10

A Branching Space-Times Perspective on Presentism

10.1 Introduction

Our commonsense metaphysics of time is, arguably, best expressed by *presentism*, which holds that "the present simply *is* the real considered in relation to two particular species of unreality, namely the past and the future" (Prior, 1970, p. 245). However, it is often claimed that presentism is in conflict with the theory of special relativity, which holds that the simultaneity of distant events is frame-relative. Is there really a conflict? The issue is complicated, to say the least. One might doubt whether relativity theory can have an impact on metaphysics or on everyday notions at all. And even if it can, what precisely is the notion of the present whose independent reality is threatened by relativity theory, and how can that threat be spelled out in a formally precise way?

Our aim in this chapter is to flesh out a notion of the present that can serve the metaphysical role that presentism requires while being relativity-friendly. To this end, we will distinguish two different notions of the present, one based on simultaneity and one based on co-presentness. Simultaneity invokes a static role of the present in singling out something like a temporal location of an event (a time coordinate). Co-presentness, on the other hand, invokes a dynamic role of the present in separating a fixed past from an open future and thereby anchoring a notion of coexistence. We hold that it is the latter role that is important for presentism as a doctrine in the metaphysics of time, and we will show that a relativity-proof notion of the present in its dynamical role can be defended by exploiting the idea that dynamic change must be based on the indeterministic realization of possibilities for the future. In working out the formal details of this idea, we will make use of the fact that BST offers a rich notion of modal correlations (see Chapter 5), based on which we will be able to extend the notion of a fixed past such that it contains more than just an event's past light cone.

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The chapter is structured as follows. We begin with Chapter 10.2 by reflecting on the supposed conflict between presentism and special relativity and by charting the options available for avoiding that conflict. Next, in Chapter 10.3, we describe the main idea of the chapter, which is to focus on a notion of dynamic time based on real indeterministic change in contrast to static coordinate time. Chapter 10.4 summarizes the formal desiderata for a notion of dynamic time and briefly motivates two different approaches to defining dynamic time. In Chapter 10.5 we describe in detail the first of these approaches, in which dynamic time is analyzed in terms of *causae causantes*. In the second approach, dynamic time is analyzed in terms of an open future; that approach is described in Chapter 10.6. In Chapter 10.7 we draw a unifying conclusion from both approaches, viz., that the fulfillment of all the desiderata on dynamic time in a BST structure can be secured by a certain strong kind of modal correlation, which we call "sticky modal funny business". In Chapter 10.8, we illustrate our results in the framework of Minkowskian Branching Structures, one which will be familiar to the reader from Chapter 9.1. As usual, we end with Conclusions and Exercises.

10.2 The problem of defining the present in special relativity

There appears to be a conflict between our manifest, intuitive notions of the past, present, and future, and what special relativity says about time. Here are seven important features of manifest time:[1] (i) it assumes a mind-independent tripartite division of worldly events into past, present, and future. (ii) These three partitions are supposed to continuously change as future events turn into present events and then into past events. (iii) There is a further difference with respect to openness vs. settledness: the future is viewed as open, in contrast to the past, which is viewed as settled, or closed. (Whether the present is settled is a subject of a small controversy, with the majority view opting for its being settled, like the past). Concerning the present, the manifest view of time suggests that (iv) it is global (so any object existing before a given present and living sufficiently long hits upon it), (v) it cannot be repeated, (vi) it does not extend in time, and (vii) no two presents overlap.

[1] For a recent characterization of manifest time, see Callender (2017, Ch. 1).

In the Minkowski space-time of special relativity, each so-called event (i.e., each element of the space-time) can be uniquely identified via its space-time coordinates, a set of four real numbers. There is, however, no unique way to divide up these coordinates into a three-dimensional spatial and a one-dimensional temporal part. Such a division is always relative to an inertial reference frame, and none of those frames is preferred—the principle of relativity states that all frames have to be treated as being on a par. Some important relations among events are frame-invariant. For example, whether one event can causally influence another one is independent of the choice of reference frame: the causal order on Minkowski space-time is frame-invariant. Accordingly, the notion of two events being space-like related is also frame-invariant. Now, one might believe that only frame-invariant properties and relations have independent, objective reality, whereas other properties and relations cannot be taken metaphysically seriously. Problematically, the simultaneity of space-like related events is frame-relative: it depends on which frame one considers whether two space-like related events occur at the same time or not.

These basic truths about the structure of Minkowski space-time can be translated into a formal claim about the definability of a notion of *simultaneity*. There is widespread agreement that such a notion of simultaneity has to be transitive, reflexive, and symmetric (i.e., it has to be an equivalence relation).[2] It follows that the simultaneity relation cannot be the relation of space-like relatedness, as that relation is not transitive. And there are no other sensible options either, as shown by Van Benthem's theorem:[3] If a relation R is definable on the basis of Minkowski space-time alone, it has to be invariant under that structure's automorphisms, which include the Poincaré group and contractions. But once there are x, y for which $x \neq y$ and xRy, one can employ suitable automorphisms to show that xRz for *any* event z. Thus, there are only two equivalence relations that can be defined on Minkowski space-time, identity and the universal relation. None of these provides a sensible notion of simultaneity: on the first option, as each event is identical only to itself, each event would be simultaneous only with itself, and on the second option, simultaneity would not discriminate among events at all. Therefore, no non-trivial equivalence relation can be defined on the

[2] See, e.g., Van Benthem (1983); Stein (1991); Clifton and Hogarth (1995); Rakić (1997b).

[3] See Van Benthem (1983, pp. 25f.).

basis of the Minkowski space-time of special relativity alone, and there is no frame-invariant notion of simultaneity.

Now, the technical notion of invariance under a structure's automorphism is meant to single out those notions that are fully objective. It seems, therefore, that the notion of simultaneity, which is not frame-independent, cannot be an objective relation. This, in turn, might mean that the present is just a subjective notion, or even an illusion, which would completely undermine presentism. This challenge concerns the tenability of an objective notion of simultaneity as a necessary, not as a sufficient condition of the tenability of the doctrine of presentism as a whole. The challenge, therefore, arises prior to and independently of the additional question of how, assuming that such an objective notion is available, one should model the phenomenon of the passage of time. In this chapter we do not discuss the latter question since there is fairly widespread agreement in the literature that an indexical treatment of the passage of time is appropriate.[4]

The metaphysical consequences of the mentioned formal result—no frame-invariant notion of objective simultaneity is definable in special relativity theory—are debatable. There appear to be four main ways of reacting:

1. *Rejection of any metaphysical status of special relativity.* It is not implausible to just shrug off any suggested metaphysical import of special relativity, pointing out that that theory is only valid within its range of applicability, which is far from universal.

   Many well established empirical facts, from the details of the orbit of the planet Mercury to gravitational effects on satellites or, recently, to gravitational waves cannot be modeled on the basis of special relativity theory alone. In this sense, special relativity is empirically refuted, and therefore it is implausible to expect to get any metaphysical mileage out of it. If we are looking for a space-time theory to provide metaphysical guidance, we need to look at the general theory of relativity, or even at a successor to that theory describing some form of quantum gravity. It may well be that such a theory will provide additional resources for defining a notion of simultaneity. For example, some cosmological models of general relativity allow for the definition of a class of fundamental observers that can anchor an absolute notion of cosmic time.

[4] See, e.g., Belnap et al. (2001, Ch. 6), Beer (1994), or Reichenbach (1952, p. 277).

Given these resources, one can then define two events to be absolutely simultaneous iff they happen at the same cosmic time.[5] So the whole discussion involving special relativity might be a non-starter.

2. *Acceptance and revision of temporal notions.* If one accepts the apparent indefinability of simultaneity as proof that the notion of the present makes no objective sense, one can try to live without it.

   While this attitude had already been recommended (for different reasons) by Spinoza,[6] it appears practically impossible: "now" is an essential indexical which has both theoretical and practical import for us.[7]

3. *Acceptance of relativization of temporal notions.* Each concrete act of communication employing temporal determinations comes from the perspective of a corporeal being. Reflecting on this fact, one can relativize temporal determinations to the rest frame of that corporeal being,[8] and one can additionally point out that relativistic effects can be neglected for most practical purposes.[9] An absolute notion of simultaneity is not needed to account for our communication practices—even in hypothetical cases in which relativistic effects become important. If I say that events e and f are simultaneous, and you, speeding by in your space-ship, deny this, then we can understand that we are not in fact disagreeing, but saying different things: I say that e and f are *simultaneous for me*, and you say that they are *not simultaneous for you*.

   Such relativizations are in fact common: if I say, "It is raining", and you say, in a different place, "It is not raining", then we are not in fact disagreeing, and we can make the compatibility of our assertions explicit by mentioning our respective locations. We can also live with

[5] See Smeenk (2013) for a discussion of results and for some pertinent qualifications.

[6] See his *Ethics*, Book IV, Proposition 62: "Insofar as the mind conceives of things by the dictate of reason, it is equally affected whether the idea is of something in the future or in the past or in the present" (Spinoza, 1677).

[7] See, e.g., Perry (1979).

[8] See Balashov (2010) for a discussion of some subtle qualifications that pertain to the definition of a relativistic object's center of mass. The resulting imprecision is negligible for our purposes. Additionally, it is enough that a speaker may provide a frame of reference in *some* way. The easiest way would certainly be via her body, but there are other possibilities. Compare the similarly imprecise "here".

[9] See Butterfield (1984) for a succinct, quantitative assessment of the practical lack of impact of relativity theory for everyday communication. It should be added that the situation has changed somewhat since 1984, at least if relativistic effects grounding everyday technology are considered as well. Most of us nowadays carry around GPS receivers whose underlying satellite infrastructure relies heavily on (special and general) relativistic effects. This technology, however, has no direct impact on our use of temporal determinations in communication.

relativization when it comes to relativistic frames of reference. In fact, employing the Lorentz transformation between our frames, we will be able to make precise sense of the apparent disagreement and come to agree on the underlying objective facts about space-time.

4. *Addition of structure.* It is possible to add some structure to plain Minkowski space-time that will allow the objective anchoring of a non-trivial equivalence relation to be read, for example, as absolute simultaneity.

   In fact, nothing about the results mentioned above rules out such additions, and Rakić (1997a) has shown precisely in which way an equivalence relation of simultaneity can be added as a conservative extension to the structure of a single Minkowski space-time.

Which of these options should a defender of presentism choose? While option (2) seems unavailable, given the importance of the notion of simultaneity, option (1) can easily be invoked. Dialectically, however, that option is not fully satisfactory: the defense of the present either becomes hostage to specific empirical facts about the actual general-relativistic space-time we inhabit, or, going beyond general relativity, the issue is deferred to a future theory of quantum gravity about which there is no consensus yet. It would be better to provide a different response, and that is what we will try in this chapter. In fact, we will provide *two* different responses, one based on option (3) and one based on option (4), which are geared toward two different questions about the present that are mostly run together, but which need to be kept apart.

As already stated in Section 10.1, the notion of the present plays a double role, one static (concerning a time coordinate) and one dynamic (concerning existence). Terminologically, we will distinguish the two relations that characterize these two different roles as simultaneity vs. co-presentness. We hold that both of these relations have to be equivalence relations,[10] but they need not be the same, and simultaneity can be relativized to a frame.

Simultaneity characterizes the present as the time of now, indicating a temporal location. Present events in this static sense are those that are

[10] We therefore do not discuss the strategy of denying that the relevant notions of simultaneity or co-presentness have to be equivalence relations. This strategy is followed by many proponents of an extended present, such as Hestevold (2008) or Baron (2012), who allow for overlapping but distinct nows, which leads to a failure of transitivity. Dialectically, denying the requirement of an equivalence relation comes with an additional burden of justification, and so it will be good if we can avoid it.

simultaneous with now, having the same temporal coordinate. This static role of the present has no immediate metaphysical or ontological import, and it should therefore not be the target of our modeling efforts in defense of presentism. In our view, the present in the sense of the time coordinate of now can be fully accounted for by the relativizing strategy (3), making it a matter of perspective. The dependence on a concrete being's rest frame is not problematic, as full agreement in communication can be ensured. As relativity theory poses no obstacle to defining an observer-relative notion of simultaneity anchoring the static present, we will not comment further on the notion of simultaneity here.[11]

Co-presentness, on the other hand, characterizes the present as that which is currently (now) real, indicating an objective, dynamic boundary between the fixed past and the open future of possibilities. These modal notions have ontological import and must not be relativized to an observer or an agent.[12] Considering the above list of options, it is clear, therefore, that we need to invoke option (4): Additional formal structure over and above that provided by a single Minkowski space-time is needed to define a dynamic relation of co-presentness among events.

Rakić's strategy of adding an equivalence relation to the basic structure of a single space-time is one route that might be used to anchor a dynamic relation of co-presentness. Following that recipe, one arrives at a relation that can in fact fulfill both the static and the dynamic requirements of a notion of the present: Rakić's (1997a) result allows for a foliation of Minkowski space-time into space-like hypersurfaces to be added conservatively, and events on the same hypersurface can then be taken to be both simultaneous *and* co-present. While this may be an advantage, one might also be critical of the combination, as there is a price to be paid: first, there can be no empirical test of the chosen equivalence relation, and second, one undercuts the independently motivated strategy (3) of accounting for the static (coordinate) notion of simultaneity via the relativization to a speaker's rest frame.[13]

In what follows we will work toward a different objective notion of dynamic co-presentness that is fully anchored in the modal notions of fixed

[11] See Müller (2006, §2) for formal details of how to work out the mentioned relativization.

[12] See, e.g., Gödel (1949, p. 258n), who says that "existence by its nature is something absolute", or Prior (1996, p. 50), who insists that "you can't have a thing existing from one point of view but not from another".

[13] In fact, such an attempt would then involve an error theory: speakers who posit the present of their rest frame as the objective present would almost certainly fail to identify the true objective notion of simultaneity, but would have no empirical means to find out about this.

past vs. open future. This relation will generally not work as a static relation of simultaneity, as the region of co-presentness will normally be extended both spatially *and* (coordinate-)temporally. The formal resources are provided by the BST notions of modal funny business (MFB), discussed in Chapter 5, of *causae causantes* as sets of transitions, analyzed in Chapter 6, and of Minkowskian Branching Structures, described in Chapter 9.1. But first we have to argue that the notion of an extended dynamic present makes good sense.

10.3 Making room for an extended dynamic present

The dynamic role of time is to account for the possibility of dynamic change, both with respect to which things exist and what their properties are. Change in that sense needs to be contrasted with so-called Cambridge change, which is just a thing's having different properties at (or with respect to) different temporal locations. Dynamic change must be more than that if it really requires a dynamic notion of time, because the static notion of temporal location is sufficient to account for Cambridge change. It is, however, notoriously difficult to spell out what dynamic change amounts to.

As announced earlier, we will explore a radical view of dynamic change: change as the indeterministic realization of one option from among a set of alternatives. Such indeterministic happenings clearly amount to change: if Alice orders fries in a Pittsburgh restaurant, or if a radium atom decays, or if a cat jumps to catch a bird, these are indeterministic events that did not have to happen, and their occurrence makes a difference to what the world is like, realizing one possibility for the future in contrast to all the others. In a nutshell, a dynamic change is a transition from open possibilities to settled facts.

Given this indeterministic notion of dynamic change, we need a corresponding notion of dynamic time to anchor the indeterministic realization of possibilities. In a second radical move, we will explore the view that, just as dynamic time is necessary for real change, so real change is necessary for real, dynamic time: No change without time, but also no time without change. In this way, we strongly dissociate the static notion of coordinate time (temporal location) from the dynamic notion of real time. This makes room for yet another move that may be perceived as radical: we will allow a moment of dynamic time to be extended not just spatially, but also coordinate-temporally.

Whether this move is really radical is debatable. A conflict between our semantic or metaphysical intuitions and an indeterminism-based notion of dynamic time could only arise if indeterminism was scarce, so that the dynamic present of an event (e.g., of an utterance) extended for so long that it included events that we would speak of as future. In such a case, the dynamic past, present, and future of our analysis might conflict with the grammatical tenses in English. Whether there is such a conflict thus depends on broadly empirical matters. We will not enter into a lengthy discussion here, but state just one observation that we take to be relevant. Consider a radioactive particle in a lab that has not decayed yet. On a standard understanding of radioactivity, at each moment in the past since the particle was brought into the lab, the particle *could* have decayed. Thus, there were many chancy events in the small spatio-temporal region of our lab—if we stick with the idealization of events as point-like, there could even have been uncountably many. In BST, these chancy events should be modeled as transitions involving choice points or elements of choice sets. Clearly, once we relax the idealization of point-like events, or stop individuating events by non-extended instants of time, we will end up with a smaller number of chancy events. But in any case, given radioactivity as standardly understood, there is really no scarcity of chancy events, and so the extension of dynamic time along the coordinate-time dimension should not pose a problem.

Our view of dynamic time needs to be distinguished sharply from other theories of an "extended present" that are neither based on indeterminism, nor on a distinction between static (coordinate) and dynamic (indeterministic) time. Taking into account indeterministic change, we have at our disposal a richer background before which to define dynamic time. This allows us to hold on to (dynamic) co-presentness as an equivalence relation, in contradistinction to theories that posit overlapping present moments (Hestevold, 2008; Baron, 2012). Before we show how, we first comment on the consequences of the assumption that there is no dynamic time without indeterministic change.

One consequence of the doctrine of dynamic time is that determinism implies no real change, and further, no dynamic time. That is, if real time and real change presuppose indeterminism, it follows that there is no real change, and no real time, in a deterministic world. This may seem outrageous. Take a simple deterministic world, modeled via a single Newtonian space-time, in which a number of point particles move about on continuous trajectories. If initial conditions are properly chosen so that there are no collisions or other problematic configurations, the motion of the particles in such a world may indeed be without physically possible alternatives, thus exhibiting determinism. According to our approach, we have to say that in such a world, there is no real, indeterministic change. There is never a non-trivial range of options from among which only one is realized; there is always and everywhere just one single option to begin with. But the particles in that world move around, changing their absolute locations, as well as their relative ones. Surely that amounts to change in that world?

Given the distinctions we are making, we can agree that such a world harbors Cambridge change: the particles have different locations at different times. But from a dynamic perspective, nothing is really happening. The temporal coordinate is just like another spatial coordinate, along which there can of course be some variation in the configuration of the particles. But it is all just one four-dimensional block without any real dynamics. Everything is accounted for by four-dimensional geometry. From the point of view of dynamic time, every event in the whole deterministic four-dimensional space-time is co-present with every other event (and, of course, with itself). The dynamic present of the deterministic world is maximally extended to the whole space-time block. Matters are only different if one introduces indeterminism.

Linking time to indeterminism has had a few well-known proponents. William James offers a strong image of determinism which deprives the world of dynamics: "The whole is in each and every part, and welds it with the rest into an absolute unity, an iron block, in which there can be no equivocation or shadow of turning" (James, 1884, p. 150). A similar position is advocated by Whitrow (1961, pp. 295f.):

> Strict causality would mean that the consequences pre-exist in the premises. But, if the future history of the universe pre-exists logically in the present, why it is not already in the present? If, for the strict determinist, the future is merely "the hidden present", whence comes the illusion of temporal succession? The fact of transition and 'becoming' compels us to recognize the existence of an element of indeterminism and irreducible contingency in the universe. The future is hidden from us—not in the present, but in the future. Time is the mediator between the possible and the actual.

Similar elaborations of this view can be found in Eddington (1949, 1953).[14] More recently, this position underlies Ellis's models of an evolving block universe:

> Things could have been different, but second by second, one specific evolutionary history out of all the possibilities is chosen, takes place, and gets cast in stone. (Ellis, 2006, pp. 1812f)

The doctrine that real time requires modal indeterminism has been vigorously opposed.[15] But neither friends nor foes of the doctrine have expressed the underlying association between time and indeterminism with enough rigor to enable a formal treatment of the doctrine. It is precisely this task to which the rest of this chapter is devoted.

10.4 The dynamic present, past, and future: Two approaches

We have agreed to link dynamic time to indeterminism; we still need to decide how this link is to be defined. Apart from the dynamic present, we need to consider the accompanying notions of the dynamic past and the dynamic future. This triad constitutes the flow of (dynamic) time, and so the past, the present, and the future should be explicated together as equally dynamic. Our distinction between the static present (defined in terms of simultaneity) and the dynamic present (defined in terms of co-presentness and, ultimately, in terms of indeterminism) carries over to the past and the future as well. We will thus contrast the static past and future, explained in terms of relations between coordinates, with the dynamic past and future, explained in terms of indeterminism. As we remarked, the dynamic present can be (coordinate-)temporally extended. The dynamic past and future can therefore be different from their static counterparts as well.

As we stated in Section 10.2, the main challenge for presentism posed by relativity theory is that the space-time of special relativity does not contain

[14] The view also bears some affinity to Reichenbach (1952, p. 276), who argues that "[t]he distinction between the indeterminism of the future and the determinism of the past has found, in the end, an expression in the laws of physics". Reichenbach refers to quantum mechanics, and claims that "[t]he consequences for the time of our experience [...] are evident" (Reichenbach, 1952, p. 276). He also advocates an indexical (or, as he says, "token-reflexive", p. 277) treatment of the passage of time.

[15] See, e.g., Gale's (1963) attempt to rebut Whitrow's and Eddington's arguments.

enough structure to define a frame-invariant notion of simultaneity. With respect to the Minkowski space-time of special relativity, our discussion of Minkowskian Branching Structures in Chapter 9.1 has shown that BST offers additional resources, viz., the formal representation of indeterminism via transitions, while the BST ordering in such structures is directly taken from the Minkowskian ordering of cause-like relatedness defined in special relativity. So in this sense, BST is relativity-friendly, and we can consider constructions that base a notion of dynamic time on the primitive notions of BST to be relativity-friendly as well. As we discussed at length in Chapter 9, it is difficult to say whether BST is relativity-friendly in the more general sense of fully capturing possible cases of indeterminism in space-times of General Relativity. But at least there are encouraging positive results also with respect to this question, such as the fact that we can represent single General Relativistic space-times as BST orderings. And the philosophical discussion about the definability of simultaneity focuses on special, not on General Relativity—not the least because some models of General Relativity arguably admit an objective notion of simultaneity (see footnote 5). With respect to our topic here, we thus feel justified to proceed from the assumption that a construction in terms of the primitive notions of BST will be relativity-friendly.

How can we characterize the notion of dynamic time precisely? Here we state the set of desiderata that the sought-for relativity-proof notions of dynamic past, present, and future should ideally satisfy. These desiderata were motivated by our preceding discussion.

Definition 10.1 (Desiderata for dynamic past, present, and future)**.** For any event e,

- D1. Past(e), Present(e), and Future(e) are defined in a relativity-friendly way;
- D2. no two of Past(e), Present(e), and Future(e) overlap;
- D3. Past(e) and Future(e) are symmetric in the sense that e' is in Past(e) iff e is in Future(e');
- D4. Past(e) is settled in the modal sense that if a possible scenario includes e, it also includes Past(e);
- D5. any possible scenario to which e belongs is fully partitioned by Past(e), Present(e), and Future(e);
- D6. in any possible scenario, the co-presentness relation (e is co-present with e' iff e is in Present(e')) is reflexive, symmetric, and transitive.

As we argued above, desideratum (D1) is satisfied once we define the notions of dynamic time in terms of the primitive notions of BST. In BST, a (maximal) possible scenario is a history, so that items (D4) and (D5) should be read in terms of histories. As we will show in what follows, there are non-trivial BST structures that satisfy all the desiderata (D1)–(D6).[16] Furthermore, we can spell out exact necessary and sufficient conditions for BST structures that fulfill these desiderata. Perhaps surprisingly, these conditions involve a strong form of modal funny business. We postpone the formal statement of these general results to Section 10.7. Before that, we provide some more motivation by characterizing two different routes to explicating co-presentness via the resources of BST.

The first route focuses on what delineates co-present events from below (i.e., on the events' causal past). It is reasonable to require that dynamically co-present events should share the same *indeterministic causal factors*. In BST, causally significant factors for a given event e are captured by the notion of the *causae causantes* for e, which are indeterministic transitions. Furthermore, the initial of a *causa causans* for e can lie in the past light-cone of e, but it can also be SLR to e if modal funny business is present (see Fact 6.4(3)). Therefore, a focus on causal factors permits us, in the end, to remove the restriction involved in focusing on factors in the causal past only. It is natural to define co-present events as those events that share the same set of *causae causantes*. With co-presentness so defined, it is straightforward to define the notions of the dynamic present, the dynamic past, and the dynamic future of an event e. The resulting notion of dynamic time is relativity-proof. As we will show, this explication of dynamic time does not guarantee that all of the desiderata of Def. 10.1 are fulfilled—the problematic item is (D5), the full partitioning of any history into past, present, and future. All the desiderata are, however, fulfilled in structures in which there is a quite specific form of modal funny business, so that the co-presentness relation reaches all across a history. The *causae causantes*-based approach is described in detail in Section 10.5. In spelling it out, we will write co-presentness of e and e' as "$\mathrm{CP}_C(e, e')$", with subscript "C" for "*causae causantes*", and similarly for "$\mathrm{Present}_C(e)$".

The second route, described in Section 10.6, takes a language-oriented turn that reflects how we link our talk about the future with indeterminism.

[16] There are also trivial structures that satisfy the desiderata, e.g., deterministic structures that contain only a single history structures, or structures that are basically linear. See Section 10.7 for details.

As not every future event is undetermined (contingent), some work is needed to spell out what the open future of a given event is. The resulting notion is used to explain the relation "e' belongs to the dynamic future of e". This relation naturally permits one to define a sister notion of "e belongs to the dynamic past of e'". Hence we arrive at the concepts of the dynamic past and dynamic future of a given event. Finally, the dynamic present of e from a given history h is identified with whatever remains from h once the dynamic past and the dynamic future of e are removed. This approach also delivers relativity-proof notions of the dynamic present, past, and future, but again, not all of the desiderata of Def. 10.1 are fulfilled automatically. Here, item (D6) is problematic: the notion of co-presentness may fail to be transitive, even when restricted to the events in one single history. Again, a strong form of modal funny business provides a sufficient condition for fulfilling all the desiderata. On this semantics-based approach, we will write co-presentness of e and e' as "$\text{CP}_S(e, e')$" and the present of e as "$\text{Present}_S(e)$", with subscript "S" for "semantic".

Both approaches suggest that our intuitive ideas about the natural features of dynamic time, as far as they can be made exact, place significant demands on the indeterministic structure of *Our World* $\mathcal{W}$: for all of our intuitive desiderata of Def. 10.1 to hold without restrictions, a strong form of MFB is required. In Section 10.7 we spell out this dependence in the form of a number of precise formal results.

As our discussion concerns the space-times of physics, it is advisable to use a BST framework in which the local Euclidicity condition can be satisfied. For this reason we use BST_{NF}. For the purposes of illustrating our results in Section 10.8, we also use the version of Minkowskian Branching Structures that builds upon BST_{NF}, as described in Chapter 9.1.[17]

10.5 Dynamic time via *causae causantes*

Given the formal background of BST_{NF}, we have at our disposal a precise candidate definition of dynamic time: time passes at exactly those events that belong to choice sets. At other events, there is no indeterminism, no dropping off of histories, no realization of one possibility in contrast to

[17] In this book we do not attempt to model any additional indeterministic *dynamics* of the passage of time. As we said in Section 10.2, we assume that an indexical treatment of the "dropping off of histories" is appropriate for our communication purposes.

others, no real change, and therefore no passing of dynamic time. Of course, as stressed above, static (coordinate) time also passes at other events—but our target here is exclusively dynamic time.

To provide the necessary background, we recall the definitions of cause-like loci and *causae causantes* in $\mathrm{BST_{NF}}$, in the version that allows for MFB. Based on Def. 5.12 of cause-like loci (cll) for outcome chains and on Def. 6.7 of the set of *causae causantes* of a transition in an MFB context (compare also Def. 6.3), we arrive at the following:

Definition 10.2 (Cause-like loci and *causae causantes* for e)**.**

$$cll(e) =_{\mathrm{df}} \{\ddot{c} \subseteq W \mid \exists h \in \mathrm{Hist}\; h \perp_{\ddot{c}} H_e\}.$$

$$CC(e) =_{\mathrm{df}} \{\ddot{c} \rightarrowtail \Pi_{\ddot{c}}\langle H_e\rangle \mid \ddot{c} \in cll(e)\}, \text{ where}$$

$$\Pi_{\ddot{c}}\langle H_e\rangle = \bigcup\{H \in \Pi_{\ddot{c}} \mid H \cap H_e \neq \emptyset\} = \bigcup\{H_c \subseteq \mathrm{Hist} \mid c \in \ddot{c} \wedge H_c \cap H_e \neq \emptyset\}.$$

That is, the second element of a *causa causans* for e is the union of the elementary outcomes of $\ddot{c}$ that permit the occurrence of e. And, to recall, if there is no MFB, every *causa causans* of e is a basic proposition-like transition with an initial in $cll(e)$, i.e., $\Pi_{\ddot{c}}\langle H_e\rangle \in \Pi_{\ddot{c}}$ (see the discussion of Fact 5.7).

A definition of co-presentness in terms of an event's *causae causantes* is not hard to come by (note the subscript "C" for "*causae causantes*"):

Definition 10.3 (Co-presentness based on *causae causantes*)**.** Events e_1, $e_2 \in W$ are *co-present*, written $\mathrm{CP}_C(e_1, e_2)$, iff $CC(e_1) = CC(e_2)$.

Being based on an identity, the relation $\mathrm{CP}_C(\cdot,\cdot)$ is clearly an equivalence relation on W. And it is well-defined no matter whether there are modal correlations or not, by the generality of Def. 10.2.

There is an alternative way of characterizing $\mathrm{CP}_C(\cdot,\cdot)$, viz., in terms of the sameness of histories, as shown by the following fact:

Fact 10.1. *We have $CP_C(e_1, e_2)$ iff $H_{e_1} = H_{e_2}$.*

Proof. "$\Leftarrow$": From the Definition 10.2 of *causae causantes*, if $H_{e_1} = H_{e_2}$, then $CC(e_1) = CC(e_2)$, and hence $\mathrm{CP}_C(e_1, e_2)$.

"$\Rightarrow$": For this direction, we show that the set of histories H_e in which an event e occurs is determined by its *causae causantes* $CC(e)$. We show that for any $e \in W$,

$$H_e = \bigcap\{\Pi_{\ddot{c}}\langle H_e\rangle \mid (\ddot{c} \rightarrowtail \Pi_{\ddot{c}}\langle H_e\rangle) \in CC(e)\}. \qquad (*)$$

Given (*), from $\mathrm{CP}_C(e_1, e_2)$, i.e., $CC(e_1) = CC(e_2)$, we immediately have $H_{e_1} = H_{e_2}$.

For the "$\subseteq$" direction, by Definition 10.2, if $h \in H_e$, then for every $\ddot{c} \in cll(e)$ there is $c \in \ddot{c}$ such that $h \in H_c$, so $h \in \Pi_{\ddot{c}}\langle H_e\rangle$, by the same definition.

For the "$\supseteq$" direction, let us assume for reductio that there is (†) $h \in \bigcap\{\Pi_{\ddot{c}}\langle H_e\rangle \mid (\ddot{c} \rightarrowtail \Pi_{\ddot{c}}\langle H_e\rangle) \in CC(e)\}$, but $h \notin H_e$. Take some $h' \in H_e$. As $e \in h' \setminus h$, by $\mathrm{PCP_{NF}}$ there is a choice set $\ddot{c}$ at which $h \perp_{\ddot{c}} h'$, and $c \in \ddot{c}$ such that $c \leqslant e$. This implies $h \perp_{\ddot{c}} H_e$, so that $\ddot{c} \in cll(e)$, and $(\ddot{c} \rightarrowtail \Pi_{\ddot{c}}\langle H_e\rangle) \in CC(e)$. Since $h \perp_{\ddot{c}} H_e$, we get $h \notin \Pi_{\ddot{c}}\langle H_e\rangle$, and hence $h \notin \bigcap\{\Pi_{\ddot{c}}\langle H_e\rangle \mid (\ddot{c} \rightarrowtail \Pi_{\ddot{c}}\langle H_e\rangle) \in CC(e)\}$, which contradicts (†). □

This is a welcome result: even though BST allows for modal correlations, there are still two differently motivated definitions of co-presentness that characterize the same relation.[18] Note that the identity (*) still obtains if we restrict $\ddot{c}$ to the *past* cause-like loci of e—this is the subject of Exercise 10.2.

A typical shape of a region of co-presentness in the absence of modal correlations is shown in Figure 10.1. Modal correlations allow for more extended regions of co-presentness. A pertinent example is shown in Figure 10.2. The generalization to larger sets of correlated choice sets is suggestive: if many space-like related, modally correlated choice sets exist, a region of co-present events can spatially extend arbitrarily far, up to universal Sticky MFB spanning all of space-time (see Def. 10.9 in Section 10.7).

We showed above that the relation $\mathrm{CP}_C(\cdot,\cdot)$ on W is an equivalence relation. It follows that the restriction of $\mathrm{CP}_C(\cdot,\cdot)$ to any history h is also an equivalence relation. We may thus use the restricted $\mathrm{CP}_{C|h}(\cdot,\cdot)$ to carve the dynamic present of an event e from a history containing e. Excluding

[18] One might perhaps criticize our definition because in the presence of modal correlations, it allows for events to be co-present while their obvious alternatives fail to be co-present. For a pertinent example, consider two ternary (outcomes 1, 2, 3) choice sets $\ddot{c}$ and $\ddot{e}$, with elements c_1, c_2, c_3 and e_1, e_2, e_3, respectively, whose 1-outcomes are strictly correlated, while the 2- and 3-outcomes are uncorrelated, leading to the five (instead of nine) histories $h^{11}, h^{22}, h^{23}, h^{32}, h^{33}$. Here c_1 and e_1 count as co-present (they occur exactly in history h^{11}), but the alternative events c_2 and e_2 do not count as co-present. We are not aware of a thorough discussion of whether the dynamic present should be modally robust, and we do not view the mentioned situation as a failure of our definition, as a fairly straightforward sharpening is available. To define co-presentness such that only modally robust pairs of events are included, we need a formal notion of alternatives, which is easily available for elements of a choice set: we just declare any two elements of a choice set to be alternatives. For such alternatives, a modified definition of co-presentness is not hard to come by; see Exercise 10.3. The hints to that exercise also provide some suggestions for how to define alternatives in the general case.

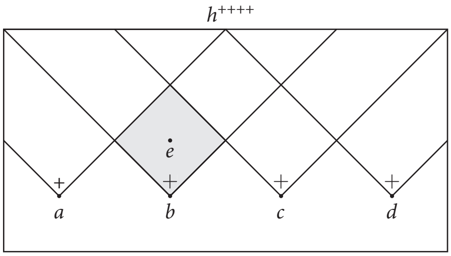

Figure 10.1 The region of events co-present with event e in one history of a BST structure. There are four binary (+/−) choice sets $\ddot{a}$, $\ddot{b}$, $\ddot{c}$, and $\ddot{d}$, and no modal correlations. Thus there are sixteen possible histories, of which h^{++++} is shown. Event e and all events in the shaded region have just a single *causa causans*, $\ddot{b} \rightarrowtail b$. They occur in exactly those eight histories in which the choice set $\ddot{b}$ has outcome +.

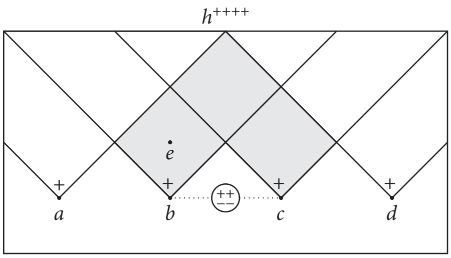

Figure 10.2 The region of events co-present with event e in one history of a BST structure. There are four binary (+/−) choice sets $\ddot{a}$, $\ddot{b}$, $\ddot{c}$, and $\ddot{d}$, and outcomes of $\ddot{b}$ and of $\ddot{c}$ are modally correlated. Thus there are eight possible histories, of which h^{++++} is shown. Event e and all events in the shaded region have a set of two *causae causantes*, $\{\ddot{b} \rightarrowtail b, \ddot{c} \rightarrowtail c\}$. They occur in exactly those four histories in which choice set $\ddot{b}$ (and thus, by modal correlation, also choice set $\ddot{c}$) has outcome +.

featureless structures (see Def. 10.4), the restricted equivalence relation $\mathrm{CP}_{C|h}(\cdot,\cdot)$ is neither the identity nor the universal relation on the history in question. We can sum up this result as Theorem 10.1, which shows that we have indeed defined a non-frame-dependent, non-trivial equivalence relation of co-presentness on relativistic space-times, based on spatio-temporal indeterminism.

To be formally precise, here is the definition that singles out those $\mathrm{BST}_{\mathrm{NF}}$ structures in which our construction only yields a trivial equivalence relation:

Definition 10.4. A BST_{NF} structure is called *featureless* if either (i) it contains just a single history (i.e., no indeterminism), or (ii) it has at least one history consisting wholly of elements of choice sets and in which there are no modal correlations.

Our Theorem then reads as follows:

Theorem 10.1. *Let $\langle W, < \rangle$ be a BST_{NF} structure that is not featureless. Then for any history $h \in \text{Hist}(W)$, the relation of co-presentness $CP_C(\cdot,\cdot)$ restricted to h, $CP_{C|h}(\cdot,\cdot)$, is a non-trivial equivalence relation on h, i.e., neither the identity nor the universal relation on h.*

Proof. Let h be a history in a BST_{NF} structure $\langle W, < \rangle$. As mentioned, $\text{CP}_{C|h}(\cdot,\cdot)$ is a restriction of an equivalence relation and thus is itself an equivalence relation. For non-triviality, assume first that $\text{CP}_{C|h}(\cdot,\cdot)$ is the universal relation on h. This means that all $e, e' \in h$ satisfy $H_e = H_{e'}$ (i.e., h contains no element of a choice set). This is only possible in a BST_{NF} structure with just one history, which is featureless according to Def. 10.4. The second type of triviality would be that $\text{CP}_{C|h}(\cdot,\cdot)$ is the identity relation on h. This implies that for any $e, e' \in h$, $H_e \neq H_{e'}$. This is only possible in a BST_{NF} structure in which each element of h is a member of an uncorrelated choice set, which again is a featureless structure according to Def. 10.4. □

Having defined co-presentness, we can define the dynamic present of a given event e as the set of events that are co-present with e; we preserve the subscript "C" to indicate that we are dealing with a notion defined in terms of *causae causantes*:

Definition 10.5 (Dynamic present based on *causae causantes*)**.** Let e be an event in a BST_{NF} structure $\mathscr{W} = \langle W, < \rangle$. The dynamic present of e is defined as

$$\text{Present}_C(e) =_{\text{df}} \{e' \in W \mid \text{CP}_C(e, e')\} = \{e' \in W \mid H_{e'} = H_e\}.$$

In order to provide a full explication of dynamic time based on *causae causantes*, so that we can match our approach with the list of desiderata given above (Def. 10.1), we need to spell out definitions of the dynamic past and the dynamic future of an event e as well. Clearly, the dynamic future of e must come after the dynamic present of e and must not overlap with it. Analogously, the dynamic past of e must be before the dynamic present of e and must not overlap with it either. Accordingly, $e' \in \text{Future}(e)$ iff $e' \notin \text{Present}_C(e)$ and there is $e'' \in \text{Present}_C(e)$ such that $e'' < e'$. The last two conditions imply $H_{e'} \subseteq H_{e''} = H_e$, so by $e' \notin \text{Present}_C(e)$, we get $H_{e'} \subsetneq H_e$. By an analogous argument we have $e' \in \text{Past}(e)$ iff $H_e \subsetneq H_{e'}$. So these are our definitions for Past(e) and Future(e) (we put no subscript on these definitions as they turn out to coincide with the definitions on the semantic approach of Section 10.6, see Fact 10.5(1,3)):

Definition 10.6 (Dynamic past and future based on *causae causantes*)**.** The dynamic future of e is $\text{Future}(e) =_{\text{df}} \{e' \in W \mid H_{e'} \subsetneq H_e\}$. The dynamic past of e is $\text{Past}(e) =_{\text{df}} \{e' \in W \mid H_e \subsetneq H_{e'}\}$.

Given these definitions, it is easy to see that any event e has a non-empty dynamic present that contains it; $e \in \text{Present}_C(e)$. It might transpire, however, that Past(e) is empty, or Future(e) is empty, or both. By Theorem 10.1, the latter case obtains in a deterministic world (i.e., in a BST structure containing just one history): in a deterministic world there is no real change, and any two events are co-present. The cases of empty dynamic past or future can be interpreted as restricted forms of determinism. Quite generally, if indeterminism is scarce, then a large region of a history contains no real change and belongs fully to the present of an appropriate event.

Past(e), Present$_C(e)$, and Future(e) as just defined have many welcome features. They are defined in purely modal terms, based on the inclusion relation among sets of histories, and they satisfy desiderata (D1)–(D4) and (D6) of our list from Section 10.4, as shown by the following Fact:

Fact 10.2. *Let e be an event in a BST_{NF} structure $\mathscr{W} = \langle W, < \rangle$. Then with respect to the notions of Present$_C(e)$, Past(e), and Future(e) from Defs. 10.5 and 10.6, the following desiderata of Def. 10.1 are fulfilled:*

- *(D1) Present$_C(e)$, Past(e), and Future(e) are defined in a relativity-friendly way,*
- *(D2) any two of Past(e), Present$_C(e)$, and Future(e) have an empty overlap;*
- *(D3) $e' \in \text{Past}(e)$ iff $e \in \text{Future}(e')$;*
- *(D4) the dynamic past and present are modally settled:*
  $e \notin \text{Past}(e)$ and for every $h \in H_e$: $\text{Past}(e) \subseteq h$; furthermore,
  $e \in \text{Present}_C(e)$ and for every $h \in H_e$: $\text{Present}_C(e) \subseteq h$; and
- *(D6) the co-presentness relation $CP_C(\cdot,\cdot)$ is reflexive, symmetric, and transitive.*

Proof. (D1) This holds because we work in terms of the primitive notions of BST, as argued in Section 10.4.

(D2) Immediate from Defs. 10.5 and 10.6.

(D3) Immediate from Def. 10.6.

(D4) From Def. 10.6 it follows that $e \notin \text{Past}(e)$ for any $e \in W$. The settledness of the past follows from the downward closure of histories: Let $h \in H_e$ and $e' \in \text{Past}(e)$. As $H_e \subsetneq H_{e'}$ by the definition of $\text{Past}(e)$, we have $e' \in h$. Hence $\text{Past}(e) \subseteq h$.

From the reflexivity of $\text{CP}_C(\cdot,\cdot)$, via Def. 10.3, we have $e \in \text{Present}_C(e)$ for every $e \in W$. Finally, to show the modal settledness of the present, pick some $h \in H_e$. Since for every $e' \in \text{Present}_C(e)$, $H_{e'} = H_e$, we have $\text{Present}_C(e) \subseteq h$.

(D6) The reflexivity, symmetry, and transitivity of $\text{CP}_C(\cdot,\cdot)$ all follow from its definition via an identity (Def. 10.3), as remarked above. □

Here are some further welcome features of our definitions: an event's dynamic future, if non-empty, is never fully contained in a single history, it is thus open. Furthermore, the dynamic past is closed downward, whereas the dynamic future is closed upward. Item (4) of the following Fact points out the conditions under which desideratum (D5) of Def. 10.1, the full partitioning of any history into past, present, and future of any of its events, is satisfied.

Fact 10.3. *Let e be an event in a BST_{NF} structure $\mathscr{W} = \langle W, < \rangle$. Then the following holds:*

(1) *If $Future(e) \neq \emptyset$, then there is no history $h \in \text{Hist}$ such that $Future(e) \subseteq h$;*
(2) *$Future(e)$ is closed upward: if $e' \in Future(e)$ and $e' \leqslant e''$, then $e'' \in Future(e)$;*
(3) *$Past(e)$ is closed downward: if $e' \in Past(e)$ and $e'' \leqslant e'$, then $e'' \in Past(e)$.*
(4) *Let $h \in H_e$. Then desideratum (D5) of Def. 10.1 holds (i.e., $h \subseteq (Past(e) \cup Present_C(e) \cup Future(e))$) iff for every $e' \in h$: $H_e \subseteq H_{e'}$ or $H_{e'} \subseteq H_e$.*

Proof. (1) Let $f \in \text{Future}(e)$. By Def. 10.6, this means that $H_f \subsetneq H_e$. Take $h_f \in H_f$ and $h \in H_e \setminus H_f$; by this choice, $h \neq h_f$. So we can pick some $e' \in h \setminus h_f$. By PCP_{NF}, there is thus some choice set $\ddot{c}$ with unique elements $c \in \ddot{c} \cap h$ and $c_f \in \ddot{c} \cap h_f$, $c \neq c_f$, for which $c < e'$. By directedness of h, there is some $f' \in h$ for which $c \leqslant f'$ and $e \leqslant f'$. The latter implies $H_{f'} \subseteq H_e$ by

Fact 2.2(2), and as $c_f \in h_f$, we have $h_f \notin H_{f'}$ (else also $c \in h_f$, contradicting unique intersection of choice sets with histories). So we have $H_{f'} \subsetneq H_e$, i.e., $f' \in \text{Future}(e)$. Similarly, by directedness of h_f, there is some $f'' \in h_f$ for which $e \leqslant f''$, $c_f \leqslant f''$, and $f \leqslant f''$. As $H_f \subsetneq H_e$ and $H_{f''} \subseteq H_f$ by $f \leqslant f''$, we have $H_{f''} \subsetneq H_e$, i.e., $f'' \in \text{Future}(e)$. Now there can be no history h' that contains both f' and f'', because any such history would have to contain the inconsistent elements c and c_f of the choice set $\ddot{c}$.

(2) and (3) follow from the observation that if $e' \leqslant e''$, then $H_{e''} \subseteq H_{e'}$, invoking the transitivity of $\subseteq$.

(4) Let $e' \in h \in H_e$. The conclusion is immediate from Defs. 10.5 and 10.6: $e' \in (\text{Past}(e) \cup \text{Present}_C(e) \cup \text{Future}(e))$ iff $(H_e \subseteq H_{e'}$ or $H_{e'} \subseteq H_e)$. □

Fact 10.3(4) shows that the only thing that our definitions of dynamic time in terms of *causae causantes* leave open in general is desideratum (D5), the full partitioning of any history $h \in H_e$ by $\text{Present}_C(e)$, $\text{Past}(e)$, and $\text{Future}(e)$. As Figure 10.3 shows, our definitions imply that in general, the dynamic present of an event need not reach all across space. And as the dynamic past and future only contain events that are properly before or after an event in the dynamic present, that situation implies that a portion of an event's so-called *elsewhere* (i.e., a portion of the events *SLR* to it), will be neither dynamically past, nor present, nor future. In terms of *causae causantes*, the reach of causation in such structures does not extend far enough across space to allow one to classify these events one way or another.

As we are working toward a characterization of structures in which all of our desiderata are fulfilled, we do not enter into a lengthy discussion of the significance of a history not being partitioned into $\text{Past}(e)$, $\text{Present}_C(e)$, and $\text{Future}(e)$. One might argue that such a failure of (D5) makes possible the following odd situation: an object is in the past of e, continues to exist for a very long time, but avoids the present of e completely. A full discussion of such cases would require a theory of enduring objects in BST and probably also an explication of the notion of self-moving agents, two topics that we have to leave to the wayside here.[19]

The exhaustiveness of $\text{Past}(e)$, $\text{Present}_C(e)$, and $\text{Future}(e)$ can be guaranteed, but there is a price to be paid for that, viz., a strong form of modal correlations reaching all across space. We will consider the general situation

[19] See Belnap (2003a, 2005a) for some pertinent remarks.

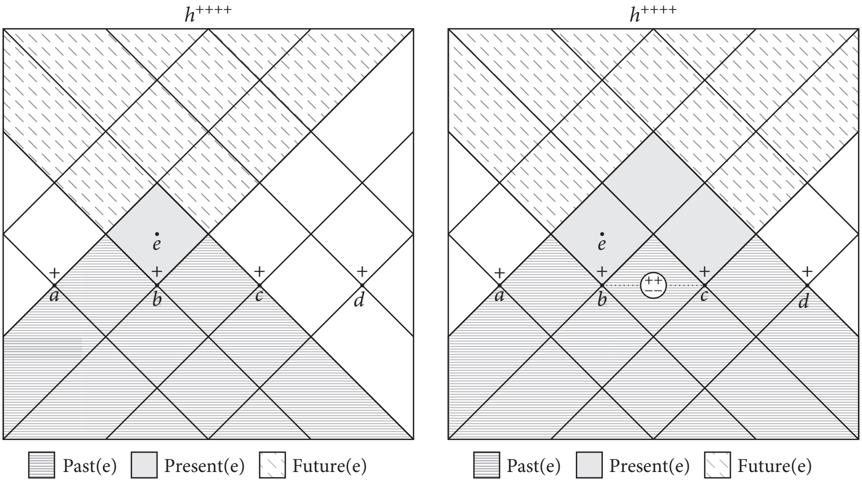

Figure 10.3 Illustration of the *causae causantes*-based notions of dynamic past, present, and future for the structures depicted in Figure 10.1 (left) and Figure 10.2 (right). The events in the white regions are neither in the dynamic past, nor in the present, nor in the future of e.

in Section 10.7. Here we show that desideratum (D5) can be violated in rather simple structures:

Fact 10.4. *Let $\mathscr{W} = \langle W, < \rangle$ be a BST_{NF} structure without MFB that contains two choice sets $\ddot{c}_1$ and $\ddot{c}_2$ such that c_1 SLR c_2. Then there is a history h in $\mathscr{W}$ and $e, e' \in h$ such that neither $H_e \subseteq H_{e'}$ nor $H_{e'} \subseteq H_e$. Thus, h is not partitioned by Past(e), Present$_C(e)$, and Future(e).*

Proof. For concreteness, take a 4-history BST_{NF} structure with no MFB, with two binary choice sets, $\ddot{c}_1 = \{c_1, c'_1\}$ and $\ddot{c}_2 = \{c_2, c'_2\}$, where $c_1\,SLR\,c_2$. (The case in which the choice sets have more members is exactly parallel.) Pick $e > c_1$ and $e\,SLR\,c_2$ and, symmetrically, $e' > c_2$ and $e'\,SLR\,c_1$. By no MFB, there are histories $h \in H_e \setminus H_{e'}$ and $h' \in H_{e'} \setminus H_e$. Thus, neither $H_e \subsetneq H_{e'}$ nor $H_{e'} \subsetneq H_e$, nor $H_{e'} = H_e$, so e' is neither in Past(e), nor in Future(e), nor in Present$_C(e)$. □

So, unless a structure is trivial, MFB is required in order to fulfill desideratum (D5) for $\text{CP}_C(\cdot,\cdot)$: the existence of pairs of histories h and h' as in the above proof must be prohibited by a specific form of MFB, by requiring that $\Pi_{\ddot{c}_1}(e) \in \Pi_{\ddot{c}_1}$ be correlated (in the sense of having a non-empty intersection) with $\Pi_{\ddot{c}_2}(e') \in \Pi_{\ddot{c}_2}$ only. So our illustration already suggests that not just *any* type of MFB is enough to secure that all histories are fully partitioned into the dynamic past, present, and future of any of their events. The construction of a structure with MFB that still violates (D5) is the subject of Exercise 10.4. To ensure the partitioning of histories, we need a specific form of MFB, which we call Sticky MFB and which is the subject of Definition 10.9 in Section 10.7.

10.6 Dynamic time via the semantics of the open future

In this Section we attempt to characterize dynamic time by reflecting how one might reasonably explain futurity in modal terms. That is, we imagine someone sympathetic to linking the dynamic future to its being open. To find the link, we look for the truth conditions for the sentence "f belongs to the future of e", which we then analyze in the semantic apparatus constructible on BST structures. We will find that this semantic take on dynamic time yields a structure similar to what we unearthed above, thus supporting our concept of dynamic time.

There are two intuitions that seem relevant to explaining the past, present, and future in modal terms: the settledness of the past intuition (**SP**), and the openness of the future intuition (**OF**). To clarify them, we turn our attention to how we speak about future events, and consider what explanations of futurity in modal terms are acceptable. So we will investigate schematic explanations of the form "event f belongs to the future of event e because ...". Once we find an acceptable explanation of this kind, we will turn it into truth-conditions for "f belongs to the future of e", and research the consequences of these truth-conditions. In our discussion we will focus on concrete token events.

Consider, therefore, two concrete events that appear to be good candidates for one being in the future of the other: the Summer Solstice in Prague in 2019 (s) and a rainy sunrise on Nov 20, 2018 in Del Mar (r). The first intuition sees the past as settled. That is, although before that particular sunrise in Del Mar things could have turned out differently (it might have been rainy, and it might have been sunny, or foggy, etc.), from the perspective of a future event, like s, it is settled (fixed, inevitable) that there was this rainy sunrise. The settledness of the past intuition suggests the following schematic explanation:

SP Event *s* belongs to the future of event *r* because at event *s* it is settled that *r* has happened.

The openness of the future intuition is more elusive; we begin tentatively with this proposal:

$\mathbf{OF}_1$ *s* belongs to the future of *r* because it is not settled at *r* that *s* will occur.

Our schematic explanation $\mathbf{OF}_1$ seems too strict, however. Perhaps this is overly optimistic, but we are inclined to think that no matter how the world evolves from its conditions in November 2018, there is the 2019 solstice in each of its possible evolutions. Answer $\mathbf{OF}_1$ sounds bad because in this case either the explanans is false, or *s* mysteriously does not lie in the future of *r* after all. Let us therefore try another one:

$\mathbf{OF}_2$ *s* belongs to the future of *r* because the way *s* will occur is not settled at *r*.

Although answer $\mathbf{OF}_2$ does not look immediately incorrect if applied to the Summer Solstice 2019, it is still counter-intuitive. Think of your grandfather's Swiss watch (mechanical, almost perfect, always wound); suppose it sits in an isolating contraption, and ask yourself if it is already settled how it will signal tomorrow's noon. Our intuition is that this is already settled, no matter what the watch's *surroundings* are—the watch is isolated, after all. Like with answer $\mathbf{OF}_1$, in this case either the explanans is false, or *s* does not lie in the future of *r*. The moral is that we need to accommodate the surroundings of *s*, which is what proposal $\mathbf{OF}_3$ does.

$\mathbf{OF}_3$ *s* belongs to the future of *r* because before *s* there is an event and some aspect of it that is not settled at *r*.

In other words, for *s* to belong to the future of *r*, one needs some, however small, contingency, like the presence of a radioactive particle that may or may not decay, to obtain before *s* (but not necessarily after *r*). According to this proposal, a small but properly located contingency makes tomorrow's event involving your grandfather's watch belong to the future of your reading these words, no matter how well the watch is isolated. $\mathbf{OF}_3$ thus commends itself as being sufficiently weak, while still linking the dynamic future to contingency.

Since the schematic explanations $\mathbf{OF}_3$ and **SP** of dynamic futurity seem acceptable, we will use them as truth-conditions for the sentence "*f* is in the dynamic future of *e*":

SPOF f is in the dynamic future of e iff at f it is settled that e has happened, and there is some event e' weakly before f and a subject matter A such that at e it is contingent that A obtains at the location of e'.

In what follows, we translate the schema **SPOF** into a regimented language amenable to a BST analysis, as introduced in Chapter 4.5. We can then find out the shape of the particular regions of the dynamic present, past, and future of a given possible event e.

We now formulate our schema **SPOF** in a semantic model based on BST_{NF} with space-time locations (see Def. 2.9); Ψ is the model's interpretation function.

Definition 10.7 (Dynamic future, semantic style)**.** Let $\mathscr{M} = \langle \mathscr{W}, \Psi \rangle$ be a semantic model based on a BST_{NF} structure with space-time locations $\mathscr{W} = \langle W, <, Loc \rangle$. For events $e, f \in W$ we say that f belongs to the future of e, written $f \in \text{Future}(e)$, iff there is an event $e' \in W$ and an atomic formula A such that

1. $e' \leqslant f$ and
2. $e \models Poss : At_{Loc(e')} : A$ and
3. $e \models Poss : At_{Loc(e')} : \neg A$, and
4. for every history h, if $f \in h$, then $e \in h$.

To explain, the first clause requires that a witness e' for f belonging to the future of e occur before f or be f itself. The meaning of clauses (2) and (3) is that from the perspective of e, it is contingent whether A is true at the location of the witness e'. The last clause encapsulates settledness of the past, **SP**. Note that we restrict our definition to atomic formulas, as other formulas might implicitly or explicitly refer to the past or the future, which would jeopardize our definition.

We next define the dynamic past and present of an event e by explicitly invoking desiderata (D3) and (D5) of Def. 10.1. We use the subscript "S" for "semantic-style":

Definition 10.8 (Dynamic past and present, semantic style)**.** An event e' belongs to the past of event e, written $e' \in \text{Past}(e)$, iff $e \in \text{Future}(e')$.
An event e belongs to the present of event e', written $e \in \text{Present}_S(e')$, iff e and e' are compatible and neither $e \in \text{Future}(e')$, nor $e \in \text{Past}(e')$. We write $\text{CP}_S(e, e')$ for co-presentness of e and e', defined as $\text{CP}_S(e, e') =_{\text{df}} e \in \text{Present}_S(e')$.

In order to spell out Defs. 10.7 and 10.8 in our BST framework, we need a semantic model based on a $\mathrm{BST_{NF}}$ structure with a set *Loc* of spatio-temporal locations, as explained in Chapter 4.5. In addition, we need to impose two constraints on the interpretation function.

To recall, a model's interpretation function Ψ maps the set *Sent* of sentences of our language $\mathscr{L}$ to the set of sets of indexes of evaluation, i.e., $\Psi : Sent \to \mathscr{P}(E/\mathrm{Hist})$, where $E/\mathrm{Hist} =_{\mathrm{df}} \{e/h \mid e \in W \wedge h \in H_e\}$. This definition thus allows for an atomic formula to be true at some e/h but false at e/h', for different histories h, h' passing through the same event e. If histories h and h' do not split at the point event e, the interpretation function should, however, not discern between the two indices for *atomic* sentence A. Now, in $\mathrm{BST_{NF}}$ structures, histories do *not* split at point events, as there are no maximal elements in the overlap of histories. Thus, in a semantic model based on a $\mathrm{BST_{NF}}$ structure it is reasonable to require the following:

Postulate 10.1. *For any atomic sentence A and for any $e \in W$:*

$$\{e/h \mid h \in H_e\} \subseteq \Psi(A) \quad or \quad \{e/h \mid h \in H_e\} \cap \Psi(A) = \emptyset.$$

Our next assumption concerns the relation between branching histories of $\mathrm{BST_{NF}}$ and qualitative differences between histories, the latter being induced by the interpretation function. Uncontroversially, two branching histories should be qualitatively different somewhere. $\mathrm{BST_{NF}}$ structures introduce a good candidate as to where the differences are to be located, as they have a well-defined notion of minimal elements in the difference of any two histories—such minimal elements form a choice set. Our second postulate says that there is some qualitative difference at those minimal elements:

Postulate 10.2. *For histories h_1, h_2 in a BST_{NF} structure, if $h_1 \perp_{\ddot{c}} h_2$, then there is an atomic sentence A such that*

$$(c_1/h_1 \in \Psi(A) \quad but \quad c_2/h_2 \notin \Psi(A)) \quad or \quad (c_1/h_1 \notin \Psi(A) \quad but \quad c_2/h_2 \in \Psi(A)),$$

where $\{c_i\} = \ddot{c} \cap h_i$ ($i = 1, 2$).

We are now in a position to prove that the dynamic future, present, and past are characterized modally, that is, in terms of the inclusion of histories, exactly like in Defs. 10.5 and 10.6:

Fact 10.5. *Let $\mathscr{M} = \langle W, <, Loc, \Psi \rangle$ be a semantic model based on a BST_{NF} structure with spatio-temporal locations $\mathscr{W} = \langle W, <, Loc \rangle$, where the interpretation function Ψ satisfies Postulates 10.1 and 10.2. Then for every $e, f \in W$:*

1. *$f \in Future(e)$ iff $H_f \subsetneq H_e$;*
2. *if $Future(e) \neq \emptyset$, then there are incompatible $f', f'' \in Future(e)$;*
3. *$f \in Past(e)$ iff $H_e \subsetneq H_f$;*
4. *$f \in Present_S(e)$ iff e and f are compatible and $\neg(H_e \subsetneq H_f)$ and $\neg(H_f \subsetneq H_e)$;*
5. *for every history $h \in H_e$: $h \subseteq (Past(e) \cup Present_S(e) \cup Future(e))$;*
6. *$e \in Present_S(e)$;*
7. *if $f \in Present_S(e)$ then $e \in Present_S(f)$;*
8. *any two of $Past(e)$, $Present_S(e)$, and $Future(e)$ have an empty overlap.*

Proof. (1) "$\Rightarrow$": ($\star$) $H_f \subseteq H_e$ is just clause (4) of Def. 10.7. To prove the strict inclusion, by clauses (2) and (3) of this definition, there are $h, h' \in H_e$ and an atomic sentence A such that $\{c\} = Loc(e') \cap h$ and $\{c'\} = Loc(e') \cap h'$ and $c/h \models A$ but $c'/h' \not\models A$. Now, if $e' \in h \cap h'$, we would have $e' = c' = c$, and hence, from the above, $e'/h \models A$ but $e'/h' \not\models A$, contradicting Postulate 10.1. Thus, $e' \notin h \cap h'$, and hence, by clause (1) of Def. 10.7, at least one of h, h' does not belong to H_f. By ($\star$), since $h, h' \in H_e$, we get $H_f \subsetneq H_e$.

(1) "$\Leftarrow$" $H_f \subsetneq H_e$ means that clause (4) of Def. 10.7 is satisfied and furthermore that there is h such that $e, f \in h$ and h' such that $e \in h'$ but $f \notin h'$. Thus, $f \in h \setminus h'$, so by PCP_{NF} there is a choice set $\ddot{c}$ such that $h \perp_{\ddot{c}} h'$ and $c \leqslant f$, where $\{c\} = \ddot{c} \cap h$ and $\{c'\} = \ddot{c} \cap h'$. By Postulate 10.2 there is an atomic sentence A such that $c/h \models A$ but $c'/h' \not\models A$. Since $c, c' \in Loc(c)$ and $h, h' \in H_e$, we have $e \models Poss : At_{Loc(c)} : A$ and $e \models Poss : At_{Loc(c)} : \neg A$, so clauses (2) and (3) of Def. 10.7 are satisfied, with c playing the role of the witness e'. Finally, clause (1) of this definition holds as well (by $c \leqslant f$).

(2) This follows by (1) as in the proof of Fact 10.3(1).

(3)–(8): As every clause is an immediate consequence of Def. 10.8 and item (1) of this Fact, the proofs are left as Exercise 10.1. □

Note that clause (2) means that unless the dynamic future of an event e is degenerate (i.e., empty), it is modally open in the sense of containing incompatible events: there is no history that fully contains it. In contrast, item (3) amounts to the settledness of the dynamic past, i.e., for every $h \in H_e$:

$\text{Past}(e) \subseteq h$. Clause (5) means that $(\text{Past}(e) \cap h)$, $(\text{Present}_S(e) \cap h)$, and $(\text{Future}(e) \cap h)$ partition each history from H_e. Clauses (6) and (7) mean that the co-presentness relation $\text{CP}_S(\cdot,\cdot)$ is reflexive and symmetric on W.

Let us compare the results of the semantics-inspired approach to defining dynamic time with our list of desiderata from Def. 10.1. The success we can announce is that all items (D1)–(D5) are fulfilled: (D1) holds in virtue of working in BST, (D2) follows from Fact 10.5(8), (D3) follows from clauses (1) and (3) of that Fact, (D4) follows from clause (3), and (D5) is implied by clause (5). The only item on the list of desiderata that we cannot tick off immediately is (D6), the transitivity of co-presentness.

It is not difficult to come up with BST_{NF} structures in which co-presentness as defined in Def. 10.8 *is* transitive, however. For the simplest case, consider a (deterministic) one-history structure, in which for any e, the dynamic past and future are empty and every event belongs to the dynamic present of e. There are also more interesting examples.

Fact 10.6. *There are BST_{NF} structures in which there are multiple histories and SLR choice sets and in which (1) the dynamic present is modally settled and (2) the notion of co-presentness is transitive.*

Proof. As an example, we can take a BST structure that has two histories h, h' that split at the two choice sets $\ddot{a} = \{a, a'\}$ and $\ddot{c} = \{c, c'\}$, $a\, SLR\, c$ such that there is maximal MFB in the structure: h contains a and c, h' contains a' and c', and there are no histories containing a and c' or a' and c (compare Figure 5.1).

Note that by Fact 10.5(4), we have $\text{CP}_S(e,e)$ iff either $H_e = H_{e'}$ or ($H_e \cap H_{e'} \neq \emptyset$ and $H_e \setminus H_{e'} \neq \emptyset$ and $H_{e'} \setminus H_e \neq \emptyset$). The second disjunct is impossible to fulfill in our two-history structure, so that we have, for all events e, e', that $e' \in \text{Present}_S(e)$ iff $H_e = H_{e'}$.

(1) We need to show that for any $e \in W$ and any history h'', if $e \in h''$, then $\text{Present}_S(e) \subseteq h''$. Pick some $e \in W$ and some $e' \in \text{Present}_S(e)$. Consider a history h''. If $e \in h''$, then $h'' \in H_e = H_{e'}$, i.e., $e' \in h''$. So indeed, $\text{Present}_S(e) \subseteq h''$.

(2) As $\text{CP}_S(e,e')$ iff $H_e = H_{e'}$, the transitivity of $\text{CP}_S(\cdot,\cdot)$ follows by the transitivity of identity. □

In general, however, we cannot guarantee that co-presentness as defined in Def. 10.8 is always transitive. There are in fact two different questions we can ask regarding the transitivity of $\mathrm{CP}_S(\cdot,\cdot)$. On the *causae causantes* analysis, the co-presentness relation $\mathrm{CP}_C(\cdot,\cdot)$ is defined via an identity (Def. 10.3), and so that relation is an equivalence relation on all of W. Its restriction to any history is then of course also an equivalence relation. Desideratum (D6) of Def. 10.1, on the other hand, only requires that the notion of co-presentness *in any given history* be an equivalence relation. It is possible that that is so while the union of all the history-relative relations is *not* transitive. In fact, the simple example of a structure with two *SLR* choice sets and no modal correlations discussed in the proof of Fact 10.4 provides an example (see Exercise 10.5).

The following Fact shows that there are also cases in which the history-relative notion of co-presentness $\mathrm{CP}_{S|h}(\cdot,\cdot)$ fails to be transitive.

Fact 10.7. *There are BST$_{NF}$ structures in which the semantics-based relation of being co-present, $CP_{S|h}(\cdot,\cdot)$, is not transitive on some history h.*

Proof. Consider a structure in which there are three compatible binary choice sets $\ddot{c}_1$, $\ddot{c}_2$, and $\ddot{c}_3$, and in which there is no MFB. The relation of the choice sets is such that in history h, the elements c_1, c_2, and c_3 occur (think of these as the '$+$' outcomes) and $c_1\,SLR\,c_2$ and $c_2\,SLR\,c_3$, while $c_1 < c_3$. Thus, for $\ddot{c}_3$ to occur, $\ddot{c}_1$ has to have outcome c_1. Given no MFB, there are six histories in this structure (mnemonically we can write them as h^{+++}, h^{++-}, h^{+-+}, h^{+--}, h^{-+}, and h^{--}). It is easy to verify that $\mathrm{CP}_{S|h}(c_1,c_2)$ and $\mathrm{CP}_{S|h}(c_2,c_3)$: the respective sets of histories do not properly nest (e.g., $h^{+-+} \in H_{c_1} \setminus H_{c_2}$). By the ordering relation $c_1 < c_3$ and as $\ddot{c}_3$ is a choice set, however, the sets of histories H_{c_1} and H_{c_3} do properly nest ($H_{c_3} \subsetneq H_{c_1}$), i.e., $c_3 \in \mathrm{Future}(c_1)$, whence $\neg\mathrm{CP}_{S|h}(c_1,c_3)$. This shows that $\mathrm{CP}_{S|h}(\cdot,\cdot)$ is not transitive. □

This example indicates the price to be paid for the transitivity of co-presentness on the semantic approach, and thus, for fulfilling all the six desiderata of Def. 10.1: the trouble here was connected to the existence of pairs of *SLR* events whose respective sets of histories do not nest by set inclusion either way. This observation is analogous to our diagnosis from the end of Section 10.5; see Fact 10.3(4). We will now show that these observations generalize to provide a useful characterization of those BST structures in which all of the desiderata for the definition of dynamic time can be satisfied.

10.7 The way to guarantee satisfactory dynamic time in BST: Sticky modal funny business

Let us start by remarking that with both of our approaches to defining dynamic time, we ended up with definitions in terms of the interrelation of sets of histories, which is to be expected given that, with a view to desideratum (D1) of Def. 10.1, we are working on the basis of the primitive notions of BST. On both of our approaches,

- $f \in \text{Past}(e)$ iff $H_e \subsetneq H_f$, and
- $f \in \text{Future}(e)$ iff $H_f \subsetneq H_e$.

These conditions are mirror images, as required by desideratum (D3). Now we can note that quite generally, for any $e, f \in W$, there are five different ways in which their sets of histories can be interrelated, which are mutually exclusive and jointly exhaustive. The first four are simple: (1) $H_e \cap H_f = \emptyset$, i.e., e and f are incompatible; (2) $H_e = H_f$, i.e., e and f occur on exactly the same histories (this was the basis for defining co-presentness on the *causae causantes*-based approach of Section 10.5); (3) $H_e \subsetneq H_f$ (analyzed to mean that f is in the dynamic past of e), and (4) $H_f \subsetneq H_e$ (analyzed to mean that f is in the dynamic future of e). It is easy to see that these four cases are mutually exclusive (note that the sets of histories H_e and H_f must be non-empty). The remaining fifth case came up in problematic cases on both of our approaches to defining the dynamic present. We will call the condition "(NN)" for "non-nesting". It is formally simply the negation of (1), (2), (3), and (4), which can also be written as follows:

$$H_e \cap H_f \neq \emptyset \quad \wedge \quad \neg(H_e \subseteq H_f) \quad \wedge \quad \neg(H_f \subseteq H_e). \qquad \text{(NN)}$$

As we showed, instances of (NN) can cause trouble for both of our analyses. We can systematize the respective observations in the form of an equivalence between three conditions on BST_{NF} structures.

Fact 10.8. *Let $\mathscr{W}\langle W, <\rangle$ be a BST_{NF} structure. Then the following three conditions are equivalent:*

1. *There are no $e, f \in W$ that satisfy condition (NN).*

2. *On $\mathscr{W}$, the notions of Present$_C(\cdot)$ and Present$_S(\cdot)$ coincide (and accordingly, the relations CP$_C(\cdot,\cdot)$ and CP$_S(\cdot,\cdot)$ coincide).*
3. *On $\mathscr{W}$, the notions of Past$(\cdot)$, Present$_C(\cdot)$, and Future$(\cdot)$ satisfy all the desiderata (D1)–(D6) of Def. 10.1.*

Proof. "(1) $\Leftrightarrow$ (2)": We have $f \in$ Present$_C(e)$ (and thus, CP$_C(e,f)$) iff $H_e = H_f$ by Def. 10.5. On the other hand, by Def. 10.8 and Fact 10.5(1,3), we have $f \in$ Present$_S(e)$ (and thus, CP$_S(e,f)$) iff e and f are compatible and neither $H_e \subsetneq H_f$ nor $H_f \subsetneq H_e$. So in any BST$_{\text{NF}}$ structure, if $f \in$ Present$_C(e)$, then $f \in$ Present$_S(e)$. A case in which $f \in$ Present$_S(e)$ but not $f \in$ Present$_C(e)$ has to be one in which (a) e and f are compatible, (b) $H_e \subsetneq H_f$, (c) $H_f \subsetneq H_e$ (by the definition of Present$_S(\cdot)$), but (d) not $H_e = H_f$. We can pull together (b) and (d) and (c) and (d), so that we can characterize such a case via the three conditions (i) $H_e \cap H_f \neq \emptyset$, (ii) $\neg(H_e \subseteq H_f)$, and (iii) $\neg(H_f \subseteq H_e)$. These are exactly the three conjuncts of (NN). So we have established that if there is no instance of (NN) on W, then Present$_C(\cdot)$ and Present$_S(\cdot)$ coincide on all of W, and if Present$_C(\cdot)$ and Present$_S(\cdot)$ coincide on all of W, then there can be no instance of (NN).

"(1) $\Leftrightarrow$ (3)": By Fact 10.2, Present$_C(\cdot)$ fulfills desiderata (D1)–(D4) and (D6) in any case, and by Fact 10.3(4), desideratum (D5) holds in addition iff for any compatible e and f, we have $H_e \subseteq H_f$ or $H_f \subseteq H_e$, i.e., iff there are no e and f for which (i) $H_e \cap H_f \neq \emptyset$, (ii) $\neg(H_e \subseteq H_f)$, and (iii) $\neg(H_f \subseteq H_e)$. These are again exactly the three conjuncts of (NN). So, desideratum (D5) holds for Present$_C(\cdot)$ iff there is no instance of (NN). □

There is also a relevant condition on Present$_S(\cdot)$ that is implied by (1) but the converse implication does not hold.

Fact 10.9. *Let $\mathscr{W}\langle W,<\rangle$ be a BST$_{\text{NF}}$ structure. Then condition (1) implies condition (4), but not vice versa:*

(1) There are no $e,f \in W$ that satisfy condition (NN).
(4) On $\mathscr{W}$, the notions of Past$(\cdot)$, Present$_S(\cdot)$, and Future$(\cdot)$ satisfy all the desiderata (D1)–(D6) of Def. 10.1.

Proof. "(1) $\Rightarrow$ (4)": If there are no instances of (NN), then by (3), Past$(\cdot)$, Present$_C(\cdot)$, and Future$(\cdot)$ fulfill all the desiderata (D1)–(D6), and by (2), Present$_C(\cdot)$ and Present$_S(\cdot)$ coincide, so Past$(\cdot)$, Present$_S(\cdot)$, and Future$(\cdot)$ fulfill all the desiderata (D1)–(D6) as well.

"(4) $\not\Rightarrow$ (1)": To show the failure of the converse implication, consider a 2-dimensional MBS specified by three labels, i.e., $\Sigma = \{\sigma, \gamma, \eta\}$. The sets of splitting points are $S_{\sigma\gamma} = \{y\}$, $S_{\sigma\eta} = \{x\}$, and $S_{\gamma\eta} = \{x,y\}$, where $x = (0, -1)$ and $y = (0, 1)$; see Figure 10.4. (Note that by our convention, the first, temporal coordinate is depicted vertically.) Accordingly $[\sigma x] = [\gamma x] \neq [\eta x]$, whereas $[\sigma y] = [\eta y] \neq [\gamma y]$. The structure contains thus three histories h_σ, h_γ, and h_η and harbors MFB, as $H_{[\eta x]} \cap H_{[\gamma y]} = \emptyset$ (note that $[\gamma x] \in [\ddot{\eta} x]$ and $[\gamma x]\, SLR\, [\gamma y]$). Now, $[\sigma x]$ and $[\sigma y]$ produce an instance of (NN) as (†) $[\sigma x], [\sigma y] \in h_\sigma$, $\neg(H_{[\sigma x]} \subseteq H_{[\sigma y]})$ (witnessed by history h_γ), and $\neg(H_{[\sigma y]} \subseteq H_{[\sigma x]})$ (witnessed by history h_η). It is relatively easy to establish that $\mathrm{CP}_S(\cdot, \cdot)$ is transitive for any given instance, but we need to check a number of cases. Here we consider explicitly only one of the harder cases. Let us attempt to falsify the transitivity claim by taking $[\sigma x], [\eta x]$, and $[\sigma y]$. Since the first two events are incompatible, we would falsify transitivity of $\mathrm{CP}_S(\cdot, \cdot)$ if $\mathrm{CP}_S([\sigma x], [\sigma y])$ and $\mathrm{CP}_S([\sigma y], [\eta x])$. Note that each pair of these events is compatible. By (†) and the observation that strict nesting implies nesting, we have $\mathrm{CP}_S([\sigma x], [\sigma y])$. However, for $\mathrm{CP}_S([\sigma y], [\eta x])$ we need $\neg(H_{[\eta x]} \subsetneq H_{[\sigma y]})$, which is false since, due to MFB, $H_{[\eta x]} = \{h_\eta\}$ and $H_{[\sigma y]} = \{h_\sigma, h_\eta\}$. We ask the reader to check the remaining cases to establish transitivity in Exercise 10.7. So (D6) is satisfied, and the remaining desiderata (D1)–(D5) hold by Fact 10.5 . Thus, we have a case in which there is an instance of (NN) despite the satisfaction of (D1)–(D6), which falsifies the implication from (4) to (1). □

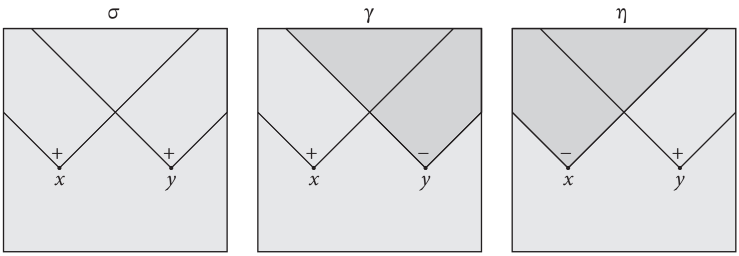

Figure 10.4 Illustration of the Minkowskian Branching Structure invoked in the proof of Fact 10.9. The structure contains three histories h_σ, h_γ, and h_η. Pluses and minuses indicate different outcomes at the splitting points x and y, and differences in shading indicate where the histories fail to overlap. Note that the structure exhibits a case of MFB, as $H_{[\eta x]} \cap H_{[\gamma y]} = \emptyset$.

We have seen in a few places above that the satisfaction of all the desiderata (D1)–(D6) for $\mathrm{CP}_S(\cdot,\cdot)$ is facilitated by the presence of certain instances of modal funny business. It is thus interesting to learn what kind of MFB it takes to secure these desiderata in any $\mathrm{BST_{NF}}$ structure. A fifth condition that we put forward below is to serve precisely this purpose: to secure the desiderata in any possible $\mathrm{BST_{NF}}$ structure. We will call this condition Sticky MFB, see Def. 10.9.

In working toward that definition, we investigate which $\mathrm{BST_{NF}}$ structures contain no instances of condition (NN). First, as one might expect, deterministic $\mathrm{BST_{NF}}$ structures are too simple to support cases of (NN):

Fact 10.10. *If a $\mathrm{BST_{NF}}$ structure $\langle W, < \rangle$ contains one history h only, then no $e, f \in W$ satisfy condition (NN).*

Proof. In a deterministic structure $\langle W, < \rangle$, for any $e, f \in W$, we have $H_e = H_f = \{h\}$, where $h = W$ is the only history in the structure. □

Somewhat more interestingly, condition (NN) also fails in a $\mathrm{BST_{NF}}$ structure without SLR choice sets. To prove this, we first establish an auxiliary fact:

Fact 10.11. *For any $e \in W$,*

$$H_e = \bigcap\{\Pi_{\ddot{c}}\langle H_e \rangle \mid \ddot{c} \in cll(e) \land \exists c \in \ddot{c}\ [c \leqslant e]\}. \qquad (\star)$$

Proof. See Exercise 10.2. □

We now show that structures without *SLR* choice sets cannot contain instances of (NN). This generalizes Fact 10.10:

Fact 10.12. *Suppose that a $\mathrm{BST_{NF}}$ structure contains no SLR choice sets. Then no two $e, e' \in W$ satisfy condition (NN).*

Proof. The absence of SLR choice sets means that in any history, choice sets induce a (possibly empty) chain, in the sense that for any choice sets $\ddot{c}_1, \ddot{c}_2$, if $\ddot{c}_1 \cap h = \{c_1\}$ and $\ddot{c}_2 \cap h = \{c_2\}$, then $c_1 \leqslant c_2$ or $c_2 < c_1$. Thus, for any events e and e', the past cause-like loci of e ($cll(e)$) and e' ($cll(e')$) constitute chains l_e and $l_{e'}$. Assume now that e and e' are compatible (else (NN) fails anyway). By the above observation, every element of l_e and every element of $l_{e'}$ are comparable, as otherwise a pair of such incomparable elements would belong to a pair of *SLR* choice sets (remember that each element of l_e or $l_{e'}$ belongs to a choice set). Thus, since by the assumption no choice sets are

SLR, (†) $l_e \subseteq l_{e'}$ or $l_{e'} \subseteq l_e$. Recall now the identity ($\star$) of Fact 10.11. By (†), the sets defining the intersection for H_e and $H_{e'}$ in this identity are related by the inclusion relation. Hence either $H_{e'} \subseteq H_e$ or $H_e \subseteq H_{e'}$, so condition (NN) is false. □

Finally, if a $\mathrm{BST_{NF}}$ structure contains SLR choice sets, there is a specific form of MFB that implies that no two events satisfy condition (NN). We call this specific form "Sticky MFB" because it binds together SLR outcomes more stringently than required for MFB alone. We define Sticky MFB as follows:

Definition 10.9 (Sticky MFB). Let $\mathscr{W} = \langle W, < \rangle$ be a $\mathrm{BST_{NF}}$ structure. We say that two SLR events $c_1 \in \ddot{c}_1$ and $c_2 \in \ddot{c}_2$, $\ddot{c}_1, \ddot{c}_2 \subseteq W$, form an instance of *Sticky MFB* in $\mathscr{W}$ iff they satisfy the following:
if $H_{c_1} \cap H_{c_2} \neq \emptyset$, then
 for every $H_1 \in \Pi_{\ddot{c}_1}$ with $H_1 \neq H_{c_1}$: $H_1 \cap H_{c_2} = \emptyset$ or
 for every $H_2 \in \Pi_{\ddot{c}_2}$ with $H_2 \neq H_{c_2}$: $H_{c_1} \cap H_2 = \emptyset$.

Here is a useful Fact that will help us to see the connection between Sticky MFB and the nesting conditions that occur in condition (NN), as well as in our definition of $\mathrm{CP}_S(\cdot,\cdot)$:

Fact 10.13. *For compatible $c_1 \in \ddot{c}_1$ and $c_2 \in \ddot{c}_2$:*

$$H_{c_1} \subseteq H_{c_2} \quad \textit{iff for every } H_2 \in \Pi_{\ddot{c}_2} \textit{ with } H_2 \neq H_{c_2} : H_{c_1} \cap H_2 = \emptyset.$$

Proof. "$\Rightarrow$": Let $H_{c_1} \subseteq H_{c_2}$. Any $h \in H_{c_1}$ must be in H_{c_2}, hence h cannot be in any outcome of $\ddot{c}_2$ other than H_{c_2}.
"$\Leftarrow$": In the opposite direction, if $h \in H_{c_1}$ and for any outcome H_2 of $\ddot{c}_2$ other than H_{c_2}, $h \notin H_2$, then $h \in H_{c_2}$ (by compatibility of c_1 and c_2). □

Now, universal Sticky MFB guarantees that no two events satisfy condition (NN). Via the equivalences stated in Fact 10.8, this guarantees that $\mathrm{Present}_C(\cdot)$ and $\mathrm{Present}_S(\cdot)$ coincide, and the dynamic Past, Present, and Future satisfy all the desiderata (D1)–(D6). Emphatically, this concerns *all* $\mathrm{BST_{NF}}$ structures: for an apparently intuitive dynamic time, universal Sticky MFB is sufficient.

Fact 10.14. *Suppose that in a $\mathrm{BST_{NF}}$ structure $\mathscr{W} = \langle W, < \rangle$, there is universal Sticky MFB, i.e., for any SLR pair of choice sets $\ddot{c}_1, \ddot{c}_2$, any two elements*

$c_1 \in \ddot{c}_1$, $c_2 \in \ddot{c}_2$ form an instance of Sticky MFB. Then no two $e, e' \in W$ satisfy condition (NN).

Proof. Observe first that in a structure without choice sets, or without SLR choice sets, the claim holds vacuously. Let us thus assume that $\mathscr{W}$ is as in the premise, and argue for the contraposition. Assume thus that there is an instance of (NN), i.e., there are two compatible $e, e' \in W$ such that $H_e \not\subseteq H_{e'} \wedge H_{e'} \not\subseteq H_e$. Thus, there are histories $h, h_1, h_2 \in$ Hist such that $e, e' \in h$ and (†), $e' \in h \setminus h_1$, $e \in h_1$ and $e \in h \setminus h_2$, $e' \in h_2$. By PCP_{NF}, there is a choice set $\ddot{c}_1$, with $c_1, c_1' \in \ddot{c}_1$ such that $h \perp_{\ddot{c}_1} h_1$ and $\ddot{c}_1 \cap h = \{c_1\}$, $\ddot{c}_1 \cap h_1 = \{c_1'\}$, and $c_1 \leqslant e'$. Accordingly, $H_{c_1} = \Pi_{\ddot{c}_1}\langle h \rangle = \Pi_{\ddot{c}_1}\langle h_2 \rangle$ and $H_{c_1'} = \Pi_{\ddot{c}_1}\langle h_1 \rangle$. There is also a choice set $\ddot{c}_2$ with $c_2, c_2' \in \ddot{c}_2$ such that $h \perp_{\ddot{c}_2} h_2$ and $\ddot{c}_2 \cap h = \{c_2\}$, $\ddot{c}_2 \cap h_2 = \{c_2'\}$, $c_2 \leqslant e$. Accordingly, $H_{c_2} = \Pi_{\ddot{c}_2}\langle h \rangle = \Pi_{\ddot{c}_2}\langle h_1 \rangle$ and $H_{c_2'} = \Pi_{\ddot{c}_2}\langle h_2 \rangle$. We claim next that $c_1\, SLR\, c_2$. For, if $c_1 \leqslant c_2$, then $c_1 \leqslant e$, and hence $e \notin h_1$, contradicting (†). For a similar reason it impossible that $c_2 \leqslant c_1$. And as $c_1, c_2 \in h$, they must be *SLR*. Furthermore, we have $\Pi_{\ddot{c}_1}\langle h \rangle \cap \Pi_{\ddot{c}_2}\langle h \rangle \neq \emptyset$, $\Pi_{\ddot{c}_1}\langle h_1 \rangle \cap \Pi_{\ddot{c}_2}\langle h \rangle = \Pi_{\ddot{c}_1}\langle h_1 \rangle \cap \Pi_{\ddot{c}_2}\langle h_1 \rangle \neq \emptyset$, and $\Pi_{\ddot{c}_1}\langle h \rangle \cap \Pi_{\ddot{c}_2}\langle h_2 \rangle = \Pi_{\ddot{c}_1}\langle h_2 \rangle \cap \Pi_{\ddot{c}_2}\langle h_2 \rangle \neq \emptyset$. This shows that two events $c_1 \in \ddot{c}_2$ and $c_2 \in \ddot{c}_2$, which are *SLR*, do not form an instance of Sticky MFB in $\mathscr{W}$. This proves the consequence of the contraposition. □

We would like to obtain an implication in the opposite direction as well: from the fact that no events satisfy (NN), to Sticky MFB. As a preparation we state and prove the following Fact. Observe that this Fact concerns a somewhat richer class of BST_{NF} structures; see Figure 10.5 for illustration.

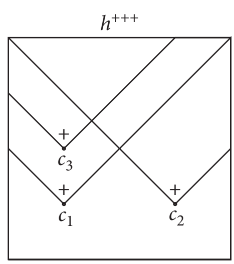

Figure 10.5 Illustration of the type of structure invoked in Facts 10.15 and 10.16: there are three choice sets $\ddot{c}_1, \ddot{c}_2$, and $\ddot{c}_3$, and $c_1\, SLR\, c_2$, $c_2\, SLR\, c_3$, but $c_1 < c_3$.

Fact 10.15. *Let $\mathscr{W} = \langle W, < \rangle$ be a BST_{NF} structure in which the notions of Past$(\cdot)$, Present$_S(\cdot)$, and Future$(\cdot)$ satisfy all the desiderata (D1)–(D6) of Def. 10.1. Let $\mathscr{W}$ contain at least three choice sets $\ddot{c}_1, \ddot{c}_2$, and $\ddot{c}_3$ such that*

$\{c_1, c_2, c_3\}$ is consistent, $c_1 < c_3$, $c_1\, SLR\, c_2$, and $c_2\, SLR\, c_3$. Then there is an instance of Sticky MFB in $\mathscr{W}$.

Proof. In our structure, $c_3 \in \text{Future}(c_1)$, so clearly $\neg\text{CP}_S(c_1, c_3)$. We must thus have $\neg\text{CP}_S(c_1, c_2)$ or $\neg\text{CP}_S(c_2, c_3)$—otherwise transitivity fails, contrary to the assumption. Consider $\neg\text{CP}_S(c_1, c_2)$ (the case of the other disjunct is analogous). To recall, $\text{CP}_S(c_1, c_2)$ is the conjunction:

$$H_{c_1} \cap H_{c_2} \neq \emptyset \wedge \neg(H_{c_1} \subsetneq H_{c_2}) \wedge \neg(H_{c_2} \subsetneq H_{c_1}),$$

so $\neg\text{CP}_S(c_1, c_2)$ is equivalent to:

$$(H_{c_1} \cap H_{c_2} = \emptyset) \vee (H_{c_1} \subsetneq H_{c_2}) \vee (H_{c_2} \subsetneq H_{c_1}).$$

As c_1 is compatible with c_2 by assumption, the first disjunct is false, and by Fact 10.13, since strict inclusion implies inclusion, $\neg\text{CP}_S(c_1, c_2)$ implies

$$\forall H_2 \in \Pi_{\ddot{c}_2}\ [H_2 \neq H_{c_2} \rightarrow H_{c_1} \cap H_2 = \emptyset] \quad \text{or}$$
$$\forall H_1 \in \Pi_{\ddot{c}_1}\ [H_1 \neq H_{c_1} \rightarrow H_1 \cap H_{c_2} = \emptyset].$$

Thus, as $c_1 \in \ddot{c}_1$ and $c_2 \in \ddot{c}_1$ are SLR, they form an instance of Sticky MFB. □

The preceding Fact, interesting on its own, helps us to clarify the relation from (NN) to Sticky MFB:

Fact 10.16. *Let $\mathscr{W} = \langle W, < \rangle$ be a BST_{NF} structure in which no $e, f \in W$ satisfy condition (NN). Let $\mathscr{W}$ contain at least three choice sets $\ddot{c}_1, \ddot{c}_2$, and $\ddot{c}_3$ such that $\{c_1, c_2, c_2\}$ is consistent, $c_1 < c_3$, $c_1\, SLR\, c_2$, and $c_2\, SLR\, c_3$. Then there is an instance of Sticky MFB in $\mathscr{W}$.*

Proof. By Fact 10.9 we have that the notions of $\text{Past}(\cdot)$, $\text{Present}_S(\cdot)$, and $\text{Future}(\cdot)$ satisfy all the desiderata (D1)–(D6) of Def. 10.1. By the proof of Fact 10.16, either $c_1 \in \ddot{c}_1$ and $c_2 \in \ddot{c}_2$, or $c_2 \in \ddot{c}_2$ and $c_3 \in \ddot{c}_3$ form an instance of Sticky MFB in $\mathscr{W}$. □

Having these Facts on the table, we can describe the logical landscape they point to as follows:

First, we have the equivalence of three conditions (by Fact 10.8): (1) There are no $e, f \in W$ that satisfy condition (NN). (2) The notions of $\text{Present}_C(\cdot)$ and $\text{Present}_S(\cdot)$ coincide. (3) The notions of $\text{Past}(\cdot)$, $\text{Present}_C(\cdot)$, and $\text{Future}(\cdot)$ satisfy all the desiderata (D1)–(D6) of Def. 10.1. Each of these conditions implies (4): The notions of $\text{Past}(\cdot)$, $\text{Present}_S(\cdot)$, and $\text{Future}(\cdot)$ satisfy all the desiderata. The converse implication, from (4) to each of (1)–(3), does not hold, however; see Fact 10.9.

Second, universal Sticky MFB is a sufficient condition of each of (1)–(3) and of (4), by Fact 10.14. As for necessary conditions, they depend on the complexity of the BST_{NF} structure under consideration. We do not offer a maximally fine-grained characterization of BST_{NF}, but we can observe the following: (a) Structures without SLR choice sets (including deterministic structures, which contain no choice sets at all) do not support cases of (NN), so they satisfy (1)–(3) trivially, and by implication also (4), the transitivity of $\text{CP}_S(\cdot,\cdot)$ as a relation on W. (b) Structures that contain choice sets that are all pairwise *SLR* need a strong form of MFB to exclude cases of (NN). Interestingly, however, in the absence of MFB, for any history h, the relation $\text{CP}_{S|h}(\cdot,\cdot)$ is transitive on such structures. In the presence of MFB, $\text{CP}_{S|h}(\cdot,\cdot)$ can be transitive or fail to be transitive—these observations are the subject of Exercise 10.6. Of course, by Fact 10.14, universal Sticky MFB is sufficient to enforce transitivity of $\text{CP}_S(\cdot,\cdot)$ in all BST_{NF} structures, including those with exclusively *SLR* choice sets. (c) If a structure lies outside of cases (a) and (b); that is, if it is neither such that it contains no *SLR* choice sets nor such that it contains choice sets all of which are *SLR*, then it has to contain instances both of *SLR* choice sets and of choice sets that are order related. The structures considered in Fact 10.15 and in Fact 10.16 are a subtype of type (c): they contain a triple of choice sets, with two pairs being *SLR* and one pair being related by $<$. For these structures we showed that for (1), (2), and (3), but also for the weaker condition (4), a necessary condition is the existence of an instance of Sticky MFB.

Having this landscape before our eyes, how should we estimate the price for having an intuitively satisfying notion of dynamic time (i.e., dynamic time that satisfies all the desiderata (D1)–(D6)), either on the $\text{Present}_C(\cdot)$ analysis or on the $\text{Present}_S(\cdot)$ analysis? It seems the response depends on whether one considers the question from the perspective of a creator of all possible BST_{NF} universes, or from the perspective of a dweller in one of such universes. For the former perspective, as the BST_{NF} postulates alone permit a large variety of structures, it seems prudent to require the constraint of universal Sticky MFB—this will guarantee that (D1)–(D6) are satisfied in all BST_{NF} structures. The dweller's perspective seems less constrained: perhaps her world is such that it tolerates some failures of (D1)–(D6). Perhaps there were such failures long ago in the past, or in remote regions—why should she care? Or, perhaps, there are such failures in her vicinity, but they are small, well-localized, and hardly visible. Our dweller might strike you as overly optimistic. The reasons for her optimism, however, are facts of dynamic time, and these can be debated. After all, such facts fully supervene on what is possible in our world.

In the next section we turn to the representation of dynamic time in Minkowskian Branching Structures (MBSs), which allow for a perspicuous representation of the welcome consequences of the types of MFB that we discussed earlier.

10.8 What does dynamic time look like in MBSs?

In Minkowskian Branching Structures, our notions of the dynamic past, present, and future of a given event are defined in terms of the Minkowski ordering $<_M$, which is clearly invariant with respect to the automorphisms of Minkowski space-time, the space-time of special relativity. Thus, the regions of histories that our definitions single out are invariant with respect to these automorphisms, and our constructions are directly relevant to the discussion of the problem of the present in special relativity (see Section 10.2).

A salient feature of our definitions is that the shape of the dynamic past, present, and future of an event e in a history h depends on the location of elements of choice sets in h. In MBSs, these choice sets are induced by the pattern of qualitative differences between histories.

In Section 10.3 we discussed and defended the general idea of making room for a dynamic present that is extended in the coordinate-temporal dimension. We will now apply our definitions to some selected MBSs to visualize what the dynamic future, present and past of a given event look like. Our focus is on cases in which the *causae causantes*-based approach and the semantics-based approach deliver the same verdict.

Example 1: Simple cases. The simplest case is a deterministic MBS (i.e., a structure with just one history). In such a structure, there is no indeterminism, and the co-presentness relation on both of our accounts turns out to be the universal relation: any event is co-present with any other event. Dynamic time is trivial under determinism. We do not provide an illustration for this case. We also leave out of our considerations another kind of structure that is featureless according to Def. 10.4 (i.e., an MBS in which there is a history that consists wholly of members of uncorrelated choice sets). In such a structure, the relation of co-presentness on that history is simply the identity relation, which is also trivial.

The first simple but non-trivial case is an MBS with two histories σ and η, which split at a single point c, so $\{c\} = S_{\sigma\eta}$ (see Figure 10.6). For $x \not\geqslant c$ (which excludes $x = c$), Present$([\sigma x]) = \{[\sigma y] \mid y \not\geqslant c\} = \{[\eta y] \mid y \not\geqslant c\}$ and Future$([\sigma x]) = \{[\sigma y], [\eta y] \mid y \geqslant c\}$. Note that Past$([\sigma x]) = \emptyset$. The situation is analogous for $x > c$: Present$([\sigma x]) = \{[\sigma y] \mid y \geqslant c\}$, Past$([\sigma x]) = \{[\sigma y] \mid y \not\geqslant c\} = \{[\eta y] \mid y \not\geqslant c\}$, and Future$([\sigma x]) = \emptyset$.

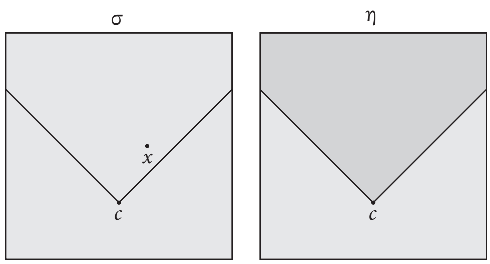

Figure 10.6 Illustration of the simplest non-trivial MBS from Example 1, which includes a single binary splitting point c. Different shading indicates where the histories differ. The present of $e = [\sigma x]$ is the future light cone above c in history h_σ; the future of e is, accordingly, empty.

Example 2: Two time-like splitting points. Consider next an MBS with three histories, $\Sigma = \{\sigma, \gamma, \eta\}$, in which there are two splitting points $c_1, c_2 \in \mathbb{R}^4$ such that $c_1 <_M c_2$ and $S_{\sigma\gamma} = S_{\sigma\eta} = \{c_1\}$ and $S_{\gamma\eta} = \{c_2\}$. That is, γ and η split from σ at c_1, and then η splits from γ at c_2. Figure 10.7 represents these three histories as squares with a common bottom region, taking the history labelled by σ to be our reference history. The shading convention is that a difference in shading indicates that the corresponding regions are not to be identified.

Now pick an event $e = [\gamma x]$ that is above c_1 but not above c_2, $c_1 <_M x$ and $x \not>_M c_2$, and ask: (1) What is the dynamic future of e? (2) What is its dynamic past? (3) And what is its dynamic present?

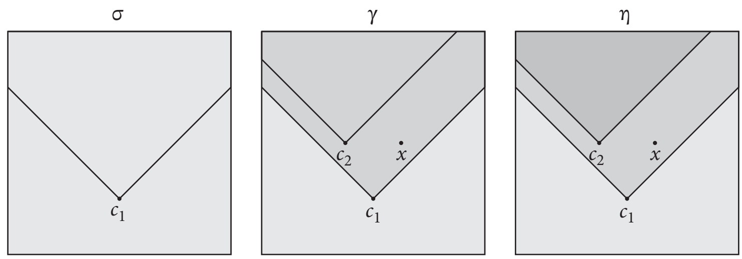

Figure 10.7 Illustration of Example 2: Two time-like splitting points. Different shading indicates where the histories differ, thus showing the past, the present, and the future of $e = [\gamma x]$. See text for details.

Fact 10.5 (as well as Fact 10.3) yields the following verdicts:

1. The dynamic future of $e = [\gamma x]$ is the set of events that are weakly above $[\gamma c_2] = [\eta c_2]$: Future$([\gamma x]) = \{[\xi z] \mid c_2 \leqslant_M z \wedge \xi \in \{\gamma, \eta\}\}$. Note that this region is the union of two future light cones of c_2, in histories labeled by γ and η. The light cones include their boundaries. And the future of $e = [\gamma x]$ is above $[\gamma c_2]$ rather than above e.
2. The dynamic past of $e = [\gamma x]$ is the set of events that are in history γ and not weakly above $[\gamma c_1]$: Past$([\gamma x]) = \{[\gamma z] \mid c_1 \not\leqslant_M z\}$. In contrast to the future, the past is shared by all of the three histories.
3. The dynamic present of $e = [\gamma x]$ is the set of events in history γ and "between" c_1 and c_2 in the sense: Present$([\gamma x]) = \{[\gamma z] \mid c_1 \leqslant_M z \wedge c_2 \not\leqslant_M z\}$. The present of e is shared by the two histories to which e belongs, γ and η.

Note that the present of $e = [\gamma x]$ turns out to be a spatially extended and temporally thick collection of events, whose temporal thickness depends on the distance between c_1 and c_2.

Example 3: Four splitting points, layered in two SLR pairs. Consider an MBS with three histories, $\Sigma = \{\sigma, \gamma, \eta\}$, with $S_{\sigma\gamma} = S_{\sigma\eta} = \{c_1, c_2\}$ and $S_{\gamma\eta} = \{c_3, c_4\}$, where $c_1 <_M c_3$ and $c_2 <_M c_4$—see Figure 10.8. We consider an event $[\gamma x]$ with x sliced between two pairs of splitting points, that is, ($c_1 <_M x$ or $c_2 <_M x$) and ($c_3 \not<_M x$ and $c_4 \not<_M x$). Applying Fact 10.5 we arrive at the following result:

$$\text{Present}([\gamma x]) = \{[\gamma y] \mid (y \geqslant_M c_1 \vee y \geqslant_M c_2) \wedge (y \not\geqslant_M c_3 \wedge y \not\geqslant_M c_4)\}.$$

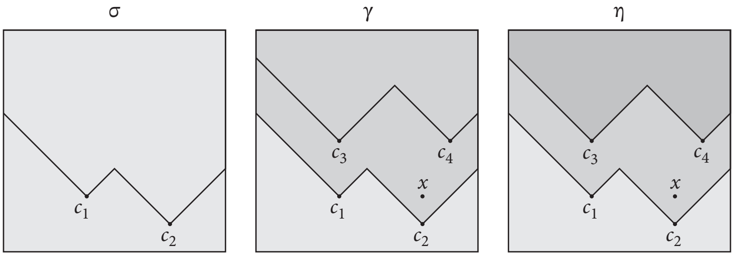

Figure 10.8 Illustration of Example 3: Three histories with four splitting points $c_1, \ldots, c_4$ and with two instances of Sticky MFB, one involving c_1 and c_2 and the other involving c_3 and c_4. The dynamic present of event $e = [\gamma x]$ is the shaded W-shaped region containing x.

Thus, the present of $[\gamma x]$ turns out to have the shape of a thick letter W. Note that by adding more splitting points to $S_{\sigma\gamma}$ and to $S_{\gamma\eta}$ we obtain as Present(e) a "generalized" letter W, with more top corners and more bottom corners. And by making the separation between $S_{\sigma\gamma}$ and $S_{\gamma\eta}$ smaller, we can make the generalized W arbitrarily thin. We can make it to maximally extend through the whole history as well. A suggestive construction can be based on universal Sticky MFB for infinitely many space-like related choice sets. Given densely packed choice sets and universal Sticky MFB, the dynamic present of an event can approximate its static present.

10.9 Conclusions

In this chapter we have investigated the concept of dynamic time that is inherently related to real change. We argued that if our world harbors no real change, there is no real time, or at least real time is thoroughly trivialized. We contrasted real, dynamic time with static coordinate time. Given this contrast between two notions of time, from the very start we acknowledged that the notions of dynamic past, present, and future might be different from their static counterparts. For instance, the dynamic present of some event might be thick along coordinate time.

Real change has a definitely modal ring to it: real change means that from among a family of alternative possible outcomes, one is actualized later on, while the remaining outcomes have ceased to be possible. BST, by being a theory of local indeterminism playing out in space and time, and with its causal and semantic resources, is perfectly suited to analyzing indeterministic real change, and hence to explicate dynamic time.

There is more than one way to link dynamic time to the phenomenon of the passing of possibilities, and one might worry that this could stand in the way of a unified account. We offered two approaches that at first glance are quite different, one based on causal concepts, and the other building on intuitions concerning our talk about future events. The former's focus is the relation of co-presentness, which is defined in terms of the identity of sets of *causae causantes*. Unsurprisingly, co-presentness so defined is an equivalence relation. The dynamic present, past, and future of a given event are then naturally defined. It turns out that the three notions can be characterized in purely modal terms, i.e., by inclusion or identity of sets of histories. Moreover, these notions are relativity-proof as the analysis is rooted in the primitive notions of BST.

The second approach begins by explicating what it means, in modal terms, that one event belongs to the dynamic (open) future of another event. It then defines the dynamic past and present of a given event by assuming certain natural desiderata on how the three notions are related. The resulting notions are relativity-proof and characterized in purely modal terms, as in the first approach. Moreover, the two approaches agree about their concepts of dynamic past and future. They can disagree, however, on the analysis of the dynamic present, and hence, of co-presentness.

When confronted with our list of intuitive desiderata for a notion of dynamic time (Def. 10.1), the two analyses show different strengths and weaknesses. In the first (causal) approach, the co-presentness relation is automatically transitive, but a history might fail to be fully carved into the past, the present, and the future of an event from this history. In the second (semantic) approach, in contrast, any history is automatically partitioned into the dynamic past, present, and future of any of its events, but the relation of co-presentness is not necessarily transitive. As our discussion showed, both issues have similar reasons: In non-trivial structures, a certain strong form of modal funny business is required to fulfill all the desiderata on any of the two analyses. A sufficient condition for a satisfying notion of dynamic time is universal Sticky MFB: If a BST structure is such that there are modal correlations all across space, then dynamic time based on indeterminism is relativity-proof whilst retaining all other intuitive features as well. This proves our main point: A BST analysis of local indeterminism in space-time leads to formal structures that are rich enough to accomplish what the structure of a single space-time cannot: to anchor a satisfying notion of real, dynamic time.

10.10 Exercises to Chapter 10

Exercise 10.1. Prove items (3)–(8) of Fact 10.5.

Exercise 10.2. Prove the strengthened version of identity ($*$) from the proof of Fact 10.1, which restricts cll of e to those lying in the past of e (i.e., prove Fact 10.11):
For any $e \in W$,

$$H_e = \bigcap \{\Pi_{\ddot{c}}\langle H_e \rangle \mid \ddot{c} \in cll(e) \wedge \exists c \in \ddot{c}\ [c \leqslant e]\}. \qquad (*)$$

Hint: A proof is provided in Appendix B.10.

Exercise 10.3. Provide formal details for the strengthened definition of co-presentness discussed in footnote 18.

Hint: As a first approximation, consider this definition:

Definition 10.10 (Co-presentness strengthened). $CP(e_1, e_2)$ iff $H_{e_1} = H_{e_2}$ and (if $\ddot{e}_i \neq \{e_i\}$ for $i = 1, 2$, then $\Pi_{\ddot{e}_1} = \Pi_{\ddot{e}_2}$).

Show how this definition resolves the problem of the non-robust co-presentness of c_1 and c_2 introduced in footnote 18. Discuss why the antecedent, "if $\ddot{e}_i \neq \{e_i\}$ for $i = 1, 2$", is needed. As a suggestion for a full-fledged notion of alternatives, consider the requirement: each alternative is above a different element of some one choice set, but in the same outcome of any other choice set.

Exercise 10.4. Construct a $\mathrm{BST_{NF}}$ structure $\mathscr{W}$ with MFB but in which desideratum (D5) of Def. 10.1 is still violated for $\mathrm{CP}_C(\cdot,\cdot)$; that is, such that for some history h and $e, e' \in h$, neither $H_e \subseteq H_{e'}$ nor $H_{e'} \subseteq H_e$.

Hint: Consider a structure with only two *SLR* binary choice sets $\ddot{e} = \{e, e_1\}$ and $\ddot{e'} = \{e', e'_1\}$, and with MFB given by $H_{e_1} \cap H_{e'_1} = \emptyset$. The remaining intersections are non-empty, so $e, e' \in h$ for some history h. And there is MFB in this structure but $H_e \not\subseteq H_{e'}$ and $H_{e'} \not\subseteq H_e$.

Exercise 10.5. Show that in the structure with two *SLR* choice sets and no modal correlations discussed in the proof of Fact 10.4, for any history, the restricted relation of co-presentness $CP_{C|h}(\cdot,\cdot)$ is transitive, while the unrestricted relation $CP_C(\cdot,\cdot)$ is not transitive.

Hint: Show that different elements of a choice set, which are incompatible and thereby not co-present, can be co-present to the same *SLR* event.

Exercise 10.6. Consider a BST_{NF} structure with exactly three pairwise *SLR* binary choice sets. Show that in this structure (a) in the absence of MFB (the case of 8 histories), the relation $CP_S(\cdot,\cdot)$ is transitive. (b) Exhibit a case in which there is MFB but $CP_S(\cdot,\cdot)$ is not transitive.

Hint: For (b), a structure with 6 histories will do.

Exercise 10.7. Check that $CP_S(\cdot,\cdot)$ is satisfied in the cases that were left open in the proof of Fact 10.9.