\,u}{v}\right|\right]$$

$$I_{24} = \rho\,q\int_0^{\tau}\frac{T}{(T\,u+v)(1+T)}dT = \frac{\rho\,q}{v-u}\left[\frac{v}{u}\ln\left|\frac{v+\tau\,u}{v}\right| - \ln|1+\tau|\right]$$

$$I_3 = 2\int_0^{\tau}\frac{a\,r+(b\,r-1)\,d}{a+b\,d}dT = \frac{p}{u}\tau^2$$

$$I_4 = \int_0^{\tau} T(1+\frac{\sigma}{1+T})\frac{\left\{a\,r+(b\,r-1)\,d(1+\frac{\sigma}{1+T})\right\}}{a+b\,d\,(1-\frac{\rho}{1+T})}dT$$

$$= \int_0^{\tau} T(1+\frac{\sigma}{1+T})\frac{T(a\,r+(b\,r-1)d)+a\,r+(b\,r-1)\,d(1+\sigma)}{T(a+b\,d)+a+b\,d(1-\rho)}dT$$

$$= \int_0^{\tau} T(1+\frac{\sigma}{1+T})\frac{pT+y}{Tu+s}dT$$

$$[\; p = ar + (br-1)d \;,\; y = ar + (br-1)d(1+\sigma) \,,\; u = a + bd \;,\; s = a + bd(1-\rho) \;]$$

$$= p\int_0^{\tau}\frac{T^2}{Tu+s}dT + y\int_0^{\tau}\frac{T}{Tu+s}dT + \sigma p\int_0^{\tau}\frac{T^2}{(Tu+s)(1+T)}dT + \sigma y\int_0^{\tau}\frac{T}{(Tu+s)(1+T)}dT$$

$$= I_{41} + I_{42} + I_{43} + I_{44}$$

$$where\; I_{41} = p\int_0^{\tau}\frac{T^2}{Tu+s}dT = \frac{p}{u^3}\left[\frac{u^2\tau^2}{2} - us\tau + s^2\ln\left|\frac{s+u\tau}{s}\right|\right]$$

$$I_{42} = y\int_0^{\tau}\frac{T}{Tu+s}dT = \frac{y}{u}\left[\tau - \frac{s}{u}\ln\left|\frac{s+\tau u}{s}\right|\right]$$

$$I_{43} = \sigma p\int_0^{\tau}\frac{T^2}{(Tu+s)(1+T)}dT = \frac{\sigma p}{u-s}\left[\tau - \ln|1+\tau| - \frac{s\tau}{u} + \frac{s^2}{u^2}\ln\left|\frac{s+\tau u}{s}\right|\right]$$

$$I_{44} = \sigma y\int_0^{\tau}\frac{T}{(Tu+s)(1+T)}dT = \frac{\sigma y}{s-u}\left[\frac{s}{u}\ln\left|\frac{s+\tau u}{s}\right| - \ln|1+\tau|\right]$$