4

Building upon the Foundations of Branching Space-Times

In this chapter we introduce a variety of events that are definable in BST and discuss in which histories these events occur. This will give rise to the concept of the occurrence proposition for events of various kinds. We also hint at how BST structures may be used to build semantic models for languages with temporal and modal operators. In this way we provide the machinery for our BST theories of causation and of propensities, and we prepare the ground for a number of further applications.

Perhaps surprisingly, a large portion of this material is independent of the choice between BST_{92} and BST_{NF}, so in Chapter 4.1 we remain in the common BST framework. The situation changes once we discuss basic transitions in Chapter 4.2, as these objects are sensitive to topology (which dictates the pattern of branching) and need to be defined differently in the two frameworks. We already introduced the notion of a transition in Chapter 3.4.4, working in BST_{92}. Here we will provide a number of additional definitions, mostly for sets of transitions which will be used extensively in later chapters. We also provide some details about basic transitions in BST_{NF}.

4.1 A variety of events and their occurrence propositions

We begin with the notion of a proposition. In line with a prominent tradition, propositions are identified with sets of histories:

Definition 4.1 (Basics of propositions)**.** H is a proposition $\Leftrightarrow_{\text{df}} H \subseteq \text{Hist}$. H is defined as true or false in a history h according to whether or not $h \in H$. H is consistent $\Leftrightarrow_{\text{df}} H \neq \emptyset$. H is universal $\Leftrightarrow_{\text{df}} H = \text{Hist}$, and H is contingent $\Leftrightarrow_{\text{df}}$ H is consistent but not universal.

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We define the notion of a necessary and of a sufficient condition in the standard way.[1]

Definition 4.2 (Necessary and sufficient conditions)**.** For propositions X and Y, we say that X is a sufficient condition for $Y \Leftrightarrow_{\mathrm{df}} X \subseteq Y$; we also say that X implies Y, or that Y is a necessary condition for X.

Although BST starts with the meagre primitive notion of a point event, it permits the introduction of a number of more complex event-like concepts. We define these together with their respective occurrence propositions. We start with initial and outcome events. The distinction is especially important for the representation of indeterministic processes. Typically, such a process can be conceived as containing an initial event and one of its multiple possible outcomes. A radioactive particle's decay or a measurement process can serve as illustrations. Our definition below reflects the intuition that for an initial event to have occurred, it needs to have come to an end. Thus, in a measurement, before an outcome occurs, the whole measurement initial event (e.g., the preparation of the apparatus in the 'ready' state) needs to have come to completion. In contrast, for an outcome to occur it is enough that it has just begun: to say that a particle has decayed, we need to witness just some arbitrarily small part of the world after the decay. Although the intuition is perhaps not crystal clear, the definitions below turn out to be fruitful in many later developments.

Definition 4.3 (Initial and outcome events and occurrence propositions)**.**

1. I is an *initial event* $\Leftrightarrow_{\mathrm{df}}$ I is a consistent nonempty set of point events (i.e., a set of point events all of which are members of some one history). The occurrence proposition for I is $H_{[I]} =_{\mathrm{df}} \{h \in \mathrm{Hist} \mid I \subseteq h\}$. Equivalently, $H_{[I]} = \bigcap_{e \in I} H_e$.
2. O is an *outcome chain* $\Leftrightarrow_{\mathrm{df}}$ O is a non-empty and lower-bounded chain. The occurrence proposition for O is $H_{\langle O \rangle} =_{\mathrm{df}} \{h \in \mathrm{Hist} \mid h \cap O \neq \emptyset\}$.
3. $\hat{O}$ is a *scattered outcome event* $\Leftrightarrow_{\mathrm{df}}$ $\hat{O}$ is a set of outcome chains all of which overlap some one history (i.e., there is one history that contains an initial segment of each of the chains).
   The occurrence proposition for $\hat{O}$ is $H_{\langle \hat{O} \rangle} =_{\mathrm{df}} \bigcap_{O \in \hat{O}} H_{\langle O \rangle}$.

[1] As we proceed to develop the BST account of causation in Chapter 6, we will need the related notion of a non-redundant part of a sufficient condition for a proposition. As it will turn out, however, this notion is subtle and not univocal. We will therefore discuss it only after we develop the framework within which this subtlety plays a role.

4. $\breve{\mathbf{O}}$ is a *disjunctive outcome event* $\Leftrightarrow_{\mathrm{df}}$ $\breve{\mathbf{O}}$ is a set of pairwise incompatible scattered outcomes (a set of sets of sets), where 'pairwise incompatible' means that for any $\hat{O}_1, \hat{O}_2 \in \breve{\mathbf{O}}$, if $\hat{O}_1 \neq \hat{O}_2$, then $H_{\langle \hat{O}_1 \rangle} \cap H_{\langle \hat{O}_2 \rangle} = \emptyset$. The occurrence proposition for $\breve{\mathbf{O}}$ is $H_{\langle \breve{\mathbf{O}} \rangle} =_{\mathrm{df}} \bigcup_{\hat{O} \in \breve{\mathbf{O}}} H_{\langle \hat{O} \rangle}$.

According to this definition, there is a hierarchy of outcome events, which differ in their complexity. Starting with an outcome chain, it is a part of a world-line that is bounded from below. A scattered outcome is a more complex spatiotemporal affair. A given result, say 'side 1 up', $\boxed{1}$, of a particular rolling of a die, is a scattered outcome. That outcome is composed of a huge number of lower bounded segments of the world-lines of the particles involved. It does not matter that, realistically speaking, no single history will contain sizable segments of all these world-lines. To constitute a scattered outcome, however, there should be at least one history in which all these segments *begin*. Next in the hierarchy come disjunctive outcomes, which are more complex, as they combine different possibilities. As an illustration, consider a set of two possible results of a particular rolling of a die, say $\{\boxed{1}, \boxed{3}\}$. Clearly, this disjunctive outcome occurs if $\boxed{1}$ occurs or $\boxed{3}$ occurs. Thus, a disjunctive outcome, in contrast to an outcome chain and a scattered outcome, can be realized in a number of different ways, as its elements belong to alternative possibilities. We thus say that disjunctive outcomes are multiply realizable. To return to our die, note that the set of *all* possible results of rolling the die, $\{\boxed{1}, \boxed{2}, \ldots, \boxed{6}\}$, counts as a disjunctive outcome as well. This disjunctive outcome has the peculiar feature that it is bound to occur once the rolling of the die occurs. Later, we will capture such cases via the concept of a deterministic transition to a disjunctive outcome: although the world is as indeterministic as you like, a deterministic transition to the disjunctive outcome including all possible outcomes is bound to occur once the initial event has occurred.

One may wonder what the initial of a particular process of rolling a die consists of. To describe a concrete happening: you put a particular die in a dice cup, give it a shake, and roll it onto a flat surface. That complex affair needs to be over before a particular result shows up. As this example makes clear, an initial event I may be extended in space and time—there are no restrictions except for consistency. As $I \subseteq h$ for some history h, $H_{[I]} \neq \emptyset$. In a similar vein, occurrence propositions for outcomes are never the empty set. Since a chain O is a directed subset of $\mathscr{W}$, it can be extended to a full history, hence $O \subseteq h$ for some $h \in \mathrm{Hist}$, and thus $H_{\langle O \rangle}$ is non-empty. With this observation it is easy to see that $H_{\langle \hat{O} \rangle}$ and $H_{\langle \check{\mathbf{O}} \rangle}$ are non-empty as well, and thus consistent in the sense of Def. 4.1.

We next turn to transition events, to be understood as a liberalized notion of change. Following von Wright (1963), we take it that a transition is of the form "something and then something", but not necessarily "...something else". "Then" has to be spelled out to mean that the beginning of a transition is appropriately below its final part. We insist, moreover, that the parts of a transition are categorically different: its beginning is to be an initial event, while its final part—an outcome event.

A paradigm example of a transition event is a choice. Before the choice there is no choice, and after the choice there is no choice. So when is the choice? Bad question: a choice, like any transition event, has no 'simple location' (Whitehead, 1925, Ch. 3). You can locate its initial in the causal order, and you can locate its outcome in the causal order; and having done that, you have done all that you can do. When a choice is made, something happens, but 'when' it happens can only be described by giving separate 'whens' to its initial and to its outcome. Exactly the same holds for any other transition event. (This thought will be applied to measurements in quantum mechanics in Chapter 8.)

In what follows, we will use a generic notation for outcomes to define transitions: we will write $\mathscr{O}^*$ for an outcome event from Definition 4.3, that is, an outcome chain O, a scattered outcome $\hat{O}$, or a disjunctive outcome $\check{\mathbf{O}}$. We obtain thus three kinds of transitions; we refer to them all by the generic notation, $I \rightarrowtail \mathscr{O}^*$. Clearly, a singleton $\{e\}$ of a point event is an initial. We simplify the unwieldy $\{e\} \rightarrowtail \mathscr{O}^*$ as $e \rightarrowtail \mathscr{O}^*$.

In Chapter 3.4.4 we already introduced a basic transition in BST_{92} as a pair $\langle e, H \rangle$ with $H \in \Pi_e$, written $e \rightarrowtail H$. In this representation, one element of a transition is an event and the other is a proposition. In contexts in which the spatio-temporal location of outcome events is irrelevant, we will often use such a "quasi-propositional notation", writing $e \rightarrowtail H_{\langle O \rangle}$ instead of $e \rightarrowtail O$, and analogously for transitions to scattered outcomes and to disjunctive outcomes. While such hybrid objects are often handy, it is also natural to represent transitions in terms of events only, as in the following definition. It will turn out that for basic transitions, both representations are equivalent; see Fact 4.3 in Chapter 4.2. Here is how we spell out that the initial of a transition of one of the kinds we consider is *appropriately below* the outcome:

Definition 4.4 (Transition events)**.** For I and $\mathscr{O}^*$ an initial event and a generic outcome event, respectively, a transition is the pair $\langle I, \mathscr{O}^* \rangle$, written $I \rightarrowtail \mathscr{O}^*$, where I is appropriately below $\mathscr{O}^*$, $I <_i \mathscr{O}^*$. "Appropriately below" is defined as follows:

$$\begin{aligned}
&e <_1 O \Leftrightarrow_{\mathrm{df}} \forall e'[e' \in O \rightarrow e < e']\\
&I <_2 O \Leftrightarrow_{\mathrm{df}} \forall e[e \in I \rightarrow e <_1 O]\\
&e <_3 \hat{O} \Leftrightarrow_{\mathrm{df}} \exists O[O \in \hat{O} \wedge e <_1 O]\\
&I <_4 \hat{O} \Leftrightarrow_{\mathrm{df}} \forall e[e \in I \rightarrow e <_3 \hat{O}]\\
&e <_5 \breve{\mathbf{O}} \Leftrightarrow_{\mathrm{df}} \forall \hat{O}[\hat{O} \in \breve{\mathbf{O}} \rightarrow e <_3 \hat{O}]\\
&I <_6 \breve{\mathbf{O}} \Leftrightarrow_{\mathrm{df}} \forall e[e \in I \rightarrow e <_5 \breve{\mathbf{O}}].
\end{aligned}$$

A welcome consequence of our definitions is that for any transition $I \rightarrowtail \mathscr{O}^*$, the occurrence of the initial is a necessary condition for the occurrence of the outcome:

Fact 4.1. *For a generic transition* $I \rightarrowtail \mathscr{O}^*$, $H_{\langle \mathscr{O}^* \rangle} \subseteq H_{[I]}$.

Proof. We consider the six cases of Def. 4.4 in turn. (1) Since histories are downward closed, $e <_1 O$ implies $H_{\langle O \rangle} \subseteq H_e$. (2) $I <_2 O$ implies that for every $e \in I$: $H_{\langle O \rangle} \subseteq H_e$, and hence $H_{\langle O \rangle} \subseteq \bigcap_{e \in I} H_e = H_{[I]}$. (3) Next, $e <_3 \hat{O}$ implies $H_{\langle O \rangle} \subseteq H_e$ for some $O \in \hat{O}$, and hence $H_{\langle \hat{O} \rangle} = \bigcap_{O \in \hat{O}} H_{\langle O \rangle} \subseteq H_e$. (4) For $I <_4 \hat{O}$, since $H_{\langle \hat{O} \rangle} \subseteq H_e$ for every $e \in I$, we get $H_{\langle \hat{O} \rangle} \subseteq \bigcap_{e \in I} H_e = H_{[I]}$. (5) And, if $e <_5 \breve{\mathbf{O}}$, then for all $\hat{O} \in \breve{\mathbf{O}}$: $H_{\langle \hat{O} \rangle} \subseteq H_e$, and hence $H_{\langle \breve{\mathbf{O}} \rangle} = \bigcup_{\hat{O} \in \breve{\mathbf{O}}} H_{\langle \hat{O} \rangle} \subseteq H_e$; thus, (6) for $I <_6 \breve{\mathbf{O}}$ we have $H_{\langle \breve{\mathbf{O}} \rangle} \subseteq H_e$ for every $e \in I$, which entails $H_{\langle \breve{\mathbf{O}} \rangle} \subseteq \bigcap_{e \in I} H_e = H_{[I]}$. □

A transition event, like any event, can occur or not occur. What, then, is the occurrence proposition for a transition event? A good guess would be that it should be an *and then* proposition: first the initial occurs, and then the outcome occurs. It turns out, however, that in BST, it is more appropriate to take the occurrence proposition for a transition event to be the material implication: *if* the initial occurs, *then* the outcome occurs.

Definition 4.5. Let $I \rightarrowtail \mathscr{O}^*$ be a transition event of one of the types allowed by Definition 4.4, and let $H_{[I]}$ and $H_{\mathscr{O}^*}$ be the occurrence propositions defined for I and $\mathscr{O}^*$ respectively. Then $H_{I \rightarrowtail \mathscr{O}^*} \Leftrightarrow_{\mathrm{df}} (\mathrm{Hist} \setminus H_{[I]}) \cup H_{\mathscr{O}^*}$ is the occurrence proposition for $I \rightarrowtail \mathcal{O}^*$, true in h iff $h \in H_{I\rightarrowtail\mathcal{O}^*}$, hence iff, if $h \in H_{[I]}$, then $h \in H_{\mathcal{O}^*}$.

The final 'if – then' must be truth-functional. Usually, in ordinary language applications, the negation of a material implication 'if A then B' seems wrong; this is of course one of the motivations for various theories of counterfactual conditionals, to say nothing of relevance logic. Here, however, there is a better fit: for the transition $I \rightarrowtail \mathcal{O}^*$ *not* to occur is for the initial to occur and then for *some other outcome* of I to occur instead. It is not merely for the outcome $\mathcal{O}^*$ not to occur. The non-occurrence proposition of $\mathcal{O}^*$ is simply Hist $\setminus H_{\mathcal{O}^*}$; the non-occurrence of the transition $I \rightarrowtail \mathcal{O}^*$ is more specific. For instance, if you understand a particular choice as a transition from a particular occasion of indecision to a settled state of having selected the tuna sandwich, then for that transition event not to occur is for the chooser to have chosen otherwise from that very same occasion of indecision. For the non-occurrence of the transition event, it does not suffice that the chooser was never born—although that would certainly be sufficient for the non-occurrence of the tuna-selection outcome. Furthermore, we naturally say that a transition $I \rightarrowtail \mathcal{O}^*$ is (historically) 'noncontingent' when the initial already deterministically guarantees the outcome; that is, when $H_{\langle\mathcal{O}^*\rangle}$ is not merely a subset of $H_{[I]}$ (as must always be the case, see Fact 4.1), but identical to $H_{[I]}$. In that case, the transition-event occurrence proposition rightly turns out to be the universal proposition: $(\text{Hist} \setminus H_{[I]}) \cup H_{\langle\mathcal{O}^*\rangle} = (\text{Hist} \setminus H_{\langle\mathcal{O}^*\rangle}) \cup H_{\langle\mathcal{O}^*\rangle} = \text{Hist}$, which signals historical noncontingency. One should not be deeply interested in transition events whose occurrence in h is merely a matter of the initial not occurring in h, and so it is good to mark this by saying that the transition event occurs *vacuously* in h if $h \notin H_{[I]}$.

As we mentioned earlier, there are deterministic (historically noncontingent) transitions to disjunctive outcomes even in an indeterministic context. To return to our example of die casting, the set of all possible results, $\check{\mathbf{O}} =_{\text{df}} \{\boxed{1}, \boxed{2} \ldots \boxed{6}\}$, of a particular act of die casting is the exhaustive disjunctive outcome. (We exclude weird cases, such as the die landing on its edge.) The particular act of casting the die is an initial event I. Given their location, I and $\check{\mathbf{O}}$ form the transition $I \rightarrowtail \check{\mathbf{O}}$. By Fact 4.1, we have $H_{\langle\check{\mathbf{O}}\rangle} \subseteq H_{[I]}$, and by exhaustiveness, $H_{[I]} \subseteq H_{\langle\check{\mathbf{O}}\rangle}$, i.e., $H_{[I]} = H_{\langle\check{\mathbf{O}}\rangle}$, which means that if I occurs, so does $\check{\mathbf{O}}$. Moreover, by Definition 4.5, $I \rightarrowtail \check{\mathbf{O}}$ occurs in every history, as $H_{I\rightarrowtail\check{\mathbf{O}}} = (\text{Hist} \setminus H_{[I]}) \cup H_{\langle\check{\mathbf{O}}\rangle} = (\text{Hist} \setminus H_{[I]}) \cup H_{[I]} = \text{Hist}$.

For the record, here we define transitions to deterministic disjunctive outcomes:

Definition 4.6. Let I be an initial event, Γ some index set, $|\Gamma| > 1$. We call $\mathbf{1}_I = \{\hat{O}_\gamma \mid \gamma \in \Gamma\}$ a *deterministic disjunctive outcome of I* iff (1) each $\hat{O}_\gamma$ is above I in the sense of Def. 4.4, (2) for $\gamma, \gamma' \in \Gamma$, $H_{\langle \hat{O}_\gamma \rangle} \cap H_{\langle \hat{O}_{\gamma'} \rangle} = \emptyset$ if $\gamma \neq \gamma'$, and (3) $\bigcup_{\gamma \in \Gamma} H_{\langle \hat{O}_\gamma \rangle} = H_{[I]}$. We call $I \rightarrowtail \mathbf{1}_I$ a *transition to a deterministic disjunctive outcome.*

By this definition, if I occurs, some $\hat{O}_\gamma$ occurs, and hence $\mathbf{1}_I$ occurs as well. Despite indeterminism, witnessed by multiple transitions to different scattered outcomes, $I \rightarrowtail \mathbf{1}_I$ is a deterministic transition.

We end this section by noting the following simple relations between the occurrence propositions of different types of transitions:

Fact 4.2. *(1)* $H_{I \rightarrowtail \hat{O}} = \bigcap_{O \in \hat{O}} H_{I \rightarrowtail O}$. *(2)* $H_{I \rightarrowtail \check{\mathbf{O}}} = \bigcup_{\hat{O} \in \check{\mathbf{O}}} H_{I \rightarrowtail \hat{O}}$.

Proof. (1) This follows by Def. 4.3(3), noting that $(\text{Hist} \setminus H_{[I]}) \cap H_{\langle O \rangle} = \emptyset$ by Fact 4.1. (2) Observe that $(\text{Hist} \setminus H_{[I]}) \cup \bigcup_{\hat{O} \in \check{\mathbf{O}}} H_{\langle \hat{O} \rangle} = \bigcup_{\hat{O} \in \check{\mathbf{O}}} ((\text{Hist} \setminus H_{[I]}) \cup H_{\langle \hat{O} \rangle})$. □

4.2 Basic transitions

In this section we develop further the theory of basic transitions that we started in Chapter 3.4.4. Our discussion in Chapter 4.1 has been phrased in the framework of common BST structures, but basic transitions look different in BST_{92} and BST_{NF}, as they are sensitive to the topology of branching. Moreover, the proof of the interchangeability of two representations of basic transition below (Fact 4.3) appeals to the prior choice principle, which works differently in BST_{92} than in BST_{NF}. So in this section we cannot work in the common BST framework. In line with our general approach of working with BST_{92} in the main text in cases where the choice matters, we will discuss basic transitions in BST_{92} structures. We will, however, also provide some details that lead to the definition of two representations of basic transitions in BST_{NF} (see Chapter 4.2.2).

4.2.1 Basic transitions in BST_{92}

Basic transitions are the irreducible local elements of indeterminism in a BST_{92} structure, consisting of a point event e and one of its immediate possible outcomes. We introduce two alternative views of basic transitions in BST_{92}, based on the observation that the outcome of a basic transition can be represented in either of two (equivalent) ways: as a proposition defined in terms of undividedness, or as a scattered outcome event consisting of outcome chains all of which begin immediately after e:

1. In Chapter 3.4.4, we introduced a basic transition as a transition from a single point event e to one of its elementary possibilities understood propositionally. The outcome of a basic transition is then a proposition $H \in \Pi_e$, where Π_e is the partition of the set of histories containing e, H_e, induced by the equivalence relation of undividedness-at-e, $\equiv_e$. We write $e \rightarrowtail H$ for a basic transition, so understood.
2. In line with Def. 4.4, we can understand a basic transition as a transition from an initial point event e to a particular scattered outcome event $\hat{O}$ that we call an immediate (basic scattered) outcome of e. What makes the scattered outcome event $\hat{O}$ an immediate outcome of e is that for every outcome chain $O \in \hat{O}$, $\inf O = e$ and $e \notin O$. There are many such chains. In the definition below, we consider maximal chains, and we divide them up to form a particular scattered outcome event via a given history, as follows:

Definition 4.7 (Basic scattered outcomes of e)**.** Let $h \in \mathrm{Hist}$, and let $e \in h$. We define $\Omega_e\langle h\rangle =_{\mathrm{df}} \{O \mid O$ is a chain maximal with respect to $\inf O = e \wedge e \notin O \wedge h \cap O \neq \emptyset\}$. $\Omega_e =_{\mathrm{df}} \{\Omega_e\langle h\rangle \mid h \in H_e\}$. Each member $\Omega_e\langle h\rangle$ of Ω_e is a *basic scattered outcome* of e.

The members of $\Omega_e\langle h\rangle$ evidently begin in the immediate future of e, so that between e and members of $\Omega_e\langle h\rangle$ there is no room for influences from the past. Since $\Omega_e\langle h\rangle$ is a scattered outcome event, which can occur or not occur, $H_{\langle\Omega_e\langle h\rangle\rangle}$ makes sense as a proposition. That proposition equals the propositional basic outcome of e determined by h:

Fact 4.3 (Interchangeability of $\Omega_e\langle h\rangle$ and $\Pi_e\langle h\rangle$)**.** *The occurrence proposition $H_{\langle\Omega_e\langle h\rangle\rangle}$ for $\Omega_e\langle h\rangle$ is the same proposition as $\Pi_e\langle h\rangle$.*

Proof. Let $h' \in H_{\langle\Omega_e\langle h\rangle\rangle}$. This implies that every $O \in \Omega_e\langle h\rangle$ intersects nonemptily with h', $O \cap h' \neq \emptyset$. Since $O \in \Omega_e\langle h\rangle$, we get $O \cap h \neq \emptyset$ as well. Since $\inf O = e$ and $e \notin O$, there is some $e' \in h \cap h'$ such that $e < e'$, so we have $h \equiv_e h'$, hence $h' \in \Pi_e\langle h\rangle$.

In the opposite direction, let $h' \in \Pi_e\langle h\rangle$ and suppose for reductio that $h' \notin H_{\langle\Omega_e\langle h\rangle\rangle}$, which implies that for some $O \in \Omega_e\langle h\rangle$, $h' \cap O = \emptyset$. Hence there would be an initial segment O' of O such that $O' \subseteq h \setminus h'$. Since $e = \inf O = \inf O'$, by PCP_{92} we have that for some $c < O'$: $h \perp_c h'$. This contradicts $h \equiv_e h'$, as $c \leqslant e$ by the definition of the infimum. □

Occurrence propositions do not in general determine outcome events. For the special outcome events of the form $\Omega_e\langle h\rangle$, however, when we are not only given the proposition but also e, we can recover the event from the proposition:

Fact 4.4. *Let $e \in W$, and let $h_1, h_2 \in H_e$. We have $H_{\langle\Omega_e\langle h_1\rangle\rangle} = H_{\langle\Omega_e\langle h_2\rangle\rangle}$ iff $\Omega_e\langle h_1\rangle = \Omega_e\langle h_2\rangle$.*

Proof. The direction from right to left is trivial. For the other direction, assume that $H_{\langle\Omega_e\langle h_1\rangle\rangle} = H_{\langle\Omega_e\langle h_2\rangle\rangle}$. By Fact 4.3, $H_{\langle\Omega_e\langle h_i\rangle\rangle} = \Pi_e\langle h_i\rangle$ $(i = 1, 2)$, so in particular, $h_1 \equiv_e h_2$. Now assume for reductio that there is $O \in \Omega_e\langle h_1\rangle$ while $O \notin \Omega_e\langle h_2\rangle$ (the case with h_1 and h_2 reversed is exactly analogous). Let $h \in \mathrm{Hist}$ be such that $O \subseteq h$. By Def. 4.7, $O \in \Omega_e\langle h_1\rangle$ implies that $O \cap h_I \neq \emptyset$, and as $e < O$, we have $h \equiv_e h_1$. By transitivity of $\equiv_e$, $h \equiv_e h_2$ as well. On the other hand, $O \notin \Omega_e\langle h_2\rangle$ implies that $O \subseteq h \setminus h_2$, so that by PCP_{92}, there is $c < O$ for which $h \perp_c h_2$. As $c \leqslant e$ by $\inf O = e$, this contradicts $h \equiv_e h_2$. □

So there is a natural one-to-one correspondence between the set of basic scattered outcomes Ω_e of e and the set of basic propositional outcomes Π_e of e.[2] As a further consequence of Fact 4.3, in the same way in which Π_e partitions the set H_e of histories containing e, Ω_e partitions the future of possibilities of e:

Fact 4.5. *Let $e \in W$, and let $F_e =_{\mathrm{df}} \{e' \in W \mid e < e'\}$. Then Ω_e is a partition of F_e: (1) the union of all the chains that make up all the basic scattered outcomes of e cover the whole future of possibilities of e, i.e., $\bigcup\bigcup\Omega_e = F_e$, and (2) the basic scattered outcomes of e do not overlap, i.e., for $\hat{O}_1, \hat{O}_2 \in \Omega_e$, if $\hat{O}_1 \neq \hat{O}_2$, then $\hat{O}_1 \cap \hat{O}_2 = \emptyset$.*

[2] It by no means follows that for two different e_1 and e_2, if $H_{\langle\Omega_{e_1}\langle h_1\rangle\rangle} = H_{\langle\Omega_{e_2}\langle h_2\rangle\rangle}$, then $\Omega_{e_1}\langle h_1\rangle = \Omega_{e_2}\langle h_2\rangle$. You must hold e constant.

Proof. If e is a maximal element of W, there is nothing to prove. So we assume that e is not maximal.

(1) "$\subseteq$": Let $e' \in \bigcup\bigcup\Omega_e$, i.e., $e' \in \bigcup\Omega_e\langle h\rangle$ for some $h \in H_e$. As $\Omega_e\langle h\rangle$ is a set of outcome chains, there must be some outcome chain $O \in \Omega_e\langle h\rangle$ for which $e' \in O$. By Def. 4.7, $e < O$, so that $e < e'$, i.e., $e' \in F_e$.

"$\supseteq$": Let $e' \in F_e$, i.e., $e < e'$. The set $\{e, e'\}$ can be extended to a maximal chain l that begins at e and contains e'. As l is directed, there is a history $h \supseteq l$. Now by Fact 3.3, $O =_{\mathrm{df}} l \setminus \{e\}$ is an outcome chain with infimum e, so that by Def. 4.7, $O \in \Omega_e\langle h\rangle$. As $e' \in O$, this means $e' \in \bigcup\Omega_e\langle h\rangle$, i.e., $e' \in \bigcup\bigcup\Omega_e$.

(2) Let $\hat{O}_i = \Omega_e\langle h_i\rangle$ $(i = 1,2)$, and assume that $\hat{O}_1 \neq \hat{O}_2$. By Fact 4.4 and 4.3, this implies $\Pi_e\langle h_1\rangle \neq \Pi_e\langle h_2\rangle$. Assume for reductio that there is some outcome chain O with infimum e for which $O \in \hat{O}_1 \cap \hat{O}_2$, and let h be a history containing O. Then, as in the proof of Fact 4.4, we have $h_1 \equiv_e h \equiv_e h_2$, which contradicts $\Pi_e\langle h_1\rangle \neq \Pi_e\langle h_2\rangle$. □

The results above show that it makes sense to extend Def. 3.9 and to define two varieties of basic transitions, Π_e-based and Ω_e-based, which are equivalent:

Definition 4.8 (Basic transitions in BST$_{92}$)**.** For $e \in h$, $e \rightarrowtail \Omega_e\langle h\rangle$ is a basic transition event, and $e \rightarrowtail \Pi_e\langle h\rangle$ is a basic propositional transition. Both $e \rightarrowtail \Omega_e\langle h\rangle$ and $e \rightarrowtail \Pi_e\langle h\rangle$ may be called *basic transitions.*

Fact 4.6. *Let $e \in W$ and $h \in H_e$. The basic transitions $e \rightarrowtail \Omega_e\langle h\rangle$ and $e \rightarrowtail \Pi_e\langle h\rangle$ are equivalent in the sense of having the same occurrence proposition, i.e.,*

$$H_{e\rightarrowtail\Omega_e\langle h\rangle} = H_{e\rightarrowtail\Pi_e\langle h\rangle}.$$

Proof. Obviously $e \rightarrowtail \Omega_e\langle h\rangle$ and $e \rightarrowtail \Pi_e\langle h\rangle$ have the same initials, and therefore Fact 4.3 implies that

$$(\mathrm{Hist} \setminus H_e) \cup H_{\langle\Omega_e\langle h\rangle\rangle} = (\mathrm{Hist} \setminus H_e) \cup \Pi_e\langle h\rangle. \qquad \square$$

We can extend the outcome selection notation for propositions given a history, $\Pi_e\langle h\rangle$, to point events and to outcome chains: Given some initial e_1, an outcome $H \in \Pi_{e_1}$ of e_1 is not just uniquely determined by some $h \in H_{e_1}$ (which motivates the notation $\Pi_{e_1}\langle h\rangle$), but also by any later event e_2 in the future of possibilities of e_1, or by any outcome chain O for which $e_1 < O$. Therefore, we can introduce the notation "$\Pi_{e_1}\langle e_2\rangle$" and "$\Pi_{e_1}\langle O\rangle$":

Fact 4.7. *(1) Let $e_1 < e_2$. Then there is exactly one basic outcome of e_1 that is compatible with e_2, which we denote by $\Pi_{e_1}\langle e_2\rangle$. (2) Let $e_1 < O$ for an outcome chain O. Then there is exactly one basic outcome of e_1 that is compatible with O, which we denote by $\Pi_{e_1}\langle O\rangle$. (3) Let $e < O$, $h_1 \perp_e h_2$, and $h_2 \in H_{\langle O\rangle}$. Then $h_1 \perp_e H_{\langle O\rangle}$.*

Proof. (1) Let $h \in H_{e_2}$. By the downward closure of histories, $e_1 \in h$, so that h determines the element $\Pi_{e_1}\langle h\rangle$ of the partition Π_{e_1} of H_{e_1}. And for any $h' \in H_{e_2}$, we have $h \equiv_{e_1} h'$, as witnessed by $e_2 \in h \cap h'$. So we can set $\Pi_{e_1}\langle e_2\rangle =_{\mathrm{df}} \Pi_{e_1}\langle h\rangle$.

The proof for (2) is exactly parallel, and is left as Exercise 4.2.

For (3), as $e < O$, for all $h, h' \in H_{\langle O\rangle}$, we have $h \equiv_e h'$. The claim follows from the transitivity of $\equiv_e$ on the set H_e. □

A similar construction for extending the notation $\Omega_{e_1}\langle h\rangle$ for basic scattered outcomes to $\Omega_{e_1}\langle e_2\rangle$ and $\Omega_{e_1}\langle O\rangle$ is left as Exercise 4.3.

In order to extend our results to disjunctive outcomes, we first need to extend our propositional notation somewhat. In analogy with the notation $\breve{\mathbf{O}}$ for disjunctive outcome events, we write $\breve{\mathbf{H}}$ for disjunctive propositional events. Typically, such disjunctive outcomes should have at least two elements. For technical reasons (see Chapter 6.4), it is useful to be more general and allow for one-element disjunctions as well, so that in the following, we only require that the disjunction be non-empty.

Definition 4.9. Let $e \in W$. A *basic disjunctive outcome event of e*, generically written $\breve{\mathbf{O}}$, is any non-empty subset of Ω_e (i.e., a set of some basic scattered outcome events of e). A *basic propositional disjunctive outcome of e*, generically written $\breve{\mathbf{H}}$, is any non-empty subset of Π_e, (i.e., a set of some basic propositional outcomes of e). The occurrence proposition for $\breve{\mathbf{H}}$ is

$$H_{\breve{\mathbf{H}}} =_{\mathrm{df}} \bigcup \breve{\mathbf{H}}.$$

Given this definition, $\breve{\mathbf{H}}$ occurs in precisely those histories in which one of its members (one of the disjuncts) occurs. In full analogy with Def. 4.5, the occurrence proposition for the transition from e to a basic propositional disjunctive outcome $\breve{\mathbf{H}}$ is

$$H_{e \rightarrowtail \breve{\mathbf{H}}} =_{\text{df}} (\text{Hist} \setminus H_e) \cup H_{\breve{\mathbf{H}}}.$$

If e is an indeterministic event, both Π_e and Ω_e are disjunctive outcomes of e, but their occurrence propositions exhaust all of H_e, and, accordingly, the occurrence propositions of the respective transitions from e are universal:

Fact 4.8. *Let $e \in W$. Then $H_{\Pi_e} = H_{\langle \Omega_e \rangle} = H_e$, and $H_{e \rightarrowtail \Pi_e} = H_{e \rightarrowtail \Omega_e} = \text{Hist}$.*

Proof. Let $h \in H_e$, then $h \in \Pi_e \langle h \rangle$, and so $h \in H_{\Pi_e}$. In the other direction, let $h \in H_{\Pi_e}$, so that $h \in \Pi_e \langle h' \rangle$ for some $h' \in H_e$, which implies $h \in H_e$. The claim for $H_{\langle \Omega_e \rangle}$ follows by Fact 4.3, and the claim about the occurrence propositions of the respective transitions is an immediate consequence of the definitions. □

In a somewhat idealized fashion, we can represent the die rolling example from the end of Chapter 4.1 as follows: the rolling of the die corresponds to a point event e with six immediate basic scattered outcome events, $\hat{O}_i = \Omega_e \langle h_i \rangle = \boxed{i}$, $i = 1, \ldots, 6$. Exhaustiveness then means that $\Omega_e = \{\hat{O}_i \mid i = 1, \ldots, 6\}$. In this representation, it is immediately clear that $e \rightarrowtail \Omega_e$ is a deterministic transition to a disjunctive outcome, as $H_e = H_{\langle \Omega_e \rangle}$.

By Fact 4.3, the equivalence between basic disjunctive and basic propositional disjunctive outcomes also holds in non-extremal cases. We therefore have two equivalent representations of basic outcomes, single or disjunctive, and consequently, of basic transitions: in terms of propositions or sets of propositions ($\Pi_e \langle h \rangle$ or $\breve{\mathbf{H}}$) and in terms of scattered outcomes or disjunctive outcomes ($\Omega_e \langle h \rangle$ or $\breve{\mathbf{O}}$). We will use these two equivalent representations almost interchangeably, mostly giving preference to the propositional version in proofs and theorems. The chief place in which we rely on the interchange is in the idea of a *causa causans*, which will be discussed extensively in Chapter 6.

In Chapter 4.3, we will discuss sets of transitions. With a view to that discussion, a note of caution may be useful: transitions to disjunctive outcomes are *not* equivalent to sets of transitions to the disjuncts. Take, for example, some indeterministic $e \in W$ and two $H_1, H_2 \in \Pi_e$, $H_1 \neq H_2$. Then $e \rightarrowtail \{H_1, H_2\}$ is a transition to a basic disjunctive outcome of e, and that outcome occurs on $H_{\{H_1,H_2\}} = H_1 \cup H_2$. On the other hand, $\{e \rightarrowtail H_1, e \rightarrowtail H_2\}$ is a set of two transitions that are incompatible local alternatives and which, therefore, cannot occur together.

4.2.2 A note on basic transitions in BST_{NF}

We end this section with an observation concerning basic transitions in the BST_{NF} framework. Recall that by Def. 3.12, $H_{\ddot{e}} = \bigcup_{e \in \ddot{e}} H_e$. Read propositionally, the occurrence proposition for a choice set $\ddot{e}$ is therefore $H_{\ddot{e}} = H_{[\mathscr{P}_e]}$ (see Theorem 3.1). Since in a BST_{NF} structure there is a minimal element in the difference of any two histories, an element of a choice set uniquely singles out a choice. It is therefore tempting to identify a BST_{NF} basic transition simply with an element of a choice set, $c \in \ddot{c}$, where trivial choice sets $\ddot{c} = \{c\}$ would obviously be allowed as well and would give rise to trivial basic transitions. Thus, in this proposal, any point event of *Our World* counts as a transition. Care is, however, needed when passing to propositional basic outcomes, since the set H_c of histories by itself carries no information about the relation of c to other point events of the structure. That is, there will generally also be point events $e \neq c$ for which $H_c = H_e$. To avoid ambiguity, we could associate the history set H_e with the point event e, and take the propositional basic transition corresponding to e to be the pair of e and H_e, which we could write $e \rightarrowtail H_e$.

That is still troublesome, however, as the occurrence proposition of $e \rightarrowtail H_e$ is universal. Recall that via Def. 4.5, we opted for an implication-like reading of occurrence propositions for transitions. But if e occurs, then H_e occurs as well. So $H_{e \rightarrowtail H_e} = \text{Hist}$, providing no information at all.

To resolve this difficulty, we take as the initial not e, but the choice set $\ddot{e}$, which is still weakly before e in the sense that $e \in \ddot{e}$. We define two kinds of BST_{NF} basic transitions in analogy to the BST_{92} case, as follows:

Definition 4.10 (Basic transitions in BST_{NF}). For $\langle W, < \rangle$ a BST_{NF} structure and for $e \in W$, any pair $\langle \ddot{e}, e \rangle$ with $e \in \ddot{e}$, written as $\ddot{e} \rightarrowtail e$, is a *basic transition event*. The pair $\langle \ddot{e}, H_e \rangle$, written as $\ddot{e} \rightarrowtail H_e$, is a *basic propositional transition*. Both $\ddot{e} \rightarrowtail e$ and $\ddot{e} \rightarrowtail H_e$ are called *basic transitions* in BST_{NF}.

Basic transition events are ordered by $(\ddot{e}_1 \rightarrowtail e_1) \prec (\ddot{e}_2 \rightarrowtail e_2)$ iff $e_1 < e_2$, and basic propositional transitions are ordered by $(\ddot{e}_1 \rightarrowtail H_{e_1}) \prec (\ddot{e}_2 \rightarrowtail H_{e_2})$ iff $e_1 < e_2$.

The occurrence proposition for a basic transition $\ddot{e} \rightarrowtail e$ (as well as for a basic propositional transition $\ddot{e} \rightarrowtail H_e$) is $(\text{Hist} \setminus H_{\ddot{e}}) \cup H_e$, which provides an analogon of Fact 4.3 about the interchangeability of basic transition events and basic propositional transitions in BST_{NF}.

Note that unless $\ddot{e} = \{e\}$ (i.e., $\ddot{e} \rightarrowtail e$ is deterministic), the occurrence proposition of $\ddot{e} \rightarrowtail e$, $(\text{Hist} \setminus H_{\ddot{e}}) \cup H_e$, is contingent, not universal. And clearly, the ordering $\prec$ of basic transitions is a strict partial ordering.

It turns out that a fact analogous to Fact 4.7 also holds for basic transitions in BST_{NF}:

Fact 4.9. *(1) Let $e_1 < e_2$. Then there is exactly one basic outcome of $\ddot{e}_1$ that is compatible with e_2. We write the corresponding basic transition as $\ddot{e}_1 \rightarrowtail \Pi_{\ddot{e}_1}\langle e_2 \rangle$. (2) Let $e_1 < O$ for an outcome chain O. Then there is exactly one basic outcome of $\ddot{e}_1$ that is compatible with O, which we denote by $\Pi_{\ddot{e}_1}\langle O \rangle$. (3) Let $e < O$, $h_1 \perp_{\ddot{e}} h_2$, and $h_2 \in H_{\langle O \rangle}$. Then for every $h \in H_{\langle O \rangle}$: $h_1 \perp_{\ddot{e}} h$.*

Proof. (1) Since $H_{e_2} \cap H_{e_1} = H_{e_2} \neq \emptyset$, e_2 and $\ddot{e}_1 \rightarrowtail e_1$ are compatible: their occurrence propositions intersect non-emptily. Since distinct elements of $\ddot{e}_1$ must be incompatible and histories are downward closed, no other basic transition from $\ddot{e}_1$ is compatible with e_2. We have $\Pi_{\ddot{e}_1}\langle e_2 \rangle = H_{e_1}$. (2) and (3) The arguments for these claims are analogous to the one just given. □

4.3 Sets of basic transitions

In later chapters we will continue to employ a dual view of transitions as either BST events of a spatio-temporal-modal kind, as introduced via Def. 4.4, or as proposition-like objects as discussed in Chapter 4.2 (see Def. 4.8). In the latter approach, the notion of a basic transition is best generalized via *sets of basic transitions*. We introduce a number of relevant notions here, retaining the BST_{92} framework for concreteness. Our definitions easily transfer to the BST_{NF} framework.

As stated in Chapter 3.4.4 (Def. 3.9), there are two kinds of basic transitions, deterministic and indeterministic ones. Deterministic basic transitions are trivial, from a deterministic point event e (a point event that is not a choice point) to the only immediate outcome of e; indeterministic transitions are from an indeterministic point event (a choice point) e to one of the several immediate outcomes of e. For some applications, it is useful to look at all transitions, deterministic and indeterministic alike—and this approach was taken to define $\mathrm{BST_{NF}}$ in Chapter 3.5. In many other contexts, however, it is most useful to disregard the trivial deterministic transitions, and to focus exclusively on indeterministic ones. This is the route we will follow in our discussion of modal funny business (Chapter 5), of causation (Chapter 6), of probabilities (Chapter 7), and in many applications. Accordingly, we develop our notation here with a view to later uses of sets of indeterministic transitions.

Working in terms of propositions, an indeterministic basic transition is of the form

$$\tau = e \rightarrowtail H, \quad \text{where } e \in W, H \in \Pi_e, \text{ and } \Pi_e \neq \{H_e\},$$

where the last clause implies that e, and thereby τ, is indeed indeterministic. In order to be able to identify initials and outcomes easily, we often write sets of such non-trivial basic transitions as

$$T = \{\tau_\gamma = e_\gamma \rightarrowtail H_\gamma \mid \gamma \in \Gamma\}, \quad \text{where } \Gamma \text{ is some index set.}$$

Here are some pertinent definitions.

Definition 4.11 (Notation for sets of transitions)**.** Let $\langle W, < \rangle$ be a $\mathrm{BST_{92}}$ structure.

- We denote the set of all basic indeterministic transitions in W by $\mathrm{TR}(W)$, as already announced on p. 59.
- For $h \in \mathrm{Hist}(W)$, we write $\mathrm{TR}(h)$ for those basic indeterministic transitions that occur non-vacuously in h. That is, we have

$$\mathrm{TR}(h) =_{\mathrm{df}} \{\tau = e \rightarrowtail H \in \mathrm{TR}(W) \mid h \in H\}.$$

- We write $H(T)$ for the set of histories admitted by the outcomes of a set of transitions T. That is, for $T = \{\tau_\gamma = e_\gamma \rightarrowtail H_\gamma \mid \gamma \in \Gamma\}$ (where Γ is some index set), we set

$$H(T) =_{\mathrm{df}} \bigcap_{\gamma \in \Gamma} H_\gamma.$$

We extend this notation to single transitions, writing $H(\tau)$ in place of $H(\{\tau\})$. That is, for $\tau = e \rightarrowtail H$, we have $H(\tau) = H$.

Given that $H(T)$ is a set of histories (i.e., a proposition), the notion of consistency of Def. 4.1 naturally applies. We extend that notion to sets of transitions in the obvious way:

Definition 4.12 (Consistency of a set of transitions)**.** We call a set of transitions T *consistent* iff $H(T)$ is consistent (i.e., iff $H(T) \neq \emptyset$). A consistent set of transitions thus admits at least one history. If $H(T) = \emptyset$, we call T *inconsistent.*

The above notation, as well as Def. 4.12, naturally extends to sets of transitions in $\mathrm{BST_{NF}}$.

4.4 Topological aspects of BST

In the following section we describe the natural topology for common BST structures, and we comment on some of the topological features of $\mathrm{BST_{92}}$ and of $\mathrm{BST_{NF}}$. To recall, a topology on a set X is given by specifying a family of subsets of X, known as "open sets", that is closed under finite intersection and arbitrary union, and which contains X as well as the empty set. (See, e.g., Munkres, 2000, for an overview.)

4.4.1 General idea of the diamond topology

BST admits a natural topology, introduced by Paul Bartha,[3] which we call the diamond topology. The topology is defined either for W, the base set of a BST structure, or for a given history $h \in \mathrm{Hist}(W)$. In the definitions below, $MC(e)$ stands for the set of maximal chains in W that contain e, whereas $MC_h(e)$ stands for the set of maximal chains in history h that contain e.

Definition 4.13 (Diamond topology $\mathscr{T}$ on W)**.** Z is an open subset of W, $Z \in \mathscr{T}$, iff $Z = W$ or for every $e \in Z$ and for every $t \in MC(e)$, there are $e_1, e_2 \in t$ such that $e_1 < e < e_2$ and the diamond $D_{e_1,e_2} \subseteq Z$, where

$$D_{e_1,e_2} =_{\mathrm{df}} \{e' \in W \mid e_1 \leqslant e' \leqslant e_2\}.^4$$

[3] Cf. note 26 of Belnap (2003b), the "postprint" of the original BST paper (Belnap, 1992).

[4] Note that the diamonds themselves are *not* open sets. It is possible to introduce borderless diamonds, which are in fact open sets in the topology defined here, but they are harder to work

Definition 4.14 (History-relative diamond topologies $\mathscr{T}_h$ on W). For $h \in$ Hist, Z is an open subset of h, $Z \in \mathscr{T}_h$, iff $Z = h$ or for every $e \in Z$ and for every $t \in MC_h(e)$, there are $e_1, e_2 \in t$ such that $e_1 < e < e_2$ and the diamond $D_{e_1,e_2} \subseteq Z$.

It is not too difficult to check that $\mathscr{T}$ and $\mathscr{T}_h$ are indeed topologies; that is, both the empty set and the base set (W or h, respectively) are open, the intersection of two open sets is open, and the union of countably many open sets is open (see Exercise 4.4). The claim of the naturalness of the diamond topology is based on the observation that for an important class of BST structures, this topology coincides with the standard open-ball topology on $\mathscr{T}(\mathbb{R}^n)$ (to be described in Def. 4.16 on p. 95) and that the notion of convergence it induces coincides with the order-theoretic notions of infima and suprema.[5] As one can see from the definition, the history-relative topologies are the so-called subspace topologies induced by the diamond topology on W, by taking a history as a subspace of W. This means that $A \in \mathscr{T}_h$ iff there is $A' \in \mathscr{T}$ such that $A = A' \cap h$.

In BST_{92}, the global topology and the history-relative topologies have different features. As we will show, this fact reflects a problem with local Euclidicity.

4.4.2 Properties of the diamond topology in BST_{92}

We review here some facts about the diamond topology in BST_{92}. The first observation is that unless $\langle W, < \rangle$ is a one-history structure, the history-relative and the global topologies disagree with respect to a topological separation property called the Hausdorff property. That property is defined as follows:

Definition 4.15 (Hausdorff property). A topological space $\langle X, \mathscr{T}(X) \rangle$ is *Hausdorff* iff for any distinct $x, y \in X$ there are disjunct open neighborhoods of x and of y (i.e., there are $O_x, O_y \in \mathscr{T}(X)$ for which $O_x \cap O_y = \emptyset$).

Putting aside BST_{92} structures that are pathological in the sense that they prohibit the construction of light-cones, it can be proved that the

with technically (Placek et al., 2014, Def. 23). Simply removing e_1 and e_2 from the definition does not help, as the borders of the respective space-time region will then be retained.

[5] For a discussion of the naturalness of the diamond topology, see Placek et al. (2014, §6). See also Fact 9.13 in Chapter 9.

history-relative topologies $\mathscr{T}_h$ on a BST_{92} structure have the Hausdorff property.[6] This fact stands in sharp contrast with the properties of the global diamond topology $\mathscr{T}$: if a BST_{92} structure has more than one history, its global topology is non-Hausdorff (again, ignoring pathological structures); see Figure 3.1. In fact, the non-Hausdorffness of the global topology is related to the existence of upper-bounded chains that have more than one history-relative supremum. As one might expect, a pair of distinct history-relative suprema of a chain provides a witness for non-Hausdorffness: if any two open sets in $\mathscr{T}$ each contain a distinct supremum, they must overlap because they share some final segment of the chain in question.

These results about Hausdorffness in the diamond topology in BST_{92} appear encouraging as far as the relation to physics in concerned. In physics it is standardly required that individual space-times be Hausdorff (see, e.g., Wald, 1984, p. 12). As individual space-times are represented by single histories in a BST_{92} structure, we take it that BST_{92} structures are not in conflict with the Hausdorffness requirement of space-time physics. The non-Hausdorffness of the global topology of a BST_{92} structure simply reflects the fact that such a structure brings together more than one history (space-time), explicitly representing a number of alternative spatio-temporal developments.

There is, however, another difference between the history-relative and the global topologies in BST_{92} that is more problematic: again, putting aside trivial one-history structures, a history h is not open in the global topology $\mathscr{T}$, whereas it is open by definition in its own history-relative topology $\mathscr{T}_h$. Generally, if an open set A from a history-relative topology $\mathscr{T}_h$ contains a choice point, then $A \notin \mathscr{T}$:

Fact 4.10. *Let $\langle W, < \rangle$, $h \in \text{Hist}(W)$ and $A \in \mathscr{T}_h$. Then, if A contains a choice point, $A \notin \mathscr{T}$. This implies that unless* Hist *has only one member, for any $h \in \text{Hist}$, $h \in \mathscr{T}_h$, but $h \notin \mathscr{T}$.*

Proof. Let $A \in \mathscr{T}_h$, and let $e \in A$ be a choice point, so that $h \perp_e h'$ for some $h' \in H_e$. Thus, e is not maximal in W, and hence, not maximal in h' (by Fact 2.1(9)). Now pick a maximal chain $t \in MC_{h'}(e)$, so that $e \in t$ and $t \subseteq h'$. By Fact 3.4, $t \in MC(e)$. As e is not maximal in h' and t is a maximal chain in h', t extends above e in h'. For A to be open in $\mathscr{T}$, by Def. 4.13 there needs to be $e_2 \in t$, $e < e_2$, such that $e_2 \in A \subseteq h$. But, since e is a choice point for h

[6] For the proofs, see Placek et al. (2014). The mentioned pathological BST_{92} structures violate one of the conditions C1–C4 discussed in that paper.

and h', which is maximal in the intersection of h and h', there is no such e_2. Thus, $A \notin \mathscr{T}$. Note that in a BST_{92} structure with more than one history, by PCP_{92}, any history contains at least one choice point. □

There is thus a systematic discrepancy between the global and the history-relative notions of openness. This result spells trouble for an important topological property called local Euclidicity. Technically, this property is defined as follows:

Definition 4.16 (Local Euclidicity)**.** A topological space $\langle X, \mathscr{T}(X)\rangle$ is *locally Euclidean of dimension n* iff for every $x \in X$ there is an open neighborhood $O_x \in \mathscr{T}(X)$ and a homeomorphism φ_x that maps O_x onto an open set $R_x \in \mathscr{T}(\mathbb{R}^n)$. Here, $\mathscr{T}(\mathbb{R}^n)$ is the standard so-called open ball topology of $\mathbb{R}^n$, which has as a basis open balls of the form $B(x,\varepsilon) =_{\mathrm{df}} \{y \in \mathbb{R}^n \mid d(x,y) < \varepsilon\}$ according to the standard Euclidean distance d.

In Chapter 3.6.1 we already noted that local Euclidicity is standardly presupposed, often without mentioning the condition by name, when the notion of a space-time manifold is introduced. On such a manifold, local coordinates are defined via so-called charts (see, e.g., Wald, 1984, pp. 12f.), and the existence of charts is guaranteed by local Euclidicity: at each point of the manifold, one can find a neighborhood that is homeomorphic to some open set of $\mathbb{R}^n$. In this way (additionally assuming some compatibility requirements between charts), coordinates can be introduced. If a topological space is not locally Euclidean, it is not possible to assign coordinates in this way.

We also noted that given the frugality of the BST_{92} postulates, BST_{92} structures can differ widely. It would not be realistic to hope that their global topology will always be locally Euclidean—but one can reasonably require that local Euclidicity should transfer from the individual histories to the whole global structure. More precisely, if for each history h of $\langle W, <\rangle$, $\mathscr{T}_h$ is locally Euclidean, then the global topology $\mathscr{T}$ should be locally Euclidean as well. If we have some collection of physically reasonable space-times, each with an assignment of coordinates, then a BST analysis of indeterminism should not destroy the coordinate assignment. Unfortunately, in BST_{92} local Euclidicity is not preserved as one moves from the history-relative topologies to the global topology. As a case in point, in Chapter 3.6.1 we already discussed the simple example from Figure 3.1 (p. 44).

4.4.3 The diamond topology in BST_{NF}

The situation of BST_{92} is unfortunate with respect to local Euclidicity. It would be better if we could have a BST framework for local indeterminism that preserves local Euclidicity: if each history (space-time) is locally Euclidean of dimension n, then the global topology should be locally Euclidean of dimension n as well. As we saw in §4.4.2, the diamond topology on BST_{92} structures does not preserve local Euclidicity when moving from the history-relative topologies to the global topology. In contrast, we can prove that the diamond topology on BST_{NF} structures preserves local Euclidicity. Working toward Theorem 4.1 about the preservation of local Euclidicity, we first need an auxiliary Lemma, which is also of interest on its own. Recall the disturbing feature of BST_{92} discussed as Fact 4.10 in Chapter 4.4.2: a set that is open in a history-relative topology need not be open in the corresponding global topology. The Lemma below states that this problem cannot occur in the diamond topology on BST_{NF} structures:

Lemma 4.1. *Let a BST_{NF} structure $\langle W, < \rangle$ be given, let $h \in \mathrm{Hist}(W)$, and let $Z \subseteq W$ be such that $Z \in \mathscr{T}_h$, i.e., Z is an open set with respect to the history-relative topology $\mathscr{T}_h$. Then $Z \in \mathscr{T}$, i.e., Z is also open with respect to the global topology on W.*

Proof. Let $Z \in \mathscr{T}_h$ for some $h \in$ Hist. Let $e \in Z$, and let $t \in MC(e)$. In order to establish the openness of Z with respect to $\mathscr{T}$, we need to show that there is an e-centered diamond with vertices on t wholly contained in Z. The openness of Z with respect to $\mathscr{T}_h$ gives us such a diamond for any $t_h \in MC_h(e)$, but not necessarily for our given $t \in MC(e)$.

We show that the given t has a segment both below and above e that is contained in some $t_h \in MC_h(e)$. The segment below e is contained in h by downward closure of histories. For the segment above e, we proceed in two steps. First, we claim that t contains some $e' \in h$ for which $e' > e$. Assume otherwise, i.e., the chain $t^+ =_{\mathrm{df}} \{e^* \in t \mid e^* > e\}$ contains no element of h. Note that by construction, $\inf t^+ = e$. As t^+ is a chain, it is directed, and thus wholly contained in some history h_2. Pulling these facts together, $t^+ \subseteq h_2 \setminus h$, and by the maximality of t and the construction of t^+, we have that t^+ is a maximal chain in $h_2 \setminus h$. The PCP_{NF} gives us a choice set $\ddot{c}$ such that (†) $h \perp_{\ddot{c}} h_2$, and for the unique $c' \in \ddot{c} \cap h_2$, we have $c' \leqslant t^+$. We observe next that from the fact that t^+ is a maximal chain in $h_2 \setminus h$, it follows that $c' = \inf t^+$.

Otherwise, for $i = \inf t^+$ we would have $c' < i \leqslant t^+$. By (†) we have $c' \notin h$, so $\{c'\} \cup t^+ \subseteq h_2 \setminus h$. As this chain extends t^+, it contradicts the maximality of t^+ in $h_2 \setminus h$. Thus, $c' = \inf t^+$, whence $c' = e$. It follows that $e \in h_2 \setminus h$, which contradicts our initial assumption that $e \in h$. So indeed, t contains some $e' \in h$ for which $e' > e$.

Second, we construct t_h by starting with an initial segment of the given t, as follows: Let $t^- =_{\text{df}} \{e^* \in t \mid e^* \leqslant e'\}$; we have $t^- \subseteq h$ and $e \in t^-$. By the Hausdorff maximal principle we can extend t^- with elements of h to form a chain t_h that is maximal in h, so that $t_h \in MC_h(e)$. The chains t and t_h share the initial segment t^-. We can now invoke the openness of Z with respect to $\mathscr{T}_h$ for e and t_h, which gives us a diamond $D_{e_1^h, e_2^h} \subseteq Z$ for which $e_1^h, e_2^h \in t_h$ and $e_1^h < e < e_2^h$. We set $e_1 =_{\text{df}} e_1^h$ and $e_2 =_{\text{df}} \min\{e', e_2^h\}$. We thus have $e_1 < e < e_2$ with $e_1 = e_1^h \in t$, and also $e_2 \in t$ because $e' \in t$. And as the diamond $D_{e_1,e_2} \subseteq D_{e_1^h, e_2^h}$, we have $D_{e_1,e_2} \subseteq Z$. So we have found the witnessing e-centered diamond with vertices on t, which establishes the openness of Z with respect to $\mathscr{T}$. □

The above Lemma immediately implies a fact about histories in BST_{NF} that shows that the consequences of Fact 4.10 are avoided in BST_{NF}:

Fact 4.11. *Let $\langle W, < \rangle$ be a BST_{NF} structures. Then for every $h \in \text{Hist}(W)$: $h \in \mathscr{T}$.*

Proof. By Lemma 4.1, since $h \in \mathscr{T}_h$. □

It is easy to see that the converse of Lemma 4.1 holds as well, that is, if $A \in \mathscr{T}$, then $A \cap h \in \mathscr{T}_h$ (see Exercise 4.5). More interestingly, $\mathscr{T}$ has a handy basis, the elements of which are subsets of histories:

Lemma 4.2. *Let $\langle W, < \rangle$ be a BST_{NF} structure with diamond topology $\mathscr{T}$. Then the set B, defined as*

$$B =_{\text{df}} \cup_{h \in \text{Hist}(W)} \{O \cap h \mid O \in \mathscr{T}\},$$

is a basis of $\mathscr{T}$.

Proof. We have to show two things: (1) Any element $O \in \mathscr{T}$ is a union of elements of B, and (2) the elements $b \in B$ are in fact open in $\mathscr{T}$.

For (1), let $O \in \mathscr{T}$. Then, since $\cup_{h \in \text{Hist}} h = W$, we have $O = \cup_{h \in \text{Hist}} O \cap h$, and any $O \cap h \in B$ by construction of B. For (2), let $b \in B$ be given, so that there is $O \in \mathscr{T}$ and $h \in \text{Hist}(W)$ for which $b = O \cap h$. By Fact 4.11, $h \in \mathscr{T}$, so that the openness of b follows by the finite intersection property of $\mathscr{T}$. $\square$

With Lemma 4.1 in hand, we can prove the sought-for theorem about the transfer of local Euclidicity from histories to the whole structure in BST_{NF}:

Theorem 4.1. *Let $\langle W, < \rangle$ be a BST_{NF} structure. If there is an $n \in \mathbb{N}$ such that for every $h \in \text{Hist}(W)$, the topological space $\langle h, \mathscr{T}_h \rangle$ is locally Euclidean of dimension n, then the topological space $\langle W, \mathscr{T} \rangle$ is also locally Euclidean of dimension n.*

Proof. We need to show that each $e \in W$ has a neighborhood $O_e \in \mathscr{T}$ that is mapped by some homeomorphism φ_e to an open set $R_e \in \mathscr{T}(\mathbb{R}^n)$. Let $e \in W$, and pick some $h \in H_e$. Since h is locally Euclidean with respect to $\mathscr{T}_h$, there is a $\mathscr{T}_h$-open neighborhood $O_e^h \subseteq h$ of e, an open set of $\mathbb{R}^n$, $R_e^h \in \mathscr{T}(\mathbb{R}^n)$, and a homeomorphism φ_e^h such that $\varphi_e^h[O_e^h] = R_e^h$. By Lemma 4.1, from $O_e^h \in \mathscr{T}_h$ it follows that $O_e^h \in \mathscr{T}$. We thus let $O_e =_{\text{df}} O_e^h$, $R_e =_{\text{df}} R_e^h$, and we can use $\varphi_e =_{\text{df}} \varphi_e^h$ as our homeomorphism between the $\mathscr{T}$-open neighborhood O_e of e and the open set $R_e \in \mathscr{T}(\mathbb{R}^n)$. $\square$

BST_{NF} thus vindicates the idea that if one starts with locally Euclidean histories (space-times) that allow for the assignment of spatio-temporal coordinates, one does not destroy that feature by analyzing indeterminism within the framework of Branching Space-Times.

4.5 A note on branching-style semantics

As we have shown in Chapters 4.1 and 4.2, BST—both in the form of BST_{92} and in the form of BST_{NF}—can be developed to provide a theory of events as well as a theory of propositions, and both approaches will be used in later chapters. When we introduce the notion of a cause-like locus for an outcome (Chapter 5.3, Def. 5.10) and, more generally, for a transition (Chapter 6.3.1, Def. 6.1), we will describe a cause-like locus as a risky juncture for the occurrence of some event. We will motivate the notion of a cause-like locus in part through some claims about the truth at such a locus of certain sentences with temporal and modal operators. We will say, for instance, that a cause-like locus for an outcome O is the last event at which both the sentences "it is possible that *O* will occur" and "it is possible that *O* will not occur" are true. These claims need not be left on a merely intuitive level, as branching structures can be used to provide formal semantics for languages with temporal and modal operators. Here we recall a few definitions and facts pertaining to how semantic models on branching structures can be constructed and how temporal-modal sentences are evaluated in the resulting models. Where the difference matters, we stick to BST_{92} for simplicity's sake.

Historically, branching structures for the combination of temporal and modal information and for the formal analysis of indeterminism were first discussed in a now famous exchange between Saul Kripke and Arthur Prior in the late 1950s (see Ploug and Øhrstrøm, 2012). Roughly, such branching structures, which became known under the name 'Branching Time' (BT),[7] use a tree-like ordering to depict the difference between an open future of possibilities and a fixed past: such an ordering is backwards-linear, thus allowing for branching toward the future, but not toward the past. Branching Time provides the formal basis for many applications in computer science, in logic, and in philosophy, including the *stit* ('seeing to it that') formal theory of agency in BT (Belnap et al., 2001). In a BT structure, a history is a maximal chain (a maximal linear subset). There is a direct connection to the notion of a history as a maximal directed set in BST; see Belnap (2012). In fact, BT structures are BST structures of a particularly simple kind, namely, BST structures without SLR elements.

It is not easy to fulfill all our intuitive requirements for the notion of an open future as incorporated in natural language expressions, let alone for the subtle natural-language interaction between tenses and modals. Faced with these problems, Prior developed two different BT-based approaches to the formal semantics for a temporal-modal language, which he called 'Peircean' and 'Ockhamist' (Prior, 1967, pp. 126ff.). It is generally acknowledged that the Peircean approach, which is less expressive than the Ockhamist one, faces such serious difficulties that the Ockhamist approach is usually taken as default. That approach was formally precisified by Thomason (1970). For a comprehensive introduction to Ockhamist semantics for BT, see Belnap

[7] As already mentioned in note 3 on p. 27, the terminology is unfortunate, suggesting perhaps that time itself is branching, while the theory clearly pictures branching *histories* before the background of a linear order of temporal instants (see also Belnap et al., 2001, p. 29).

et al. (2001, Ch. 8). In what follows, we will stick to the Ockhamist approach.[8] The Ockhamist language $\mathscr{L}$ based on BT has temporal operators for the past ("it was the case that", P) and for the future ("it will be the case that", F) as well as the dual modal operators of settledness ("it is settled that", Sett), which is sometimes also called "historical necessity", and real possibility ("it is really possible that", Poss).

Note that in generalizing from BT to BST, the addition of space-like related events in BST permits the introduction of additional spatio-temporal operators. It is challenging to work out how these new operators should be defined and especially how they should interact with the mentioned temporal and modal operators. This is a large topic that will not be discussed in this book.[9] Our explicit motivational claims that refer to semantics concern only the combination of tenses and possibilities, and therefore we focus on the BT-based Ockhamist language only. That is, we work with BST structures, but we assume that the formal language $\mathscr{L}$ that we are dealing with contains only the usual propositional connectives (written $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$) and the temporal and modal operators mentioned above, P, F, Sett, and Poss. In later applications we will need to consider sentences of the form "At st-location x it is φ", like "The value of electromagnetic field at x is such-and-such." The truth conditions for such sentences (i.e., with At_x as the main operator) can only be formulated with respect to a BST model with set S of spatio-temporal locations (see Definition 2.9), so that $x \in S$.

The most salient feature of Ockhamist logic is that sentences are evaluated as true or false at an index of evaluation that specifies an event and a history containing that event. To have a handy notation, we will write e/h to stand for a pair of exactly this kind, i.e., $e \in W$, $h \in \text{Hist}(W)$, and $e \in h$. A semantic model $\mathscr{M}$ based on a BST structure $\mathscr{W} = \langle W, < \rangle$ for our language $\mathscr{L}$ is a pair $\mathscr{M} = \langle \mathscr{W}, \Psi \rangle$, where Ψ is an interpretation function from the set *Sent* of sentences of $\mathscr{L}$ to the set of sets of indexes of evaluation, i.e., $\Psi : Sent \rightarrow \wp(E/\text{Hist})$, where $E/\text{Hist} =_{\text{df}} \{e/h \mid e \in W \wedge h \in H_e\}$.

[8] Recently, a more general framework for BT semantics, the so-called transition semantics, has been proposed by Rumberg (2016a), building upon, but going far beyond earlier work of Placek (2011) and Müller (2014). For a more thorough investigation of formal semantics based on BST, the transition framework would be the ideal starting place, and we strongly encourage its use. Here we only provide a brief introduction of simple BT semantics for purely motivational purposes, so that a detailed introduction to transition semantics seems unwarranted at this point.

[9] For an approach to alternative space-times that centers on operators, see Strobach (2007).

The interpretation function Ψ is required to satisfy the following semantic clauses, where $e/h \models \varphi$ means that $e/h \in \Psi(\varphi)$, to be read as "φ is true at e/h".

$$e/h \models \varphi \text{ iff } e/h \in \Psi(\varphi), \text{ for } \varphi \text{ an atomic sentence of } \mathscr{L};$$
$$e/h \models \neg\varphi \text{ iff it is not the case that } e/h \models \varphi;$$
$$e/h \models (\varphi \vee \Psi) \text{ iff } e/h \models \varphi \text{ or } e/h \models \psi$$
(and similarly for the other propositional connectives);
$$e/h \models \mathrm{F} : \varphi \text{ iff for some } e' \in h \text{ such that } e < e' : e'/h \models \varphi;$$
$$e/h \models \mathrm{P} : \varphi \text{ iff for some } e' \in h \text{ such that } e' < e : e'/h \models \varphi;$$
$$e/h \models \mathrm{Poss} : \varphi \text{ iff for some } h' \in H_e : e/h' \models \varphi;$$
$$e/h \models \mathrm{Sett} : \varphi \text{ iff for all } h' \in H_e : e/h' \models \varphi;$$
$$e/h \models At_x : \varphi \text{ iff } \exists e' : e' \in h \cap x \wedge e'/h \models \varphi, \text{ where } x \in S.$$

Note that, generally, the truth of a sentence depends on both parameters, the event e and the history $h \in H_e$. However, since in the clauses for modal operators one quantifies over histories, a sentence beginning with a modal operator is evaluated the same on any history $h \in H_e$, so that we may set:

$$e \models \mathrm{Sett} : \varphi \Leftrightarrow_{\mathrm{df}} \exists h \in H_e \; [e/h \models \mathrm{Sett} : \varphi];$$
$$e \models \mathrm{Poss} : \varphi \Leftrightarrow_{\mathrm{df}} \exists h \in H_e \; [e/h \models \mathrm{Poss} : \varphi].$$

Having provided the background for the semantics, let us return to the motivational use of the semantics in the definition of a cause-like locus later on. Let $e \in W$, and let O be an outcome chain for which $e < O$. We need to explain why e appears to be decisive for the occurrence of O, given that at e, all histories h on which O remains possible ($h \in H_{\langle O \rangle}$) split from some history h' on which O does not occur. Given such e and O, by the clauses above, we have both

$$e \models \mathrm{Poss} : F : (O \text{ is occurring}) \quad \text{and}$$
$$e \models \mathrm{Poss} : \neg F : (O \text{ is occurring}),$$

where the proposition "O is occurring" is formally represented as $H_{\langle O \rangle}$. Furthermore, these two sentences are true at any e' below e. In contrast, for any $e' > e$, we have:

$$e' \models \text{Sett} : \neg F : (O \text{ is occurring}) \quad \text{or}$$
$$e' \models \text{Sett} : F : (O \text{ is occurring}).$$

This justifies our later claim that a cause-like locus e for O such that $e < O$ is decisive for the occurrence of O in the following sense: e is the last event at which both the future occurrence as well as the future non-occurrence of O is possible. At each event in the future of possibilities of e, depending on its location, it is either settled that O will occur, or it is settled that O will not occur.

4.6 Exercises to Chapter 4

Exercise 4.1. Prove that $H_{[I]} =_{\text{df}} \{h \in \text{Hist} \mid I \subseteq h\} = \bigcap_{e \in I} H_e$.

Exercise 4.2. Provide an explicit proof of Fact 4.7(2).

Exercise 4.3. Prove a variant of Fact 4.7 for basic scattered outcomes; that is, show that given $e_1 < e_2$ [$e_1 < O$], there is exactly one basic scattered outcome of e_1 that is compatible with e_2 [with O], which we therefore denote by $\Omega_{e_1}\langle e_2\rangle$ [$\Omega_{e_1}\langle O\rangle$].

Exercise 4.4. Prove that the diamond topology of Def. 4.13 and the history-relative diamond topologies of Def. 4.14 are indeed topologies for both BST_{92} and BST_{NF}, i.e., prove that both (1) the base set (W or h, respectively) and (2) the empty set are open, (3) arbitrary unions of open sets are open, and (4) finite intersections of open sets are open.

Hint: For intersection (4), identify the relevant maximal chains and appropriate limits on them. (A full proof is given in Appendix B.4.)

Exercise 4.5. Let $\langle W, <\rangle$ be a BST_{92} or BST_{NF} structure. Then for every $h \in \text{Hist}(W)$, if $A \in \mathscr{T}$, then $A \cap h \in \mathscr{T}_h$.

Hint: This establishes that $\mathscr{T}_h$ is a subset topology of $\mathscr{T}$. For the proof, consider the respective maximal chains in W and in h.