2

The Foundations of Branching Space-Times

2.1 The underlying ideas of BST

In this chapter we guide the reader through the construction of the core theory of Branching Space-Times. This discursive approach culminates in proposing a set of postulates that a structure of the core theory of Branching Space-Times (BST) is to satisfy (see Chapter 2.6). The rigorous theory commences with Postulate 2.1. We begin with an informal gloss, explaining the main ideas of our construction as clearly as we can.

The fundamental element of the construction is the set W of all point events, ordered by a certain pre-causal relation $<$. What postulates hold for the pre-causal order? For Minkowski space-time, Mundy (1986) describes the results of Robb (1914, 1936) and gives additional results for the light-like order. That research, however, does not immediately help here because a Minkowski space-time does not contain incompatible point events. We shall need to proceed more slowly. The first postulate is so natural and vital that without it we would not know what to say next.

Postulate 2.1 (BST Strict Partial Order). Our World $\mathscr{W}$ *is a nontrivial strict partial ordering* $\langle W, < \rangle$*, i.e.:*

1. *Nontriviality:* W *is nonempty.*
2. *Nonreflexivity: For all* $e \in W$, $e \not< e$.
3. *Transitivity: For all* $e_1, e_2, e_3 \in W$*, if* $e_1 < e_2$ *and* $e_2 < e_3$*, then* $e_1 < e_3$.

We read $e_1 < e_2$ as "e_2 can occur after e_1". Recall that asymmetry (i.e., if $e_1 < e_2$, then $e_2 \not< e_1$) follows from nonreflexivity and transitivity. Note that asymmetry incorporates the prohibition of repeatable events. Transitivity can be motivated by reflecting on how *Our World* $\mathscr{W}$ is

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constructed: if e_2 is a future possibility of e_1 and e_3 is a future possibility of e_2, then e_3 is a future possibility of e_1.

For convenience, we add the following simple definition of weak partial ordering:

Definition 2.1 (BST weak companion of strict order)**.** The symbol $\leqslant$ stands for the companion weak partial ordering: $e_1 \leqslant e_2$ if $e_1 < e_2$ or $e_1 = e_2$. We also write $\geqslant$, naturally defined as $e_1 \geqslant e_2$ iff $e_2 \leqslant e_1$.

We mark the stronger relation with 'proper', as in 'e_1 is properly earlier than e_2'.

In *Our World* possible point events are related by the pre-causal relation $<$; some such events are compatible with others. Here is an idealized illustration. There is an ideally small event, e_m, at which a certain electron's spin is measured along a certain direction. There are two possible outcomes: measured spin up or measured spin down. Take a possible point event, e_u, at which it is true to say, 'It has been measured spin up', and another, e_d, at which it is true to say 'It has been measured spin down'. The point events e_u and e_d are incompatible, though each is compatible with e_m. How precisely can two incompatible point events both fit into *Our World*? Answer: By means of the pre-causal order. In the next three paragraphs we explain the intuition that leads to a criterion of which events are compatible.[1]

Let e_1, e_2, and e_3 be three events in *Our World*, with pre-causal ordering $<$. How can these three events be ordered, and how are the resulting patterns of ordering to be interpreted? Here are three paradigms.

Causal dispersion: Causal order can hold between a given point event, e_3, and two space-like separated (space-like related) future point events, e_1 and e_2, in a single Minkowski space-time, just as one might expect: $e_3 < e_1$ and $e_3 < e_2$. The three events form an "up-fork".

Causal confluence: Causal order also can hold between two given space-like related point events, e_1 and e_2, in a single Minkowski space-time, and a single future point event, e_3, as one might equally expect: $e_1 < e_3$ and $e_2 < e_3$. The three events form a "down-fork".

[1] We say of two or more events that they are compatible or not; the corresponding set of events is then, accordingly, consistent or not. That is, we use "consistent" as a unary predicate for sets of entities such as events or histories, and we use "compatible with" for the relation. An alternative for "compatible with" is "co-possible with".

Causal branching: Causal order can also hold between a given e_3 and two possible future point events e_1 and e_2 that might be said to be alternative possibilities: $e_3 < e_1$ and $e_3 < e_2$. The three events form an "up-fork."

Observe that an "up-fork" allows for two interpretations: if you see just an "up-fork" (i.e., without seeing how it is related to other events), you can read its upper events either as (incompatible) modal alternatives, or as (compatible) spatially separated events. It seems to us that the analogous dichotomy is absent in the case of "down-forks", at least on the understanding of point events as concrete, non-disjunctive objects. The distinction between states and concrete events is highly relevant here: the same state can result from incompatible starting points, and a state can be repeated, whereas a concrete event, with its identity being tied to its past, is not repeatable. (At least this is a plausible and intuitive view; see a relevant caveat below.) By saying "non-disjunctive" we refer to the second relevant distinction, that between Lewis's fragile and non-fragile events (Lewis, 1986b, Ch. 23), which relates to the question of whether an event might occur in more than one alternative way. In our basic construction we rely on non-disjunctive events, that is, those events that cannot occur in more than one alternative way. In the Lewisian terminology, concrete events are therefore fragile. (In Chapter 4.1 we study other types of events, including disjunctive events.)

We thus deny *backward branching*: we deny that incompatible point events can lie in the past (i.e., that some events could have incompatible 'incomes' in the same sense that some have incompatible possible outcomes). To put it differently, we deny that two events that cannot occur together somehow combine to have a common future successor. No backward branching is part of common sense, including that of scientists when speaking of experiments, measurements, probabilities, some irreversible phenomena, and the like. We need a caveat, however: some models of the general theory of relativity (GR) allow for so-called causal loops, which are typically interpreted as involving repetitions of a concrete event (Hawking and Ellis, 1973). We return to this topic in Chapter 9.3.6, in which we suggest how to modify BST to accommodate causal loops of GR.[2]

The 'no backward branching' intuition gives rise to a semi-criterion as to which events are compatible (co-possible). The fact that a "down-fork" has a univocal meaning provides a recipe for the criterion of a later witness: if two events are seen as past from the perspective of some third event,

[2] An attempt to generalize BST so that it accommodates causal loops is provided in Placek (2014).

these two events are compatible. We will later strengthen this recipe to provide a criterion that will deliver possible histories (as maximal subsets of compatible events) of *Our World*.

2.2 Histories

How does one further describe the way that point events fit together in *Our World*? The crucial concept is that of a *history*, which will help us keep track of the compatibilities and incompatibilities between events in *Our World*. In BST, which adds a spatial aspect, we generalize the concept of a history as defined in Prior's theory of branching temporal histories, which is often called "Branching Time" (Prior, 1967; Thomason, 1970). The basic blocks of this theory are spatially maximal instantaneous objects, somewhat misleadingly called "moments".[3] The theory arranges such moments into a tree (see Figure 2.1): incompatible moments have a lower bounding moment in the tree (a feature we will call "historical connection"), but never a common upper bound (no backward branching). The formal definition of a tree gives expression to the openness of the future in contrast to the settledness of the past. A key point to always bear in mind is that in Branching Time, the entire tree is 'the world'. In addition there is the concept of a 'history', defined as a maximal chain of moments. Locate yourself at a moment in the tree, perhaps at the moment at which the spin measurement occurs. You will easily visualize that in this picture your 'world' is unique, whereas you belong to many 'histories'. Until and unless branching ceases before your expiration, there is no such thing as 'your history'. Of course in Branching

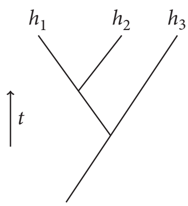

Figure 2.1 A tree-like structure of Branching Time with three histories.

[3] The construction of moments requires a frame-independent simultaneity relation, and for that reason Branching Time is not a relativity-friendly theory. The terminology "Branching Time" is doubly misleading, suggesting that moments are just instants of time and that time branches. For a discussion of what branches (viz., histories rather than time or space-time), see p. 10.

Time 'your historical past' makes perfect sense, in contrast to 'your entire history'. Branching Time takes uniqueness to fail when histories are taken as stretching into the future. Significantly, in this usage a 'world' contains incompatible possibilities, while a 'history' does not. A history represents a choice between incompatible possibilities, a resolution of all disjunctions unto the end that presumably never comes.

The present development retains Prior's idea of *Our World* as involving many possible histories. A novelty is that a history can be isomorphic to a relativistic space-time (e.g., to Minkowski space-time),[4] the latter being understood as the set of quadruples of reals, $\mathbb{R}^4$, together with Minkowskian ordering $<_M$, defined as follows:

$$x <_M y \quad \text{iff} \quad -(x^0 - y^0)^2 + \sum_{i=1}^{n-1} (x^i - y^i)^2 \leqslant 0 \text{ and } x^0 < y^0. \tag{2.1}$$

(Here $<$ and $\leqslant$ are, respectively, the strict and weak orderings of reals and it is assumed that the zero coordinate is temporal. Space-times of dimensions other than 4 are defined analogously.) Given our aim to accommodate Minkowski space-time, histories cannot be defined as maximal chains of point events; the latter are mere paths without a spatial dimension. But, as we have just seen, there is the notion of a later witness on which we may base our attempt: if two point events share a later point event, the three events must be in one history. For this to yield a viable criterion of being a history in BST, however, we had better have an opposite implication as well, in the form "if two point events do not have a common later point event, then they do not share a history". The geometry of Minkowski space-time certainly guarantees that this latter implication holds, yet it is easy to construct a truncated Minkowski space-time, or a more sophisticated space-time of general relativity, which violates this implication (cf. Müller, 2014). Nevertheless, we assume here the strengthened intuition of a later witness in the form of an equivalence: "two point events have a common later point event iff they share a history".[5]

Since the structural feature underlying the later witness intuition has a name, we will use it: a history must be a 'directed' set, defined as follows.

[4] We mean this as a promissory note; in Section 9.1 we will exhibit BST structures in which all histories are indeed isomorphic to Minkowski space-time.

[5] It is possible to work with only a one-way implication in order to properly analyze the truncated space-times that were just mentioned (Placek, 2011). This approach is, however, too complicated for the pay-off it might bring.

Definition 2.2. A subset E of *Our World* is *directed* just in case for all e_1 and e_2 in E there is a point event e_3 in E that is their common upper bound: $e_3 \in E$ and $e_1 \leqslant e_3$ and $e_2 \leqslant e_3$.

Not every directed set should be counted as a history; in line with a respectable tradition, we expect a history to be maximal.[6]

Definition 2.3. A subset h of *Our World* is a *history* just in case h is a maximal directed subset of *Our World*: h itself is a directed subset of *Our World*, and no proper superset of h has this feature.

Histories are a key conceptual tool.[7] Each history might be a Minkowski space-time; but typically *Our World* is no such thing, because a single Minkowski space-time, unlike *Our World*, fails to contain any incompatible possible point events.

Here are some elementary facts about histories.

Fact 2.1. *Let $\mathscr{W} = \langle W, < \rangle$ be a partially ordered set satisfying Postulate 2.1. Then:*

1. *Every finite set of points contained in a history, h, has an upper bound in h.*
2. *Infinite subsets of a history, for example a history itself, need not have a common upper bound.*
3. *Every directed subset of W can be extended to a history. In particular, every chain is a subset of some history, and every point event belongs to some history.*
4. *$\mathscr{W}$ has at least one history.*
5. *Histories are closed downward: if $e_1 \leqslant e_2$ and $e_2 \in h$, then $e_1 \in h$.*
6. *The complements of histories are closed upward: if $e_1 \leqslant e_2$ and $e_1 \in W \setminus h$, then $e_2 \in W \setminus h$.*
7. *No history is a subset of a distinct history.*

[6] Against branching theories, Barnes and Cameron (2011) object that they cannot accommodate the intuition that it "may be open whether or not reality will continue beyond tonight". The intuition, if rendered as a history possibly being a proper segment of another history, contradicts the maximality of histories. However, one may save this intuition (if one needs to) by drawing a distinction between a history and "its" space-time and appealing to an assignment of properties to spatio-temporal points. One may then consider two histories such that none is a segment of the other, yet the space-time of one is a segment of the space-time of the other, and such that they are qualitatively the same in the shared segment of space-time.

[7] Note that the ordinary use of "history" is relational, as in "the history of Pittsburgh". The monadic use appears to be technical. For example, in physics one identifies possible histories with possible evolutions. A different name for Branching-Time histories is "chronicles"; see Øhrstrøm (2009).

8. *No history, h, is a subset of the union of a finite family, H, of histories of which it is not a member. Provided H is a finite set of histories, if $h \subseteq \bigcup H$ then $h \in H$.*
9. *If e is not maximal in* Our World, *then neither is e maximal in any history to which it belongs.*
10. *If e is maximal in W, then there is exactly one history in $\mathscr{W}$ containing e, and e is the unique maximum of h.*

Proof. (1) follows from the definition of a directed subset by finite induction, starting with a common upper bound for the first two points and then consecutively constructing the upper bounds with the next points. As a witness for (2), take for instance the real line with its natural ordering. For (3), let A be a directed subset of W; note that any chain, including the singleton set of a point, is directed. To arrive at a maximal directed set (i.e., a history) containing A, we apply the Zorn-Kuratowski lemma to the family of directed supersets of A in W, ordered by set inclusion.[8] That partial ordering satisfies the premise of the lemma, as for any chain C of directed subsets of W, the union $\bigcup C$ is a directed subset of W that contains every element of C as a subset, so that $\bigcup C$ is an upper bound of C. The lemma then implies that the partial ordering of directed supersets of A contains a maximal element, which is the sought-for history. (4) follows from (3), as W is non-empty. (5) follows from the definition of history as a maximal directed subset, and (6) by taking the contraposition of (5). A maximality of histories suffices to prove (7). As for (8), let $H = \{h_1, ..., h_n\}$ and $h \neq h_i$ for every $h_i \in H$. By (7) $h \setminus h_i \neq \emptyset$ for each $i = 1, ..., n$, so for each i there is $x_i \in h \setminus h_i$. The set $\{x_i \mid i = 1, ..., n\}$ is a finite subset of h, so by (1), there is $y \in h$ upper-bounding all of the x_i. Now, it cannot be that $y \in \bigcup H$, as then $y \in h_j$ for some $h_j \in H$, and hence every $x_i \in h_j$, including $x_j \in h_j$, which contradicts the construction above. As $y \in h$, it follows that $h \not\subseteq \bigcup H$. For (9) let us suppose that $e \in h$ and that $e < e_1$. Then $\{e_2 \mid e_2 \leqslant e\} \cup \{e_1\}$ is a directed subset and a proper superset of $\{e_2 \mid e_2 \leqslant e\}$, so that the latter subset of h is not a history, so not identical with h, so a proper subset of h. Let $e_3 \in h \setminus \{e_2 \mid e_2 \leqslant e\}$. Since h is directed, there must be some element e^* in h that upper-bounds

[8] For future reference, here are the formulations of the versions of the Axiom of Choice that we will use: (1) The Axiom of Choice: for every set X of non-empty sets, there exists a choice function defined on X. (2) The Zorn-Kuratowski lemma: a partially ordered set containing upper bounds for every chain contains at least one maximal element. (3) Hausdorff's maximal principle: in any partially ordered set, every totally ordered subset is contained in a maximal totally ordered subset.

both e and e_3. The element e^* is distinct from e (since otherwise $e_3 < e$), hence e^* is properly later than e, and thus e is not a maximal element of h. (10) Let e be maximal in W (i.e., there is no $e' \in W$ for which $e < e'$). Then the set $h =_{\mathrm{df}} \{x \in W \mid x \leqslant e\}$ is directed (e being an upper bound for any two of its members), and e is its unique maximum. To see that h is maximal directed, suppose for reductio that there is some directed proper superset $h' \supsetneq h$ and pick some $e' \in h' \setminus h$. Since h' is directed, there is some element e'' that upper-bounds the elements e and e'. Since e is maximal in W, we get $e = e''$. Then $e' \leqslant e$, and thus $e' \in h$ by (5), contradicting that $e' \in h' \setminus h$. Thus, h is indeed a maximal directed subset. □

Two point events evidently share some history, just in case they have a common upper bound in Our World. In contrast, two point events fail to have any history in common just in case they have no common upper bound. It would be right to mark such a fundamental matter with a definition.

Definition 2.4. Point events e_1 and e_2 are *compatible* if there is some history to which both belong, and otherwise are *incompatible*.

One may wonder if BST models, and BST histories in particular, incorporate some sense of temporal direction. Much simpler Branching Time models incorporate it, since they have a form of a tree starting from a single trunk. With respect to BST, it may have crossed the reader's mind that each Minkowski space-time appears the same upside down: each is not only directed, but also 'directed downward' in the following sense.

Definition 2.5. A subset E of *Our World* is *directed downward* just in case for all e_1 and e_2 in E there is a point event e in E that is their common lower bound: $e \in E$ and $e \leqslant e_1$ and $e \leqslant e_2$.

That each Minkowski space-time is an upside-down image of itself is of course true,[9] but this should not lead one to think that the way in which we define a 'history' makes no difference. Consider, for instance, the following. While a Minkowski space-time is indeed downward directed, it would be truly peculiar if it were maximal downward directed. For it can be proved that if a subset of a partially ordered set is maximal downward directed, then it is upward closed (see Exercise 2.1). So, if a history were maximal downward directed, it would be upward closed. And if it were upward closed, then if

[9] More precisely, $\langle \mathbb{R}^4, \leqslant_M \rangle$ and its image by time-reflection are order-isomorphic.

there were any incompatible possible point events in the future of any one of its members, the history would have to contain both of them, which would run counter to the idea of compatibility.

In this way, the concepts of Branching Space-Times provide a natural, unforced articulation of the 'direction of time' without complicated physics. They do so by looking beyond the properties of a single history so as to take account of how distinct histories fit together, something that becomes really clear only later in the context of further postulates.

A definition and a fact here shift our attention from single to multiple histories.

Definition 2.6. We write $\text{Hist}(W)$ for the set of all histories in $\langle W, < \rangle$,[10] and for $E \subseteq W$, we write $H_{[E]}$ for the set of those histories containing all of E (i.e.: $H_{[E]} =_{\text{df}} \{h \in \text{Hist} \mid E \subseteq h\}$). We abbreviate $H_{[\{e\}]}$ as H_e, i.e., $H_e =_{\text{df}} \{h \in \text{Hist} \mid e \in h\}$.

Fact 2.2. *(1) H_e is never empty. Also, (2) if $e_1 \leqslant e_2$, then $H_{e_2} \subseteq H_{e_1}$.*

Proof. (1) follows from Fact 2.1(3), while (2) follows from histories being downward closed; that is, Fact 2.1(5). □

One should not generally expect the converse of (2) presented earlier, as two compatible though incomparable point events may belong to exactly the same histories.

With the criterion of historicity we are able to carve out histories from *Our World* $\mathscr{W}$. Typically $\mathscr{W}$ has more than one history, and in that case, if BST is right, the phrase 'our history' or 'the actual history' is meaningless.[11] Scientists, for instance, no matter how hardheaded and downright empirical they wish to be, cannot confine their attention to 'our history' or to 'the actual history'. It is not just that they ought not. Rather it is that if Branching Space-Times is correct, scientists cannot confine their attention to 'the actual history' for precisely the same reason for which mathematicians cannot confine their attention to 'the odd prime number': there is more than one odd prime number, and there is more than one history to which we belong. On the other hand, just as a mathematician can deal with 'the odd

[10] We omit "(W)" if the reference is clear.

[11] This remark pertains to the debate on the doctrine that there is a distinguished history (or a distinguished future): the actual history (future), or our history (future). This doctrine is known as the "Thin Red Line" view and is defended, e.g., by Malpass and Wawer (2012). For an extended argument against this view, see Belnap et al. (2001).

prime numbers' (plural), so a scientist could manage to deal only with 'our histories' (plural); that is, with the set of all histories to which this indexically indicated context of utterance belongs.[12]

With the notion of compatibility in place, we can now define space-like separation.

Definition 2.7. If e_1 and e_2 are (i) incomparable by $\leqslant$ but (ii) compatible, then they are space-like related (SLR), written as $e_1\, SLR\, e_2$. We may also call the events causal contemporaries (provided we bear in mind the failure of the transitivity of SLR).

In this definition, condition (ii) is essential. That is why it was not possible to become clear on space-like separation without the definitions of this Section. We can call events that are related by the pre-causal ordering $<$ "cause-like related". Such events are compatible because histories are closed downward. Using this terminology, we can state the following Fact:

Fact 2.3. *Incompatible points have neither a cause-like nor a space-like relation. They are thus with respect to each other neither causally future nor causally past nor causally contemporaneous.*

This fact is a trivial, albeit helpful, consequence of the definitions. Note that it leaves open the possibility that the spatio-temporal *positions* of incompatible point events are spatio-temporally related, given that a notion of spatio-temporal position of events is available.[13] Even if such a spatio-temporal concept becomes available, however, this does not imply a cause-like relation, and one cannot infer a spatio-temporal relation between the spatio-temporal positions of two point events from the mere fact that they are incompatible. Incompatibility, although defined from the pre-causal order, is not itself a spatio-temporal relation in this sense.

2.3 Historical connection

What we said thus far leaves open the question of how histories in a BST structure are to be related. That histories should somehow overlap follows from the idea of *Our World* as the totality of events accessible from a given

[12] Perhaps physics also considers worlds other than ours, such as those postulated by Lewis (1986a); it is important to recognize this as an entirely different question.

[13] See Chapter 2.5 for some discussion of the introduction of spatio-temporal positions in BST.

actual event by the pre-causal relation. A history that is completely severed from the rest of the model is in conflict with this construction. In the same spirit, in the theory of Branching Time, where histories are chains, one postulates that every two histories overlap; we call this property 'historical connection'.

The property should hold in BST, although the notion of 'history' now has a wider meaning:

Postulate 2.2 (Historical Connection). *Every pair of histories has a nonempty intersection.*

In the theory of Branching Time, this would be the equivalent of saying that every two moments have a lower bound. Here, where the topic is point events instead of moments, the 'common lower bound' principle is not equivalent to Historical Connection, and is not postulated. For more detail, see Fact 2.3. Note that Historical Connection, unlike Postulate 2.1, does not imply the result of replacing $<$ by its converse, and is thus sensitive to the direction of time. That is, if $\langle W, < \rangle$ satisfies Postulate 2.1, so does $\langle W, <\cdot \rangle$, where $<\cdot$ is the relation on W converse to $<$, i.e., $e <\cdot e'$ iff $e' < e$. However, since replacing $<$ by $<\cdot$ can change the set Hist of histories, it may happen that $\langle W, < \rangle$ satisfies Postulate 2.2, but $\langle W, <\cdot \rangle$ does not. A case in point is depicted in Figure 2.2.[14]

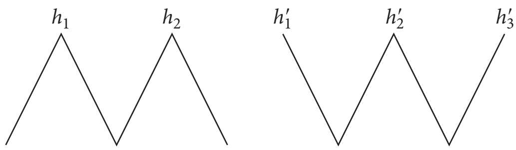

Figure 2.2 Since $h_1 \cap h_2 \neq \emptyset$, $\langle W, < \rangle$ (on the left) satisfies Postulate 2.2, but $\langle W, <\cdot \rangle$ (on the right) does not, as $h'_1 \cap h'_3 = \emptyset$.

The following consequence of historical connection supplies a good account of Lewis's notion of a "suitable external relation": *Our World* is connected by $\leqslant$, the suitable external relation, since the trip from one point to another in *Our World* may be long, but it need not have a complicated shape:

Fact 2.4 (The M property). *Let $\mathscr{W}$ satisfy Postulates 2.1 and 2.2. Then every pair e_1, e_5 of point events can be connected by a $\leqslant / \geqslant$—path no more complex*

[14] Thanks to A. Barszcz for this model.

than the shape of an M, i.e., there are e_2, e_3, e_4 in $\mathscr{W}$ such that $e_1 \leqslant e_2$, $e_5 \leqslant e_4$ and $e_3 \leqslant e_2$, $e_3 \leqslant e_4$.

Proof. Left as Exercise 2.2. □

The M property gives a unity to *Our World* of really possible events, as we explained in Section 1.3.

Note that Historical Connection does not generalize to a larger number of histories. Figure 2.3 provides an illustration: There are three-point events in each history. You see that each pair of histories overlaps (historical connection), but that no point event belongs to all three. A later postulate will rule this out as a possible model, as it will require any finite number of histories to have an overlap.

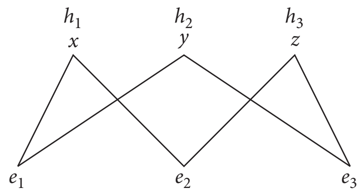

Figure 2.3 Historical Connection does not imply the existence of the overlap of more than two histories.

This observation clearly indicates that a postulate stronger than Historical Connection is needed. The core theory is too frugal to use BST to address some issues in general philosophy as well as in the philosophy of physics. In particular, a stronger postulate should govern what the overlap of two branching histories looks like. This question touches upon a contentious issue. At the bare minimum, we need a local notion of possibility, like alternative possibilities open at a junction, but to define it, we need to decide how to understand "junction" in this context: is it an event, possibly idealized as a point event, or a specific collection of point events, or some other structure? Only after deciding this can we define local alternative possibilities. These should be possibilities open at a junction, in the relevant sense, and should be defined in terms of a suitable partition of the relevant histories. The technical definition of local alternative possibilities is crucial for BST: it will later be used to analyze non-local correlations, singular causation, propensities, Bell's inequalities, tense, and the question of determinism in general relativity. Here we can note that the core of BST can be developed in two different ways. One option leads to the original theory of BST_{92}, the other to a new theory providing "new foundations" for BST, BST_{NF}. Technically, Historical Connection will be strengthened to one of two alternative Prior Choice Principles, one defining BST_{92} and the other defining BST_{NF}. We will discuss these two options in detail in Chapter 3. In the rest of this chapter, we provide a full exposition of the core theory that is common to both approaches, leading to the definition of a common BST structure (Def. 2.10).

2.4 Density and continuity

In this section we reflect further on the pre-causal relation $<$. As of now, we postulated that $\mathscr{W} = \langle W, < \rangle$ is a non-trivial strict partial ordering (Postulate 2.1). The next three postulates come from our desire to arrive at histories that are somewhat similar to space-times of physics. A world-line of a spatially non-extended object is naturally modeled in BST as a maximal chain. To be in accordance with the physics of space and time, maximal chains should be dense and continuous. Hence our next postulates:

Postulate 2.3 (Density). *If $e_1 < e_2$, then there is a point event properly between them.*

We will discuss continuity in the sense of the existence of infima (maximal lower bounds) and suprema (suitable minimal upper bounds). Each of our respective postulates implies Dedekind continuity; see Appendix A.1. Here are the formal definitions of infima and suprema:

Definition 2.8. For $E \subseteq W$, where $\langle W, < \rangle$ is a partial order, a lower bound for E is a point e such that $e \leqslant e_1$ for every $e_1 \in E$. A maximal lower bound for E is a lower bound for E such that no lower bound for E is strictly above it. If there is a lower bound e for E such that $e_1 \leqslant e$ for every lower bound e_1 of E, it will be unique. One calls it $\inf E$, the infimum of E. Similarly for upper bound, for minimal upper bound, and for supremum, written '$\sup E$' when it exists.

We postulate the existence of infima for lower bounded chains:

Postulate 2.4 (Existence of infima for chains). *Every nonempty lower bounded chain of point events has an infimum.*

It can be shown that the infima postulate does not decide the question of suprema for upper bounded chains in BST (see Exercise 2.4). We therefore need yet another postulate. In formulating the suprema postulate, we cannot just use the mirror image of Postulate 2.4, because an upper bounded chain may end in different ways depending on the history that one considers. As a supremum has to be unique, we do not postulate the existence of suprema, but rather of history-relative suprema. This shows how temporal directedness is modally grounded: a BST structure with just one history allows for unique suprema, but once there is more than one history (i.e., once there is indeterminism), the different behavior of infima and suprema signals temporal asymmetry.

Postulate 2.5 (Existence of history-relative suprema for chains)**.** *Each non-empty upper bounded chain l has a supremum $\sup_h l$ in each history h such that $l \subseteq h$, where $\sup_h l$ is characterized by the following three conditions: (1) $\sup_h l \in h$; (2) $e_1 \leqslant \sup_h l$ for every $e_1 \in l$, and (3) if $e_2 \in h$ and $e_1 \leqslant e_2$ for every $e_1 \in l$, then $\sup_h l \leqslant e_2$.*

Note that a corresponding reformulation of the infima postulate would change nothing, as by the downward closure of histories, any lower bound of a lower-bounded chain belongs to all histories to which the chain belongs.

The definition of history thus entails that infima of lower bounded chains exist independently of histories, while suprema of upper bounded chains exist only relative to a history. These features are essential features of Branching Space-Times. Take a 'process' as represented by a bounded causal interval without a first or last point event, and interpret the following tenses from the standpoint of a point event within it. 'How this process will end' (i.e., the supremum of the process) is historically contingent, depending as it does on (perhaps metaphorical) choices made in the neighborhood of the process. 'How this process began' (i.e., the infimum of the process) is, in contrast, independent of histories.

2.5 Weiner's postulate and spatio-temporal locations

There is one further postulate, suggested by M. Weiner, that needs to be added to the core theory of BST in order to exclude unwanted structures.[15]

[15] Weiner's postulate (see also Def. 2.10(6)), which was not included in early BST papers such as Belnap (1992), was added later as it was found to be necessary to develop a useful probability theory in BST_{92}; see Weiner and Belnap (2006) and Müller (2005).

Postulate 2.6 (Weiner's postulate). *Let $l, l' \subseteq h_1 \cap h_2$ be upper bounded chains in histories h_1 and h_2. Then the order of the suprema in these histories is the same:*

$$\sup_{h_1} l \leqslant \sup_{h_1} l' \quad \textit{iff} \quad \sup_{h_2} l \leqslant \sup_{h_2} l'.$$

Note that hereby we also have

$$\sup_{h_1} l = \sup_{h_1} l' \quad \text{iff} \quad \sup_{h_2} l = \sup_{h_2} l'.$$

Weiner's postulate has philosophical significance as it helps to clarify which objects branch in branching-style theories. Histories represent alternative possible scenarios. In Branching Time, histories capture merely temporal aspects of a possible scenario, whereas in BST histories accommodate spatio-temporal aspects. Perhaps misled by the names, an objection has been leveled that these theories assume the branching of time or of space-time itself. But what branches, according to these theories, is spatio-temporal (temporal) histories and *not* space-time (time). Weiner's postulate provides an argument for this verdict, as it makes the following construction possible.

Having a branching model of possible histories, one might want to coordinate, in a temporal or spatio-temporal sense, events belonging to alternative histories. The motivation for this is clearly seen in a temporal case, as we often wonder what would have occurred at a given instant of time (e.g., 9 am this morning) if things had gone differently in the past. One might thus want to view some incompatible events as possibly occurring at the same instant of time. The result of introducing such temporal coordinates would be a theory of Branching Time with temporal instants, with the set of instants naturally identified with time. Since the time so constructed is common to all histories, it cannot branch, though histories do branch.[16] The same motivation, if applied to spatio-temporal histories, yields the concept of spatio-temporal (point-like) locations, with the underlying idea that events from alternative histories belong to one and the same location of this sort. The set of spatio-temporal (point-like) locations is analogously read as space-time.

The significance of Weiner's postulate is that it provides a necessary (but not sufficient) condition for spatio-temporal locations to be definable in BST structures. To prove this claim we first define space-time locations:

[16] For a theory of Branching Time with Instants, see Belnap et al. (2001, Ch. 7A.5).

Definition 2.9 (BST with space-time locations)**.** Let $\langle W, < \rangle$ satisfy Postulates 2.1–2.5. A partition S of W is a set of *spatio-temporal locations* of $\langle W, < \rangle$ iff

1. For each history h in W and for each $s \in S$, the intersection $h \cap s$ contains exactly one element.
2. S respects the ordering, i.e., for $s, s' \in S$ and $h_1, h_2 \in \text{Hist}$, $s \cap h_1 \leqslant s' \cap h_1$ iff $s \cap h_2 \leqslant s' \cap h_2$.[17]

An auxiliary result is that the history-relative suprema of upper bounded chains, guaranteed to exist by Postulate 2.5, belong to the same location—if $\langle W, < \rangle$ admits spatio-temporal locations.

Fact 2.5. *Let $\langle W, <, S \rangle$ satisfy Postulates 2.1–2.5, and let S be a set of spatio-temporal locations of $\langle W, < \rangle$. Let l be an upper-bounded chain in W such that $l \subseteq h_1 \cap h_2$, where $h_1, h_2 \in \text{Hist}$. Then for some $s \in S$, $\{\sup_{h_1} l, \sup_{h_2} l\} \subseteq s$.*

Proof. Let $c_1 = \sup_{h_1} l$ and $s \in S$ be such that $c_1 \in s$. We take the unique $c_2 \in s \cap h_2$. We need to prove that $c_2 = \sup_{h_2} l$. Since S preserves the ordering, and c_1 upper bounds l in h_1, c_2 must be an upper bound of l in h_2. Moreover, c_2 must be the least upper bound of l in h_2, for if there were a c_2' such that $l \leqslant c_2' < c_2$, then c_1 would not be the least upper bound of l in h_1, contradicting our assumption. Thus, c_2 is an h_2-relative supremum of l, i.e., $c_2 = \sup_{h_2} l$, as required. □

Then as a corollary we get that Postulate 2.6 is a necessary condition for definability of locations S on $\langle W, < \rangle$:

Corollary 2.1. *If $\langle W, < \rangle$ satisfies Postulates 2.1–2.5 and admits spatio-temporal locations S, then it satisfies Postulate 2.6.*

Proof. Let l and l' be two upper bounded chains, and let h_1, h_2 be two histories to which both l and l' belong. Then by Fact 2.5, their history-relative suprema will be at the same space-time locations, and the claim follows by order preservation of S. □

Further, the model discussed in Exercise 2.5 shows that definability of locations S is not a trivial matter, as evidenced by the following:

[17] We are using an extension of the ordering notation to singletons here, as $s \cap h_1$ is the singleton set of an event, not an event. The analogous claims for "$=$" and for "$<$" follow directly.

Fact 2.6. *There is a strict partial ordering $\langle W, <\rangle$ that satisfies Postulates 2.1–2.5, but does not admit spatio-temporal locations.*

Proof. By the model given in Exercise 2.5 and by Corollary 2.1, using contraposition. □

2.6 Axioms of the common core of BST

We end this chapter with the "official definition" of a common BST structure, pulling together all of the above Postulates, with BST histories defined by Definition 2.3:

Definition 2.10 (Common BST structure)**.** A *common BST structure* is a pair $\langle W, <\rangle$ that fulfills the following conditions:

1. W is a non-empty set of possible point events.
2. $<$ is a strict partial ordering denoting a pre-causal relation on W.
3. The ordering $<$ is dense;
4. The ordering contains infima for all lower bounded chains;
5. The ordering contains history-relative suprema for all upper bounded chains;
6. Weiner's postulate: Let $l, l' \subseteq h_1 \cap h_2$ be upper bounded chains in histories h_1 and h_2. Then the order of the suprema in these histories is the same:
$$\sup_{h_1} l \leqslant \sup_{h_1} l' \quad \text{iff} \quad \sup_{h_2} l \leqslant \sup_{h_2} l'.$$
7. Historical connection: Any two histories have a non-empty intersection, i.e., for $h_1, h_2 \in \text{Hist}$, $h_1 \cap h_2 \neq \emptyset$.

2.7 Exercises to Chapter 2

Exercise 2.1. Complete the sketch of the proof of Fact 2.1(5). Then prove that, if a subset of a partially ordered set is maximal downward directed, then it is upward closed.

Hint: Rework the proof of Fact 2.1(5) in an appropriate way.

Exercise 2.2. Prove the M property (Fact 2.4):
For every pair e_1, e_5 of point events in $\mathscr{W}$, there are e_2, e_3, e_4 in $\mathscr{W}$ such that $e_1 \leqslant e_2$, $e_5 \leqslant e_4$ and $e_3 \leqslant e_2$, $e_3 \leqslant e_4$.

Hint: Use historical connection and the directedness of histories. An explicit proof is given in Appendix B.2.

Exercise 2.3. Let $\langle W, \leqslant \rangle$ be a partially ordered set satisfying Postulates 2.1 and 2.2. Prove that if every history of W is downward directed, then so is W as a whole. (Note that the assumption is true, for example, if each history is isomorphic to Minkowski space-time.)

Hint: Pick any $e_1, e_5 \in W$. If these events share a history, we are done. If not, invoke the M property to find appropriate lower bounds. An explicit proof is given in Appendix B.2.

Exercise 2.4. Let $\langle W, \leqslant \rangle$ be a partially ordered set satisfying the infima postulate 2.4. Show that this postulate does not imply the existence of history-relative suprema for upper bounded chains (i.e., it does not imply Postulate 2.5).

Hint: Take a maximal chain in history h and its Dedekind cut $\{A, B\}$, where $A < B$. (See Appendix A.1 for the definition.) The upper sub-chain B thus has an infimum. Suppose that it belongs to B. Then this infimum is also a minimal upper bound for A. One can show, however, that the infima postulate does not prohibit the existence of another minimal upper bound for A in h. Then in h there is no unique minimal upper bound of A, and so A has no supremum in h (and hence no supremum *simpliciter*). Although a topology for BST has not yet been introduced (see Chapter 4.4), one can reasonably suspect that the structure in question violates the Hausdorff property. The postulate of history-relative suprema accordingly assures that all histories are Hausdorff.

Exercise 2.5. Show that Postulate 2.6 is independent of the remaining postulates 2.1–2.5.

Hint: For a structure that satisfies Postulates 2.1–2.6, one can pick a simple two-histories structure in which each history h_1 and h_2 is isomorphic to the two-dimensional real plane with the Minkowskian ordering $\leqslant_M$ (see Eq. 2.1).

For a structure that satisfies Postulates 2.1–2.5 but violates Postulate 2.6, consider the essentially linear two-history structure depicted in Figure 2.4.[18]

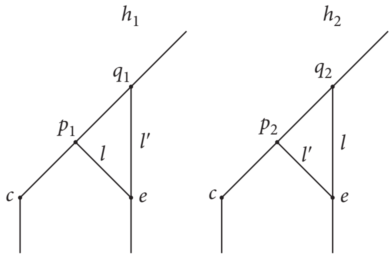

Figure 2.4 Postulate 2.6 is violated by chains l and l'.

[18] The figure is based on a drawing by M. Weiner. For a different example that is essentially two-dimensional, see Appendix A of Müller (2005).