5
Modal Funny Business

In this chapter we deal with a type of correlation that is not much discussed and which only becomes properly analyzable by means of BST's focus on possibilities that are localized in both space and time.

There is much interest in correlations, and for good reasons. Correlations are often a good guide to causal dependencies, and we are naturally inclined to look for explanations behind observed correlations. To give a trivial example, the light is on in this room 20% of the time, and the light switch in this room is thrown 20% of the time. And these events are correlated perfectly: the probability of the switch being thrown, given that the light is on, is 100%, as is the probability of the light being on given that the switch is thrown. The explanation for this correlation is a simple direct causal connection: the throwing of the switch causes the light to turn on. Famously, not all correlations are due to direct causation between the correlated events. Sometimes we can explain a correlation conceptually. For example, it is day 50% of the time, and it is night 50% of the time, but whenever it's day it's not night, and vice versa, whereas if day and night were uncorrelated, it should be day half of the time that it is night. This correlation is just due to the fact that day *is* when it's not night, by the very concept of day. No causal link is involved here. In other cases, there is a causal link, but not a direct one. The standard example is a barometer falling and the coming of a storm: these events are highly correlated, but neither causes the other. Rather, there is a common cause, low atmospheric pressure, which causes both the barometer's fall and the storm.

Such probabilistic correlations have been much discussed. BST allows us to dig one level deeper and to unearth and analyze an unspoken assumption behind most talk of probabilistic correlations. As we will discuss at length in Chapter 7, probabilities need to be analyzed on the basis of possibilities; probabilities are graded possibilities. So to understand to what to assign a probability in the first place, we need to understand the underlying possibilities. Standard expositions of probability theory usually assume

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that one can combine probability spaces smoothly by forming Cartesian products. Behind this construction there is the assumption that the underlying possibilities combine smoothly such as to form Cartesian products—but this is not always the case. BST has the resources to analyze this issue: The spatio-temporal anchoring of possibilities in BST structures allows us to make sense of what we will call *modal correlations*. Roughly, a modal correlation is present whenever possibilities do not combine in the simplest imaginable way. It turns out, interestingly, that BST can distinguish two types of modal correlations. Many modal correlations are to be expected upon a simple reflection on the fact that possiblities in BST are concrete possibilities in space and time. Paying attention to these modal correlations is crucial for the theory of causation and of probabilities that we will build up in Chapters 6 and 7. But these modal correlations are not by themselves strange at all. Pointing to their existence is just highlighting an important lacuna in the general discussion of combining possibilities. The other type of modal correlations for which BST makes conceptual room *is* strange, and we do not take it to be a settled matter that such modal correlations exist in our world. BST does, however, provide a formally precise picture of what our world would have to be like in order for such strange modal correlations, which we call *modal funny business*, to exist.

We will use the term "modal correlation" in exact analogy with the notion of a probabilistic correlation: some joint occurrences are not independent. In probabilistic terms, dependence is expressed in terms of the probability of the joint occurrence vis-à-vis the individual occurrences. Thus, given events A and B in a probability space with probability measure pr, a correlation between A and B means that

$$pr(A \cap B) \neq pr(A) \cdot pr(B).$$

Such correlated individual events do not combine smoothly in the probabilistic sense of independence.

In the modal case, there are no probabilities assigned to the occurrences (yet), so a correlation has to be expressed solely in terms of the presence and absence of combined possibilities. Our main idea is the following: a modal correlation is present whenever two individually possible outcomes do not combine to yield a possible joint outcome. In terms of transitions, the simplest case of a modal correlation consists of two basic transitions that are individually possible, but not jointly possible. That is:

Definition 5.1. Two basic transitions $\tau_1 = e_1 \rightarrowtail H_1$ and $\tau_2 = e_2 \rightarrowtail H_2$ constitute a case of *modal correlation of the simplest kind* iff they are individually possible but not jointly so; that is, iff $H_1 \neq \emptyset$ and $H_2 \neq \emptyset$, but $H_1 \cap H_2 = \emptyset$. In terms of the histories admitted by (sets of) transitions (see Def. 4.11), we can also write this as: $H(\tau_1) \neq \emptyset, H(\tau_2) \neq \emptyset, H(\{\tau_1, \tau_2\}) = \emptyset$.

5.1 Motivation for being interested in modal correlations

Earlier we said that some modal correlations are to be expected, while others would be strange. It is "scientifically natural" to be puzzled by modally correlated transitions whose initials are space-like related—in a way, that is the gist of the famous "EPR" argument against the completeness of quantum mechanics (Einstein et al., 1935, see Chapters 5.2 and 8). On the other hand, we can show that if the initials of two transitions are *not* space-like related, then there is no deeper interest in their modal correlation.

As usual, we work in BST_{92} for concreteness, deferring a discussion of $\mathrm{BST}_{\mathrm{NF}}$ to Chapter 5.4. Consider then two basic transitions $e_1 \rightarrowtail H_1$ and $e_2 \rightarrowtail H_2$, where $H_i \in \Pi_{e_i}$ $(i = 1,2)$. There are three ways (1)–(3) in which their initials can fail to be space-like related. In each of these cases, we indicate why the question of modal correlation is obviously uninteresting.

1. If the initials e_1 and e_2 are incompatible, there is an inevitable and indeed rampant modal correlation, since every member of H_1 must contain e_1, whereas in virtue of the inconsistency of e_1 and e_2, no member of H_2 can contain e_1. So in this case, modal correlation is trivially inescapable. Since the existence of incompatible point events is a direct consequence of indeterminism (that is, it follows from the bare existence of more than one history), such modal correlations do not by themselves warrant our interest.
2. If $e_1 = e_2$, then intuitively we might not even speak of "correlation". But it is illuminating to spell out the two equally uninteresting cases. Case (a). If $H_1 \neq H_2$ (where $H_1, H_2 \in \Pi_{e_1}$ because $\Pi_{e_1} = \Pi_{e_2}$), then $H_1 \cap H_2 = \emptyset$ as Π_{e_1} partitions H_{e_1}, and modal correlation cannot be avoided. It is a conceptual truth that different immediate outcomes of the same event are incompatible and cannot occur together. Case (b). On the other hand, if $H_1 = H_2$, then the absence of modal correlation is vacuous and of equal lack of interest.

3. If e_1 is in the causal past of e_2 ($e_1 < e_2$), then according to BST_{92} theory, the very occurrence of e_2 is consistent with one and only one basic outcome of e_1, viz., $\Pi_{e_1}\langle e_2\rangle$—see Fact 4.7. There are two cases. Case (a): Perhaps e_1 has H_{e_1} as its single vacuous basic outcome. This is evidently a case of uninteresting absence of modal correlation. Case (b): e_1 has more than one basic outcome. We know that e_2 is compatible with only one of them, so that *each* of the *other* outcomes of e_1 is incompatible with *all* of the outcomes of e_2; which is equally uninteresting. So if $e_1 < e_2$, "modal correlation" is in either case uninteresting. And, of course, the case is the same if e_2 lies in the causal past of e_1.

What happens when one eliminates these three uninteresting cases? In BST_{92} that is *exactly* to say that e_1 and e_2 are "space-like related": A modal correlation between transitions $e_1 \rightarrowtail H_1$ and $e_2 \rightarrowtail H_2$ is puzzling only if e_1 is space-like related to e_2. In our view, such correlations are interesting in precisely the same way that EPR-like phenomena are, which is why we call such modal correlations a kind of "funny business", an opinion built into the wording of the following *definiendum*.

Definition 5.2. Two basic transitions $\tau_1 = e_1 \rightarrowtail H_1$ and $\tau_2 = e_2 \rightarrowtail H_2$ constitute a case of *space-like-related modal funny business of the simplest kind* $\Leftrightarrow_{\text{df}}$ $e_1\, SLR\, e_2$ and $H_1 \cap H_2 = \emptyset$.

5.2 Modal funny business

The basic message of our discussion of modal correlations so far is this: Quite a number of modal correlations are to be expected, but given the BST framework, there is a class of modal correlations that constitute funny business. Empirically, such funny business appears to be present (at least ideally) in EPR-like scenarios,[1] in which an entangled two-partite quantum

[1] So-called because such a scenario figures prominently in the famous 1935 article by Einstein, Podolsky, and Rosen (EPR), "Can quantum-mechanical description of reality be considered complete?" (Einstein et al., 1935). In that paper the authors argue that space-like correlations need an explanation in terms of an expanded description of reality that goes beyond the quantum mechanical formalism. John Bell (1964) showed a way of deriving empirical predictions from this assumption of (certain forms of) quantum-mechanical incompleteness (the existence of so-called hidden variables), and the predictions of a number of hidden variable theories have been found to be empirically violated, providing an argument for the completeness of quantum mechanics *pace* Einstein et al. These issues will be discussed in more detail in Chapter 8.

system shows perfect correlations of outcomes for space-like separated measurements. For example, the maximally entangled singlet state of two spin-1/2 particles, $|\psi\rangle$ (written in the basis $|\pm,\pm\rangle = |\pm\rangle_1 \otimes |\pm\rangle_2$)

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+,-\rangle - |-,+\rangle)$$

is such that upon measuring the first particle in the $\pm$ basis, the result of a measurement on the second particle in that basis is already determined, even if the measurement events are space-like separated. In modal terms, while for the measurement on the first particle, both the outcomes $+$ and $-$ are possible, and the same holds for the measurement on the second particle, it is impossible that the measurements give the same results, and the only possible joint outcomes are $(+,-)$ and $(-,+)$. The outcomes $(+,+)$ and $(-,-)$, which appear to be combinatorially possible given the individual possibilities, are missing from the set of joint possibilities. The large literature on quantum correlations and their possible causal interpretation is witness to the fact that people find such phenomena strange or funny, or even "spooky" (to use the common translation of Einstein's phrase, "spukhaft").[2]

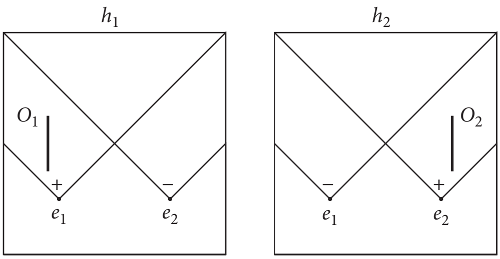

Figure 5.1 BST diagram for the EPR scenario. Choice points e_1 and e_2 represent the measurement events. O_1 and O_2 are outcome chains that represent measurement results. Note that there are only two rather than four histories, even though there are two binary choice points.

The scenario discussed so far has just two binary choice points, thus constituting the simplest possible case in which modal funny business can occur. There are a number of ways of generalizing the notion of modal funny business from this base case to arrive at more encompassing notions of

[2] The phrase occurs in a letter from Einstein to Max Born dated March 3, 1947; see Einstein et al. (1971, p. 158). We defer a discussion of the literature to Chapter 8.

modal funny business. We present the two most general definitions in what follows.[3] With a view to the prominence of possibilities described via sets of basic transitions, esp. in Chapters 6 and 7, we phrase our definitions in terms of sets of basic transitions.

5.2.1 Expected inconsistencies in sets of basic transitions

Modal correlations, generally speaking, are present whenever there is a set of transitions that are individually possible but jointly inconsistent in the sense of Def. 4.12 (i.e., not admitting a joint outcome). The individual consistency of a single basic transition is guaranteed as a matter of definition: A basic transition in BST_{92} is of the form

$$\tau = e \rightarrowtail H, \quad H \in \Pi_e,$$

and as e is always a member of some histories ($H_e \neq \emptyset$) and Π_e partitions H_e into non-empty subsets, we have that $H(\tau) = H \neq \emptyset$.

Accordingly, we link the notion of modal correlation to the inconsistency of a set of transitions:

Definition 5.3. A set of transitions

$$T = \{\tau_i = e_i \rightarrowtail H_i \mid i \in \Gamma\}, \text{ where } \Gamma \text{ is some index set}$$

constitutes a case of *modal correlation* iff it is inconsistent, i.e., iff $H(T) = \bigcap_{i \in \Gamma} H_i = \emptyset$.

Our preceding discussion has shown that certain cases of modal correlations are to be expected once one acknowledges the spatio-temporal nature of individual possibilities (transitions). Thus, to mention the simplest case, a set T containing two different transitions from the same initial has to be inconsistent. Such an inconsistency is of the clearest and most easily discernible variety: the set T runs together different local alternatives. We call such an inconsistency "blatant inconsistency":

[3] Four different notions of modal funny business were initially introduced by Belnap (2002, 2003c). They all turn out to be special cases of the definitions given here, as shown in Müller et al. (2008). Note that while Theorem 2 in the latter paper is indeed correct as stated, its proof is faulty, as kindly pointed out to us by Leszek Wroński. See Theorem 5.1 for a corrected version.

Definition 5.4 (Blatant inconsistency). A set $T = \{\tau_i = e_i \rightarrowtail H_i \mid i \in \Gamma\}$ of transitions is *blatantly inconsistent* iff there are $\tau_1, \tau_2 \in T$ such that $e_1 = e_2$, but $H_1 \neq H_2$.

Apart from blatant inconsistency, there are two other forms of inconsistency that are not surprising, as the above discussion in Chapter 5.1 has shown. The first form is that the initials of two transitions $\tau_1, \tau_2 \in T$, e_1 and e_2, are incompatible to begin with, that is, that these initials are incomparable and do not belong to any one history. In that case, no outcomes of e_1 and e_2 can share a history as the initials do not share a history to begin with (i.e., $H_1 \cap H_2 = \emptyset$ because $H_{e_1} \cap H_{e_2} = \emptyset$). This is not surprising. The second form of unproblematic inconsistency can occur when the initials e_1 and e_2 of two transitions τ_1 and τ_2 are consistent and in fact comparable (let us assume $e_1 < e_2$ for concreteness). In that case, there is exactly one outcome of e_1 that is compatible with the occurrence of e_2, namely, $\Pi_{e_1}\langle e_2 \rangle$ (see Fact 4.7(1)). So if we have $\tau_1 = e_1 \rightarrowtail H_1 \in T$, where $H_1 \neq \Pi_{e_1}\langle e_2 \rangle$, the transition τ_1 excludes the occurrence of τ_2 by already excluding the occurrence of its initial e_2. Again, it is obvious why the whole set T is inconsistent; the blame is on τ_1 selecting the wrong outcome. Interestingly, this case can be linked to blatant inconsistency in the following way, and which we will make use of later in Chapter 5.2.3. In BST, the occurrence of any point event implies the previous occurrence of its complete spatio-temporal past; histories are closed downward. We thus do not add anything modally substantial (we do not add new choices) if we complete a transition set toward the past. But given that $\tau_2 \in T$ and $e_1 < e_2$, any history $h \in H_2$ contains e_1 *and* fixes the outcome of e_1 to be $\Pi_{e_1}\langle h \rangle = \Pi_{e_1}\langle e_2 \rangle$, a superset of H_2. So adding the transition $\tau_1' =_{\text{df}} e_1 \rightarrowtail \Pi_{e_1}\langle e_2 \rangle$ to T should be an innocuous downward extension that just adds a part of the story that was implied by the downward closure of histories anyway.[4] Yet it turns out that the set of transitions $T' =_{\text{df}} T \cup \{\tau_1'\}$ is not just inconsistent (as was T), but even blatantly inconsistent, containing now two different transitions τ_1 and τ_1' with the same initial e_1. In this way, we see that inconsistency due to consistent, order-related initials with incompatible outcomes is quite close to blatant inconsistency.[5]

Pulling together the various strands of the discussion so far, we can repeat the observation from Chapter 5.1 that the only surprising cases of modal

[4] The case for the term "downward extension" is further strengthened by noting that in terms of the ordering of transitions (Def. 3.10), $\tau_1' \prec \tau_2$.

[5] See Chapter 5.2.3 for more details on downward extensions and the related idea of explanatory funny business as absence of blatant inconsistency in all downward extensions.

correlations—the ones whose presence has no immediate explanation—are those linked to space-like related initials which have incompatible outcomes.

5.2.2 Combinatorial funny business

Summing up the discussion of the expected inconsistencies of sets of basic transitions above, we can say that transition sets of the following form are well-behaved in the sense of not containing a direct case of obvious inconsistency. We call such sets combinatorially consistent:

Definition 5.5 (Combinatorial consistency). A set $T = \{\tau_i = e_i \rightarrowtail H_i \mid i \in \Gamma\}$ of basic transitions is *combinatorially consistent* iff for any $\tau_i, \tau_j \in T$:

1. if $e_i = e_j$, then $H_i = H_j$ (i.e., $\tau_i = \tau_j$);
2. if $e_i < e_j$, then $H_{e_j} \subseteq H_i$ (i.e., $\tau_i \prec \tau_j$);
3. if $e_j < e_i$, then $H_{e_i} \subseteq H_j$ (i.e., $\tau_j \prec \tau_i$);
4. if e_i and e_j are incomparable, then $e_i \, SLR \, e_j$.

Thus, in a combinatorially consistent set of transitions, there is no blatant inconsistency (1), there is no order-related inconsistency (2, 3), and there is no inconsistency related to inconsistent initials (4). It is indeed the last clause (4) that relies on combinatorics, as it suggests that the compatibility of two SLR initials should be enough for a pair of transitions starting with these initials to be consistent.

A combinatorially consistent set looks well behaved. One aspect of this well-behavedness is that any two initials from transitions from such a set T share some history—so T is, in some sense, almost consistent:

Fact 5.1. *If $T = \{\tau_i = e_i \rightarrowtail H_i \mid i \in \Gamma\}$ is combinatorially consistent, then for any two initials e_i and e_j ($i, j \in \Gamma$), there is a history h containing them both.*

Proof. Let $i, j \in \Gamma$. We argue by cases. (1) If $e_i = e_j$, then any history h from H_{e_i} serves as a witness. (2) If $e_i < e_j$, any history h from H_{e_j} serves as a witness (note that $h \in H_i$, and thus in particular, $e_i \in h$). (3) If $e_j < e_i$, then similarly, take any history h from H_{e_i}. (4) The fact that $e_i \, SLR \, e_j$ implies, by definition, that there is a history h containing them both. □

So there is no apparent reason why a combinatorially consistent set of transitions should not be consistent. To support this idea, we can note that sets of transitions that are in fact consistent are also well-behaved according to the definition:

Lemma 5.1. *If a set of basic transitions T is consistent, then it is also combinatorially consistent.*

Proof. Assume that T is not combinatorially consistent. Thus, there are $\tau_1, \tau_2 \in T$ ($\tau_i = e_i \rightarrowtail H_i$, $i = 1,2$) constituting a counterexample to one of the four clauses from Def. 5.5. For each of these cases, the discussion of Section 5.2.1 has shown that $H_1 \cap H_2 = \emptyset$, so that T is not consistent. □

The other direction does not hold in general but, if it fails, something at least mildly counterintuitive is going on: The set T is well-behaved, but the combinatorics do not work out as expected. Some histories that should witness the consistency of T are, as it were, missing. Thus we define:

Definition 5.6 (Combinatorial funny business)**.** A set of basic transitions T constitutes a case of *combinatorial funny business* (CFB) iff T is combinatorially consistent (Def. 5.5), but inconsistent ($H(T) = \emptyset$).

It should be reassuring that the simplest EPR-like case of funny business modal correlations, discussed earlier, falls under this definition.

Fact 5.2 (The EPR scenario exhibits CFB)**.** *The EPR scenario of Figure 5.1, discussed at the beginning of Section 5.2, constitutes a case of combinatorial funny business according to Definition 5.6.*

Proof. Let e_1, e_2 denote the two binary choice points in the EPR scenario (measurement in the left and right wing of the experiment, respectively). Each of these choice points has two basic scattered outcomes, $\Omega^+_{e_i}$ and $\Omega^-_{e_i}$, so that for $i = 1,2$, there are two transitions each, $\tau_i^+ = e_i \rightarrowtail \Omega^+_{e_i}$ and $\tau_i^- = e_i \rightarrowtail \Omega^-_{e_i}$. The scenario thus contains four basic indeterministic transitions in total, τ_1^+, τ_1^-, τ_2^+, and τ_2^-. The measurement outcomes $\Omega^{\pm}_{e_i}$ are correlated to allow only the combination of one $+$ outcome with one $-$ outcome. Thus, out of the four (2×2) histories one would combinatorially expect, only two, $h_1 = h^{+-}$ (containing the transitions τ_1^+ and τ_2^-) and $h_2 = h^{-+}$ (containing τ_1^- and τ_2^+), are possible. Thus, in particular, the joint $++$ outcome cannot happen, meaning that the set of transitions $T =_{\text{df}} \{\tau_1^+, \tau_2^+\}$ is inconsistent. This set is, however, combinatorially consistent, as the initials of the two transitions, e_1 and e_2, are *SLR*: they are incomparable, and their compatibility is witnessed, for example, by the history h^{+-} featuring the $+-$ joint outcome. □

5.2.3 Explanatory funny business

According to the preceding discussion, an instance of modal correlations constitutes funny business iff the spatio-temporal layout of the possibilities (transitions) that are combined does not provide a reason for the inconsistency of the whole set of transitions. On the face of it, such a set T looks consistent, and it is a raw additional fact that it is nevertheless inconsistent.

Another line of looking at inconsistency and funny business is opened up by not just looking at the transition set T as given, but by attempting to come up with a satisfactory explanatory account of the inconsistency. As we said earlier, an immediately understandable form of inconsistency is blatant inconsistency (i.e., the running together of incompatible local alternatives). This form of incompatibility is also acknowledged in standard probability theory: If we have two members A, B of the event algebra of a probability space and $A \cap B = \emptyset$, then we know that $pr(A \cap B) = 0$ even if $pr(A) \neq 0$ and $pr(B) \neq 0$.

In probabilistic contexts, correlations are often interesting, and an account for a correlation can often be found by looking at circumstances in the past of the correlated events. Thus, in the case of the correlation between the falling barometer and the impeding storm described at the beginning of this chapter, we find a previous event, the advent of a low pressure weather system, that explains the correlation. Technically, we often assume that given two correlated contemporaneous events A and B,

$$pr(A \cap B) \neq pr(A) \cdot pr(B),$$

there is an additional past event C such that conditional on the occurrence of C, A and B become probabilistically independent, i.e.,

$$pr(A \cap B \mid C) = pr(A \mid C) \cdot pr(B \mid C).$$

In this spirit, Reichenbach (1956) proposed his common cause principle, which has been the subject of much recent research and discussion.[6] When transferred to the context of modal correlations in sets of transitions in BST, we cannot, of course, expect the whole idea of the common cause principle to carry over, but the idea of "explaining a surprising phenomenon

[6] See, e.g., Hofer-Szabó et al. (2013) and Wroński (2014).

by looking in the past" generalizes in a useful way. Earlier, in Section 5.2.2, we already pointed out that in case a set of transitions $T = \{\tau_1, \tau_2\}$ (with $\tau_i = e_i \rightarrowtail H_i$, $i = 1,2$) is inconsistent ($H(T) = \emptyset$) and $e_1 < e_2$, we can provide a local account of that inconsistency by adding in the seemingly innocuous downward extension $\tau_1' =_{\text{df}} e_1 \rightarrowtail \Pi_{e_1}\langle e_2 \rangle$, which, lying in the past of e_2, has to have occurred for the initial of τ_2 to occur. The extended set $T' =_{\text{df}} T \cup \{\tau_1'\}$ is blatantly inconsistent, which readily explains why it is inconsistent.

It is interesting to inquire as to whether we can always explanatorily extend a given inconsistent transition set such that the blame for the inconsistency is ultimately on some blatant inconsistency (i.e., such that the extended set is blatantly inconsistent). It turns out that the answer is no. The EPR-like example of Figure 5.1 that was discussed in the context of Fact 5.2 already suffices as an illustration: In this structure, there are only two choice points, e_1 and e_2, so that the inconsistent set of transitions $T =_{\text{df}} \{\tau_1^+, \tau_2^+\}$ cannot be extended to the past in any way. And that set is inconsistent, but not blatantly inconsistent. So we can have cases of modal correlations that do not have a (local) explanation. We call such cases of modal correlations, accordingly, *explanatory funny business*.

To make this notion formally precise, we start with the notion of downward extension. The idea is that, in searching for the explanation of the inconsistency of a given set of transitions T, we may add transitions in the past, as these cannot get in the way modally speaking (their occurrence is implied by the occurrence of a later transition), and they may add explanatory detail.

Definition 5.7 (Downward extension)**.** The set of basic transitions T^* is a *downward extension* of T iff (1) $T \subseteq T^*$ and (2) for any (new) $\tau^* \in (T^* \setminus T)$, there is some $\tau \in T$ for which $\tau^* \prec \tau$.

We can prove the following facts about downward extensions:

Lemma 5.2. *Let T^* be a downward extension of a given set of basic transitions T. Then the following holds: (1) T^* is consistent iff T is consistent. (2) T^* is combinatorially consistent iff T is combinatorially consistent.*

Proof. (1) "$\Rightarrow$" Let T be inconsistent, i.e., $H(T) = \cap_{\tau \in T} H(\tau) = \emptyset$. Then we have

$$H(T^*) = \cap_{\tau \in T^*} H(\tau) = H(T) \cap (\cap_{\tau^* \in T^* \setminus T} H(\tau^*)) = \emptyset,$$

so that T^* is also inconsistent.

"$\Leftarrow$" Let T be consistent, i.e., $H(T) = \cap_{\tau \in T} H(\tau) \neq \emptyset$. Again we can write

$$H(T^*) = \cap_{\tau \in T^*} H(\tau) = H(T) \cap (\cap_{\tau^* \in T^* \setminus T} H(\tau^*)).$$

Now take some $\tau^* \in T^* \setminus T$. By definition of downward extension, there is some $\tau \in T$ for which $\tau^* \prec \tau$, and this implies that $H(\tau^*) \supseteq H(\tau)$. So (as $H(T) \subseteq H(\tau)$) we have for any of the new τ^* that $H(\tau^*) \supseteq H(T)$. This implies that

$$H(T^*) = \cap_{\tau \in T^*} H(\tau) = H(T) \cap (\cap_{\tau^* \in T^* \setminus T} H(\tau^*)) = H(T) \neq \emptyset,$$

establishing the consistency of T^*.

(2) "$\Rightarrow$" Let T^* be combinatorially consistent, which means that for any two $\tau_1 = e_1 \rightarrowtail H_1, \tau_2 = e_2 \rightarrowtail H_2 \in T^*$, one of the four clauses from Definition 5.5 applies. This implies in particular that for any two transitions from $T \subseteq T^*$, one of the four clauses applies, so that T is also combinatorially consistent.

"$\Leftarrow$" Let T be combinatorially consistent, which means that for any two transitions from T, one of the four clauses from Definition 5.5 applies. We have to show that that feature transfers to T^*, which is a downward extension of T. Thus, take some $\tau_1 = e_1 \rightarrowtail H_1, \tau_2 = e_2 \rightarrowtail H_2 \in T^*$. There are three cases, depending on whether or not τ_1 and τ_2 already belong to T.

Case 1. If both $\tau_1, \tau_2 \in T$, then the combinatorial consistency of T alone suffices to show that one of the four clauses holds for τ_1 and τ_2.

Case 2. Assume that $\tau_1 \in T$, but $\tau_2 \in T^* \setminus T$. (The case with τ_1 and τ_2 reversed is exactly analogous.) As τ_2 is part of a downward extension of T, there is some $\tau_3 = e_3 \rightarrowtail H_3 \in T$ for which $(*)$ $\tau_2 \prec \tau_3$ (i.e., $e_2 < e_3$ and $H_{e_3} \subseteq H_2$; see Fact 3.11). As $\tau_1, \tau_3 \in T$ and T is combinatorially consistent, by Fact 5.1 there is $h \in$ Hist such that $e_1, e_3 \in h$, and since $e_2 < e_3$, (†) $h \in H_{e_1} \cap H_{e_2} \cap H_{e_3}$. We show that one of four clauses applies for τ_1 and τ_2. (1) It cannot be that $e_1 = e_2$. Assume otherwise, then $\tau_2 \in T^* \setminus T$ implies that $H_1 \neq H_2$. Now $e_1 < e_3$ (as $e_1 = e_2$), so the combinatorial consistency of T implies $H_{e_3} \subseteq H_1$. But by $\tau_2 \prec \tau_3$, we also have $H_{e_3} \subseteq H_2$. This is a contradiction, because H_1 and H_2 are by assumption different elements of the partition Π_{e_1}, and $H_{e_3} \neq \emptyset$. (2) If $e_1 < e_2$, we have to show that $H_{e_2} \subseteq H_1$. By combinatorial consistency of T, we have $\tau_1 \prec \tau_3$ as $e_1 < e_3$, so for the h witnessing (†), we have $H_1 = \Pi_{e_1}\langle e_3 \rangle = \Pi_{e_1}\langle h \rangle$. Pick an arbitrary $h' \in H_{e_2}$.

As $e_2 > e_1$, we have $h' \equiv_{e_1} h$, i.e., $h' \in H_1$. (3) If $e_2 < e_1$, we have to show that $H_{e_1} \subseteq H_2$. Analogously to case (2), we have $H_2 = \Pi_{e_2}\langle e_3\rangle = \Pi_{e_2}\langle h\rangle$. Pick an arbitrary $h' \in H_{e_1}$. Since $h, h' \in H_{e_1}$ and $e_2 < e_1$, we have $h \equiv_{e_2} h'$, and hence $h' \in H_2$. Finally, (4) if e_1 and e_2 are incomparable, then $e_1\, SLR\, e_2$ by (†).

Case 3. Assume that $\tau_1, \tau_2 \in T^* \setminus T$. Then there are $\tau_1', \tau_2' \in T$, $\tau_i' = e_i' \rightarrowtail H_i'$ such that (i) $\tau_i \prec \tau_i'$ $(i = 1,2)$. Hence by Fact 5.1, there is $h \in$ Hist such that $e_1', e_2' \in h$, and hence (ii) $h \in H_1 \cap H_2$ by (i). We show that τ_1, τ_2 satisfy one of the four clauses (1)–(4). (1) If $e_1 = e_2$, then (ii) implies that $H_1 = H_2$. (2) If $e_1 < e_2$, then for any $h' \in H_{e_2}$ we have $h \equiv_{e_1} h'$ since $e_2 \in h$ and $e_1 < e_2$, so $h' \in H_1$. Case (3) follows analogously. And (4) if e_1 and e_2 are incomparable, since by (ii) $e_1, e_2 \in h$, it follows that $e_1\, SLR\, e_2$. □

If a given set T is inconsistent, one can hope that it will be possible to find a downward extension of T that is blatantly inconsistent. This would plainly make the inconsistency intelligible. If that hope is frustrated, something funny is going on. Thus we define:

Definition 5.8 (Explanatory funny business). A set T of transitions is a case of *explanatory funny business* (EFB) iff (1) T is inconsistent ($H(T) = \emptyset$) and (2) there is no downward extension T^* of T that is blatantly inconsistent.

Note that the verdict of explanatory funny business can be checked by considering a single structure, viz., the maximal downward extension of a given transition set. The maximal downward extension of T is the set

$$T^{*max} =_{\mathrm{df}} \{\tau' \in \mathrm{TR}(W) \mid \exists \tau \in T\, [\tau' \preccurlyeq \tau]\},$$

where $\mathrm{TR}(W)$ is the set of basic indeterministic transitions introduced in Chapter 3.4.4, $\mathrm{TR}(W) = \{e \rightarrowtail H \mid e \in W, \Pi_e \neq \{H_e\}, H \in \Pi_e\}$.

Obviously, if some downward extension T^* of T is blatantly inconsistent, then so is T^{*max}, which is a superset of any T^*, and if no downward extension of T is blatantly inconsistent, then this excludes T^{*max} from being blatantly inconsistent as well.

5.2.4 On the interrelation of combinatorial and explanatory funny business

Given the result about the preservation of (combinatorial) consistency and inconsistency from Lemma 5.2, it turns out that in any case in which there is combinatorial funny business, there is also explanatory funny business:

Lemma 5.3 (Combinatorial funny business implies explanatory funny business)**.** *If a set of transitions T is an instance of combinatorial funny business (i.e., if T is inconsistent, but combinatorially consistent), then T is also an instance of explanatory funny business (i.e., no downward extension of T is blatantly inconsistent).*

Proof. Let T be an instance of combinatorial funny business (i.e., inconsistent but combinatorially consistent), and let T^* be some downward extension of T. By Lemma 5.2(1), T^* is also inconsistent. By part (2) of that Lemma, T^* is also combinatorially consistent. Thus, for any $\tau_1 = e_1 \rightarrowtail H_1, \tau_2 = e_2 \rightarrowtail h_2 \in T^*$, if $e_1 = e_2$, then by clause (1) of Definition 5.5, we have $H_1 = H_2$, i.e., $\tau_1 = \tau_2$. So T^* is not blatantly inconsistent. As T^* was an arbitrary downward extension of T, we have established that T is an instance of explanatory funny business. $\square$

The other direction of Lemma 5.3, however, fails to hold. The reason is that the notion of combinatorial funny business cannot detect funny business in cases in which transitions with inconsistent initials are present: by Def. 5.6, a transition set that includes inconsistent initials cannot be combinatorially consistent, and therefore cannot be a case of combinatorial funny business. For a relevant example, consider the two structures shown in Figure 5.2.

In each of these two BST_{92} structures, the possible point events e_3 and e_4 are incompatible; they are not order related and do not share a history. Therefore, the set of transitions $T =_{\mathrm{df}} \{e_3 \rightarrowtail \Omega^+_{e_3}, e_4 \rightarrowtail \Omega^+_{e_4}\}$ is inconsistent, but it is also combinatorially inconsistent—the initials e_3 and e_4 fail clause (4) of Definition 5.5. So the definition of combinatorial funny business in both cases gives the verdict that there is nothing funny going on. The notion of explanatory funny business, however, requires us to consider the downward extensions of the transition set T, and these lead to different verdicts. In case (a), the maximal downward extension of T is $T_a = \{e_3 \rightarrowtail \Omega^+_{e_3}, e_4 \rightarrowtail \Omega^+_{e_4}, e_1 \rightarrowtail \Omega^+_{e_1}, e_1 \rightarrowtail \Omega^-_{e_1}\}$, and this set is blatantly inconsistent as it combines different basic outcomes of e_1. So for case (a), the verdict of the definitions of explanatory and combinatorial funny business agree—nothing funny going on. In case (b), however, the maximal downward extension of T is $T_b = \{e_3 \rightarrowtail \Omega^+_{e_3}, e_4 \rightarrowtail \Omega^+_{e_4}, e_1 \rightarrowtail \Omega^+_{e_1}, e_2 \rightarrowtail \Omega^-_{e_2}\}$, and this set is not blatantly inconsistent. So the definition of explanatory funny business signals, correctly, that something funny is going on—the outcomes of the *SLR* initials e_1 and e_2 are, after all, correlated. The definition of combinatorial funny business does not signal anything funny, however, as the initials e_3

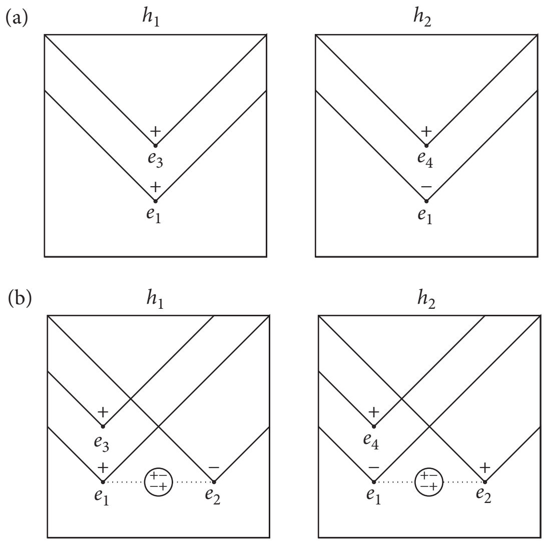

Figure 5.2 Two BST structures with four histories each and with inconsistent initials e_3 and e_4. (Histories h_3 and h_4 are not shown; they correspond to copies of h_1 and h_2 with the "$-$" outcome of e_3 and e_4, respectively.) Upper panel (a): no funny business, lower panel (b): explanatory funny business.

and e_4 are incompatible, which is enough of a combinatorial reason for the inconsistency of T. We can describe the upshot of this discussion as a useful fact:

Fact 5.3. *In the BST structure of Figure 5.2(b), the transition set* $T =_{\text{df}} \{e_3 \rightarrowtail \Omega^{+}_{e_3}, e_4 \rightarrowtail \Omega^{+}_{e_4}\}$ *exhibits explanatory funny business, but no combinatorial funny business.*

We sum up our results about the interrelation of combinatorial and explanatory funny business so far:

Lemma 5.4. *The notion of explanatory funny business properly extends the notion of combinatorial funny business. That is: (1) If a set of transitions T is an instance of combinatorial funny business, then T is also an instance of explanatory funny business. (2) There are instances of transition sets T exhibiting explanatory funny business that are not instances of combinatorial funny business.*

Proof. (1) is the content of Lemma 5.3. (2) has been shown via Fact 5.3. □

Does this mean that the notion of combinatorial funny business is too narrow and therefore inadequate? It would be good if we could salvage the notion of combinatorial funny business in some way, as it captures an important intuition behind the notion of funny business, viz., puzzlement over SLR correlations. In the example of Figure 5.2 (b), starting with the given transition set T, we can exhibit such underlying SLR correlations by downward extending T and then stripping away unnecessary transitions with inconsistent initials. That is: in the given transition set T, there is only explanatory, not combinatorial funny business, but in the BST_{92} structure from which T is taken, there *is* an instance of combinatorial funny business to be found. We will show that this idea generalizes in a useful way: we can show that *at the level of BST structures rather than transition sets*, our two notions of funny business agree.

Theorem 5.1 (There is combinatorial funny business iff there is explanatory funny business). *Let* $\langle W, < \rangle$ *be a* BST_{92} *structure. For its set of basic indeterministic transitions,* $\text{TR}(W)$*, the following holds: There is a subset* $T_1 \subseteq \text{TR}(W)$ *exhibiting combinatorial funny business iff there is a subset* $T_2 \subseteq \text{TR}(W)$ *exhibiting explanatory funny business.*

Proof. The "$\Rightarrow$" direction has been established via Lemma 5.3: we can take $T_2 = T_1$. The proof of the "$\Leftarrow$" direction is rather lengthy, so we provide full details in Appendix A.3. The main idea of that proof direction is to work under the assumption that there is no combinatorial funny business in the given structure. We then assume for reductio that there is a set of transitions T that witnesses explanatory funny business. Its maximal downward extension T^* thus contains no instance of blatant inconsistency. Given that there is no combinatorial funny business by assumption, T^* must be inconsistent, but combinatorially inconsistent. The bulk of the proof consists in using T^* to construct a transition set T_C that, contrary to our main assumption, exhibits combinatorial funny business. This proves that our reductio assumption (the existence of explanatory funny business in T) must be false, so that from the assumption of no combinatorial funny business in a given BST_{92} structure, we can show that there can be no explanatory funny business either. $\square$

The two notions of explanatory and combinatorial funny business are thus equivalent at the level of BST_{92} structures. We take this result to support the view that we have thereby provided a stable explication of the notion of a puzzling modal correlation, or modal funny business, in branching space-times.[7] In the rest of the book, we will therefore always speak of "modal funny business" as a unified concept. Here is our official definition, for which we use CFB—which can be replaced by EFB if one so wishes.

Definition 5.9. A BST_{92} structure $\langle W, < \rangle$ exhibits *modal funny business (MFB)* iff among its set of basic indeterministic transitions $\mathrm{TR}(W)$ there is a set $T \subseteq \mathrm{TR}(W)$ that constitutes combinatorial funny business according to Def. 5.6.

5.3 Some consequences of modal funny business

In this section, we prove some facts related to the BST_{92} prior choice principle that will be used later on. Recall that PCP_{92} guarantees that for any two histories h_1, h_2 and for any lower bounded chain O, if $O \subseteq h_I \setminus h_2$, then there is at least one choice point c for h_1 and h_2 that is in the past of O, $c < O$. Such a choice point c is maximal in $h_1 \cap h_2$ (i.e., $h_1 \perp_c h_2$). There can be other choice points for h_1 and h_2, and these may not lie in the past of O. The existence of such choice points not in the past of O is a common feature, independent of whether there is modal funny business in a structure or not. Compare Figure 3.3 (p. 57), in which there are two uncorrelated *SLR* choice points. In the structure depicted in that figure, the outcome chain O_I belongs to h_2 but not h_3, and these histories split both at the choice point e_1 in the past of O_I and at the choice point e_2, which is not in the past of O_I. It is not the case, however, that *all* histories in which O_I begins split off from h_2 at e_2 (the counterexample is h_1). This is different in the EPR scenario of Figure 5.1 (p. 107), which exhibits a related, slightly different notion that is of greater interest here: we may look at choice points at which *all* histories from $H_{\langle O \rangle}$ (all histories in which the chain O begins to occur) split from a given history h. As $H_{\langle O \rangle}$ is the occurrence proposition for O, a choice point of this kind constitutes a decisive event as discussed in our introduction to semantic notions in Chapter 4.5: at such a choice point, the occurrence of O is still contingent (it is possible, but not settled, that O will occur), but immediately after this event, in some histories such as h, the occurrence of O is prohibited (see Fact 4.7). We call such choice points *cause-like loci* for

[7] This attitude is further strengthened by noting that EFB and CFB extend the previous notions of funny business due to Belnap (2002, 2003c). See also note 3.

O, written $cll(O)$. Cause-like loci will figure prominently in our discussion of causation and probabilities and in related applications.

Definition 5.10 (Cause-like locus)**.** Let O be a lower bounded chain in a BST_{92} structure $\langle W, < \rangle$. The set of cause-like loci for O, written as $cll(O)$, is defined as:

$$cll(O) = \{e \in W \mid \exists h\, [h \in \mathrm{Hist} \wedge h \perp_e H_{\langle O \rangle}]\},$$

where $h \perp_e H_{\langle O \rangle} \Leftrightarrow_{\mathrm{df}} \forall h' [h' \in H_{\langle O \rangle} \to h \perp_e h']$. We simplify $cll(\{e\})$ to $cll(e)$.

It turns out that whether or not all elements of $cll(O)$ are in the past of O depends on the presence of modal funny business. Indeed, look at the EPR scenario from Figure 5.1, which was already formally discussed in the proof of Fact 5.2. There are two SLR choice points, e_1 and e_2, with two immediate outcomes $\Omega^+_{e_i}$ and $\Omega^-_{e_i}$ each ($i = 1, 2$), and thus the scenario contains the four indeterministic transitions τ_1^+, τ_1^-, τ_2^+, and τ_2^-. As the initials are SLR, one would expect there to be four histories ($h^{++}, h^{+-}, h^{-+}, h^{--}$), but due to modal funny business, there are only the two histories h^{+-} and h^{-+}. That is, the only consistent two-element sets of transitions are $\{\tau_1^+, \tau_2^-\}$ and $\{\tau_1^-, \tau_2^+\}$. Let now O_I be a chain in $h_1 = h^{+-}$ above e_1 (in $\Omega^+_{e_1}$) that does not lie above e_2, as indicated in Figure 5.1. Intuitively, the occurrence of this chain means that in the left wing of the apparatus, the outcome "$+$" has been registered. The chain O_I does not occur in $h_2 = h^{-+}$, in which the left measurement registers outcome "$-$". It is obvious that $h^{+-} \perp_{e_1} h^{-+}$, and as $H_{\langle O_I \rangle} = \{h^{+-}\}$, we have $e_1 \in cll(O_I)$, as is to be expected: e_1 is a decisive event for the occurrence of O_I. Yet, as one can easily verify, we have $e_2 \in cll(O_I)$ as well, even though $e_2 \not< O_I$. Thus, something strange is going on.[8] Since the notion of $cll(O_I)$ has causal connotations, it appears that some causal factors responsible for keeping O_I possible are not in the past of O_I. For further analysis of this weird feature, see Chapter 8. Luckily, however, we can show that this weirdness will not occur if there is no modal funny business in a BST_{92} structure. The following Fact has variants for

[8] In Belnap (2002), scenarios like this are called "Some-cause-like-locus-not-in-past funny business". As we said earlier (note 3), our notion of MFB generalizes this notion. Examples proving that MFB properly extends the earlier notions are quite involved; see Müller et al. (2008) for a pertinent construction with infinitely many SLR choice points.

scattered outcomes and disjunctive outcomes, which are left as Exercises 5.3 and 5.4. Here we state and prove the relevant Fact for outcome chains.

Fact 5.4. *Assume that in a BST_{92} structure $\langle W, < \rangle$ there is no modal funny business. For $O \subseteq W$ a lower bounded chain, $h \in \text{Hist}$, and $e \in W$: if $h \perp_e H_{\langle O \rangle}$, then $e < O$. Thus, for every $e \in cll(O)$, we have $e < O$.*

Proof. Let (i) $h \perp_e h_O$, where $h_O \in H_{\langle O \rangle}$ and $h \notin H_{\langle O \rangle}$. Since O is by definition lower bounded, there is $i = \inf(O)$, and we have $i \in h_O$. We will consider the possible ordering relations between e and i.

Consider first (ii) $e \leqslant i$; in that case it is impossible that both $i \in O$ and $e = i$ since then $e \in O$, $e \in h$, and hence $h \in H_{\langle O \rangle}$; then (i) implies $h \perp_e h$, which contradicts the reflexivity of $\equiv_e$. Accordingly, $[i \notin O$ or $e \neq i]$, which together with (ii) implies $e < O$, so we are done. We next show that all other order relations between e and i lead to a contradiction.

Let us then suppose that (iv) $i < e$. Then we claim that $\forall h\ [h \in H_e \rightarrow h \in H_{\langle O \rangle}]$, which contradicts (i). For reductio, let (v) $h' \in H_e$ but $h' \notin H_{\langle O \rangle}$, hence $h' \cap O = \emptyset$. On the other hand, there is a non-empty chain $O' = O \cap h_O$ for some $h_O \in H_{\langle O \rangle}$, so $O' \subseteq h_O \setminus h'$. By PCP of BST_{92} there is $c < O'$, hence (vi) $c < O$, such that (vii) $h_O \perp_c h'$. By (vi), since i is the infimum of O, $c \leqslant i$. Together with (iv) this implies $c < e$, and we also have $e \in h_O \cap h'$ from (i) and (v), which contradicts (vii). We thus established the above claim, which proves that the option (iv) is not possible.

The remaining case is that i and e are incomparable. Since $e, i \in h_O$, we get (viii) $i\, SLR\, e$. Consider transitions $\tau_1 = i \rightarrowtail \Pi_i\langle h_O \rangle$ and $\tau_2 = e \rightarrowtail \Pi_e\langle h \rangle$. Since by (viii) the set $\{\tau_1, \tau_2\}$ is combinatorially consistent, and by assumption there is no MFB, the set cannot exhibit CFB, so is consistent as well; that is, $\Pi_i\langle h_O \rangle \cap \Pi_e\langle h \rangle \neq \emptyset$. Let (†) $h^* \in \Pi_i\langle h_O \rangle \cap \Pi_e\langle h \rangle$, so (ix) $h^* \equiv_e h$. We claim now that (x) $h^* \in H_{\langle O \rangle}$. Otherwise there would be $c' < O$ such that (xi) $h^* \perp_{c'} H_{\langle O \rangle}$, by PCP_{92} and an argument like for the case above. By the definition of infimum, $c' \leqslant i$; yet if $c' = i$, then $h^* \perp_i H_{\langle O \rangle}$, hence $h^* \perp_i h_O$, which contradicts $h^* \in \Pi_i\langle h_O \rangle$, where the last follows from (†). And if $c' < i$, then since $i \in h^*$ and $i \in h_O$, we have $h^* \equiv_{c'} h_O$, in contradiction to (xi). We thus established (x), which given (i) implies $h \perp_e h^*$, whereas (ix) says $h^* \equiv_e h$: a contradiction. We have thus shown that incomparability of i and e is not a possible option either, and the only possible ordering is $e < i$, which implies $e < O$, as required. □

We mention two helpful consequences of $cll(O)$ being in the past of O, which follow directly from Fact 4.7(2) and (3): First, given $e < O$, there is a unique element of Π_e that is consistent with $H_{\langle O \rangle}$, for which we write $\Pi_e\langle O \rangle$.

Second, if every cause-like locus $e \in cll(O)$ is in the past of O, every history from $H_{\langle O \rangle}$ selects the same (unique) outcome $\Pi_e\langle h \rangle = \Pi_e\langle O \rangle$ of e. One may wonder what happens to these two results if for some $e \in cll(O)$, $e \not< O$. Very briefly, in such a case O might be compatible with more than one basic outcome of e, thus singling out a basic *disjunctive* outcome of e. We will return to this topic in our analysis of causation in BST in the presence of MFB (Chapter 6.4).

Here is a further consideration that illustrates the effects of modal funny business. Assume there is a set of point events E that consists of pairwise compatible points (i.e., any two elements of E share some history). Does it follow that E itself is consistent, that there is a history containing all of E? The following Fact shows that this depends on the presence or absence of MFB.

Fact 5.5. *Let $E \subseteq W$ be a set of events that are pairwise compatible, i.e., for any $e_1, e_2 \in E$, there is some $h \in$ Hist for which $e_1, e_2 \in h$. (For example, such a set E could consist of pairwise SLR events.) If there is no history h for which $E \subseteq h$, then W exhibits MFB.*

Proof. Let $E \subseteq W$ be as above, and assume for reductio that no history contains all of E while there is no MFB. Let $C =_{\text{df}} \bigcup_{e \in E} cll(e)$ be the set of cause-like loci for all members of E. As by assumption no history contains all of E, the set C is non-empty: Let $E = E_1 \cup E_2$ for non-empty subsets $E_1, E_2 \subseteq E$ such that $E_1 \cap E_2 = \emptyset$ and such that for some history h_1, $E_1 \subseteq h_1$, while $E_2 \cap h_1 = \emptyset$. Then for any $e_2 \in E_2$, H_{e_2} splits off from h_1 at some choice point $c < e_2$, and $c \in cll(e_2) \subseteq C$ (see Fact 3.8).

Consider now the set of transitions

$$T =_{\text{df}} \{c \rightarrowtail \Pi_c\langle e \rangle \mid e \in E \wedge c \in cll(e) \wedge c < e\}.$$

We can show that this set is combinatorially consistent: Let $\tau_i = c_i \rightarrowtail \Pi_{c_i}\langle e_i \rangle \in T$ $(i = 1, 2)$. As E consists of pairwise compatible events, there is some history h_{12} containing both e_1 and e_2. It is easy to check that that history serves as a witness for the relevant clause from the definition of combinatorial consistency (Def. 5.5): (1) If $c_1 = c_2$, then $h_{12} \in \Pi_{c_1}\langle e_1 \rangle \cap \Pi_{c_1}\langle e_2 \rangle$, which implies $\Pi_{c_1}\langle e_1 \rangle = \Pi_{c_1}\langle e_2 \rangle$. (2, 3) If $c_1 < c_2$, we have to show that $H_{c_2} \subseteq \Pi_{c_1}\langle e_1 \rangle$. We have $h_{12} \in \Pi_{c_1}\langle e_1 \rangle \cap \Pi_{c_2}\langle e_2 \rangle$. Let now $h' \in H_{c_2}$; as $c_2 > c_1$, we have $h' \equiv_{c_1} h_{12}$, so $h' \in \Pi_{c_1}\langle e_1 \rangle$, which implies $\tau_1 \prec \tau_2$. (4) In case c_1 and c_2 are incomparable, h_{12} witnesses their being SLR.

So, given our assumptions, T is combinatorially consistent. We can now use our assumption that there is no MFB to show that there must be some history containing all of E, which is a contradiction, and thus finishing our proof.

As there is no MFB, the combinatorially consistent set T is in fact consistent: there is some history $h \in H(T)$. We claim that $E \subseteq h$. Assume for reductio that for some $e \in E$, $e \notin h$, and let $h_e \in H_e$. Then by PCP$_{92}$, there is a choice point $c < e$ for which $h \perp_c h_e$, and indeed $h \perp_c H_e$ (Fact 3.8). So $c \in cll(e)$, and by the definition of T, we have $c \rightarrowtail \Pi_c\langle e\rangle \in T$. But this implies $h \in \Pi_c\langle e\rangle \subseteq H(T)$, contradicting $h \perp_c H_e$. □

5.4 On MFB in BST$_{\rm NF}$

In this section we outline an account of MFB in BST$_{\rm NF}$ structures. A full account of this kind requires one to formulate counterparts of the proofs from Section 5.2 in BST$_{\rm NF}$. By focusing on BST$_{92}$ structures without maximal elements, however, we can use our powerful translatability results between BST$_{\rm NF}$ and BST$_{92}$ (see Chapter 3.6.2) to prove formal results concerning MFB-related notions in BST$_{\rm NF}$ much more easily.

Let us thus begin by recalling the format of basic indeterministic transitions in BST$_{\rm NF}$: an event-type basic indeterministic transition is a pair $\langle \ddot{e}, e\rangle$, written as $\ddot{e} \rightarrowtail e$, where $\ddot{e}$ is a choice set and $e \in \ddot{e}$. On the other hand, a proposition-type basic indeterministic transition is a pair $\langle \ddot{e}, H_e\rangle$, written as $\ddot{e} \rightarrowtail H_e$, where again $\ddot{e}$ is a choice set and $e \in \ddot{e}$. The set $H_{\ddot{e}}$ is defined as $H_{\ddot{e}} = \bigcup_{e\in\ddot{e}} H_e$, and $\Pi_{\ddot{e}}$ is the partition of $H_{\ddot{e}}$ whose elements are the individual H_e, $e \in \ddot{e}$, so that $H_e \in \Pi_{\ddot{e}}$. We define an ordering of choice sets by

$$\ddot{e}_1 < \ddot{e}_2 \text{ iff } \exists e_1' \in \ddot{e}_1 \exists e_2' \in \ddot{e}_2\ [e_1 < e_2],$$

so that we need only small modifications to formulate a parallel to our Definition 5.5 of combinatorial consistency:

Definition 5.11 (Combinatorial consistency in BST$_{\rm NF}$). A set $T = \{\tau_i = \ddot{e}_i \rightarrowtail H_i \mid i \in \Gamma\}$ of basic transitions is *combinatorially consistent* iff for any $\tau_i, \tau_j \in T$:

1. if $\ddot{e}_i = \ddot{e}_j$, then $H_i = H_j$ (i.e., $\tau_i = \tau_j$);
2. if $\ddot{e}_i < \ddot{e}_j$, then $H_{\ddot{e}_j} \subseteq H_i$ (i.e., $\tau_i \prec \tau_j$);

3. if $\ddot{e}_j < \ddot{e}_i$, then $H_{\ddot{e}_i} \subseteq H_j$ (i.e., $\tau_j \prec \tau_i$);
4. if $\ddot{e}_i$ and $\ddot{e}_j$ are incomparable (i.e., if neither $\ddot{e}_i = \ddot{e}_j$ nor $\ddot{e}_i < \ddot{e}_j$ nor $\ddot{e}_i > \ddot{e}_j$), then $H_{\ddot{e}_i} \cap H_{\ddot{e}_j} \neq \emptyset$ (i.e., for some $e_1 \in \ddot{e}_1$ and some $e_2 \in \ddot{e}_2$, $e_1\, SLR\, e_2$).

Thus, in a combinatorially consistent set of transitions, there is no blatant inconsistency (1), there is no order-related inconsistency (2, 3), and there is no inconsistency related to inconsistent initials (4). Having modified the definition of combinatorial consistency for BST_{NF}, the remaining definitions of this chapter remain intact. That is, combinatorial funny business (CFB) is defined as in Def. 5.6, downward extensions are defined as in Def. 5.7, explanatory funny business (EFB) is defined as in Def. 5.8, and at the level of BST_{NF} structures, MFB is defined exactly as in Def. 5.9.

Importantly, thanks to our translatability results (see Chapter 3.6.2 and Appendix A.2, esp. A.2.5), the BST_{NF} analogues of all MFB-related results investigated in Section 5.2 hold. As the use of the translatability results is rather mechanical, we illustrate their working by proving just the BST_{NF} analogon of Theorem 5.1. This should illustrate how to produce counterparts of the remaining (and less difficult) proofs.

Theorem 5.2. *Let $\langle W, < \rangle$ be a BST_{NF} structure without maximal elements. For its set of basic indeterministic transitions, $\mathrm{TR}(W)$, the following holds: There is a subset $T_1 \subseteq \mathrm{TR}(W)$ exhibiting combinatorial funny business iff there is a subset $T_2 \subseteq \mathrm{TR}(W)$ exhibiting explanatory funny business.*

Proof. "$\Rightarrow$" For reductio, assume that there is a $T_1 \subseteq \mathrm{TR}(W)$ that is a case of CFB, but no set of transitions is a case of EFB. By Lemma A.3(4–5), in $\mathscr{W}' =_{\mathrm{df}} \Lambda(\mathscr{W})$, which is a BST_{92} structure, there is some $T_1' = \tilde{\Lambda}(T_1)$ that is a case of EFB, but no set of transitions is a case of CFB ($\tilde{\Lambda}$ is introduced in Def. A.1). This is, however, impossible by Theorem 5.1, which proves the claim.

"$\Leftarrow$" For reductio, we assume that there is a $T_2 \subseteq \mathrm{TR}(W)$ that is a case of EFB, but no set of transitions is a case of CFB. By Lemma A.3(4–5), in $\mathscr{W}' =_{\mathrm{df}} \Lambda(\mathscr{W})$, which is a BST_{92} structure, there is some $T_2' = \tilde{\Lambda}(T_2)$ that is a case of EFB, but no set of transitions is a case of CFB. This is impossible by Theorem 5.1, so the claim is proved. □

Turning to the implications of MFB, the absence of MFB in a BST_{NF} structure again has a welcome consequence for the location of cause-like loci. These are defined as follows:

Definition 5.12 (Cause-like locus in BST_{NF})**.** Let O be a lower bounded chain in a BST_{NF} structure $\langle W, < \rangle$. The set $cll(O)$ is:

$$cll(O) = \{\ddot{e} \subseteq W \mid \exists h\, [h \in \text{Hist} \wedge h \perp_{\ddot{e}} H_{\langle O \rangle}]\},$$

where $h \perp_{\ddot{e}} H_{\langle O \rangle} \Leftrightarrow_{\text{df}} \forall h' [h' \in H_{\langle O \rangle} \rightarrow h \perp_{\ddot{e}} h']$.

Note that in BST_{NF}, a chain O may begin at a member of $\ddot{e} \in cll(O)$, so that in place of the strict ordering relation $<$, we have to work with the weak ordering relation, $\leqslant$, in what follows. As in the BST_{92} case, PCP_{NF} guarantees that *some* elements of $cll(O)$ are (weakly) below O, but this does not necessarily hold for all of them—consider again the EPR scenario depicted in Figure 5.1, which indicates a cause-like locus e_2 that is not in the past of the chain O_1. If the EPR scenario is reconstructed as a BST_{NF} structure, it is clearly an example of MFB, now witnessing Def. 5.11. Again, in all cases in which there is no MFB, each element of $cll(O)$ lies in the (weak) past of O, so we have Fact 5.7, a parallel of Fact 5.4. For the proof, we need an additional Fact about outcome chains in BST_{NF}:

Fact 5.6. *Let O be a lower bounded chain and let $i =_{\text{df}} \inf O$. In BST_{NF}, for any history $h \in \text{Hist}$, if $i \in h$, then $h \in H_{\langle O \rangle}$.*

Proof. Assume for reductio that $i \in h$, but $h \notin H_{\langle O \rangle}$. Let $h_O \in H_{\langle O \rangle}$, and consider $O' =_{\text{df}} O \cap h_O$, which is non-empty by the definition of $H_{\langle O \rangle}$. As O' is an initial segment of O, we have $i = \inf O'$ as well. By our assumption, $O' \subseteq h_O \setminus h$, so by PCP_{NF}, there is some choice set $\ddot{c}$ with $c \in \ddot{c}$, $c \leqslant O'$, and for which $h_O \perp_{\ddot{c}} h$. We have $\ddot{c} \cap h_O = \{c\} \neq \ddot{c} \cap h = \{c'\}$. As $c \leqslant O'$, it must be that $c \leqslant i$ (i being the infimum of O'), so as $i \in h$, we have $c \in h$. But then $\{c, c'\} \subseteq h$, which contradicts Fact 3.13(1) about the unique intersection of choice sets with histories. □

Fact 5.7. *Assume that in a BST_{NF} structure $\langle W, < \rangle$ there is no MFB. Let O be a lower-bounded chain. Then for every $\ddot{e} \in cll(O)$, we have $e \leqslant O$ for some unique $e \in \ddot{e}$.*

Proof. If $cll(O) = \emptyset$, the claim holds vacuously. Let us thus assume $cll(O) \neq \emptyset$, and pick some $\ddot{e} \in cll(O)$. Then there is $h^* \in \text{Hist}$ such that $(*)$ $h^* \perp_{\ddot{e}} H_{\langle O \rangle}$, which means that $h^* \cap \ddot{e} = \{e^*\}$ while for any $h \in H_{\langle O \rangle}$, $h \cap \ddot{e} \neq \emptyset$ and $h \cap \ddot{e} \neq \{e^*\}$.

Since O is lower bounded, it has an infimum $i \leqslant O$. If there is $e \in \ddot{e}$ for which $e \leqslant i$, then $e \leqslant O$, and that member of $\ddot{e}$ is unique (take some $h \in H_i$ and apply Fact 3.13(1)), and so we are done.

We can show that given no MFB, this is in fact the only consistent option. So, assume for reductio that there is no such $e \in \ddot{e}$. It cannot be that *all* $e \in \ddot{e}$ are incompatible with i, due to $(*)$, which implies that any $h \in H_{\langle O \rangle}$ (which contains i) is compatible with some member of $\ddot{e}$. So there has to be some $e \in \ddot{e}$ and some history h such that $\{e, i\} \subseteq h$, and as $e \nleqslant i$ by our assumption, either (1) $e > i$ or (2) $e\, SLR\, i$.

Case (1) leads to a contradiction: Given that $e > i$, there is a chain $l \in \mathscr{C}_e$ for which $i \in l$, and so by the definition of a choice set, $i < e^*$ as $e^* \in \ddot{e}$. Thus for the history h^* witnessing $(*)$, $i \in h^*$, whence by Fact 5.6, $h^* \in H_{\langle O \rangle}$, which contradicts $(*)$.

Case (2) is also excluded. Given that $e\, SLR\, i$, consider the transitions $\tau_1 =_{\mathrm{df}} \ddot{e} \rightarrowtail H_{e^*}$ and $\tau_2 =_{\mathrm{df}} \ddot{\imath} \rightarrowtail H_i$. The set $T =_{\mathrm{df}} \{\tau_1, \tau_2\}$ is combinatorially consistent (see clause 4 of Def. 5.11), and so, by the assumption that there is no MFB, T is consistent. This means in particular that there is a history $h' \supseteq \{e^*, i\}$. Again, by Fact 5.6, $h' \in H_{\langle O \rangle}$, which contradicts $(*)$, because as $e^* \in h'$, it must be that $h' \cap \ddot{e} = \{e^*\}$. □

An analogous fact is true for scattered outcomes—see Exercise 5.5.

5.5 Exercises to Chapter 5

Exercise 5.1. Let O be an outcome chain. Prove that if for all $e \in cll(O)$, we have $e < O$, then $H_{\langle O \rangle} = \bigcap_{e \in cll(O)} \Pi_e \langle O \rangle$.

Hint: The "$\subseteq$" direction is simple. For the other direction, argue indirectly. An explicit proof is given in Appendix B.5.

Exercise 5.2. Suppose that not all elements of $cll(O)$ are in the past of O. Will there still be some formula analogous to the identity established in Exercise 5.1?

Hint: Assume that $e \in cll(O)$. Instead of a basic propositional outcome $\Pi_e \langle O \rangle$, use the basic disjunctive propositional outcome of e that is consistent with O, i.e., the set $\breve{\mathbf{H}} =_{\mathrm{df}} \{H \in \Pi_e \mid H \cap H_{\langle O \rangle} \neq \emptyset\}$. Note the extra set-theoretic layer.

Exercise 5.3. Prove a version of Facts 5.4 and 4.7(2) for $\hat{O}$ a scattered outcome. That is, prove the following facts:
Let $\hat{O}$ be a scattered outcome. (1) If there is no MFB, then for all $e \in cll(\hat{O})$, we have $e < \hat{O}$. (2) If $e < \hat{O}$, then there is a unique basic outcome of e that is consistent with $H_{\langle \hat{O} \rangle}$, which we denote $\Pi_e \langle \hat{O} \rangle$.

Hint: For (1), use Fact 5.4 and Exercise 5.1. For (2), invoke Fact 4.7(2) for $O \in \hat{O}$. An explicit proof for (1) is given in Appendix B.5.

Exercise 5.4. Formulate and prove a version of Facts 5.4 and 4.7(2) for $\check{\mathbf{O}}$ a disjunctive outcome.

Hint: Use Exercise 5.3 for the disjuncts $\hat{O} \in \check{\mathbf{O}}$.

Exercise 5.5. Prove the following parallel to Exercise 5.3 in $\mathrm{BST_{NF}}$:
Let $\hat{O}$ be a scattered outcome in a BST_{NF} structure $\mathscr{W}$. (1) If there is no MFB in $\mathscr{W}$, then [if $h \perp_{\ddot{e}} H_{\langle \hat{O} \rangle}$, then $e \leqslant \hat{O}$ for a unique $e \in \ddot{e}$]. (2) If $e \leqslant \hat{O}$ for some $e \in \ddot{e}$, then no other $e' \in \ddot{e}$ is consistent with $H_{\langle \hat{O} \rangle}$.

Hint: Use the $\mathrm{BST_{NF}}$ versions of the Facts that are useful for Exercise 5.3, (i.e., Fact 5.7 for (1) and Fact 4.9(2) for (2)).