8
On the Innocence and Determinacy of Plural Quantification

8.1 Introduction

Plural logic has undoubtedly become an important component of the philosopher's toolkit.* Many of its applications depend on two alleged virtues: ontological innocence and expressive power. In this chapter, we want to assess whether plural logic has these virtues and thus whether those applications are ultimately justified.

It is commonly assumed that plural logic is ontologically innocent in the sense that plural quantifiers do not incur ontological commitments beyond those incurred by the ordinary first-order quantifiers. This alleged virtue of plural logic is supported by the plurality-based model theory pioneered by Boolos (1985a) and further developed by Agustín Rayo and Gabriel Uzquiano (1999). (For an overview and discussion of this form of model theory, see Chapter 7.) On this model theory, the value of a plural variable is not a set (or any kind of set-like object) whose members are drawn from the ordinary, first-order domain. Rather, a plural variable has *many values* from this ordinary domain and thus *ranges plurally* over this domain. Of course, in ascribing to a plural variable many values, the plurality-based model theory makes essential use of the plural resources of the metalanguage. In a nutshell, on the traditional set-based model theory, a plural variable ranges in an ordinary way over a special domain reserved for variables of its type, whereas on the new kind of plurality-based model theory, a plural variable ranges in a special, plural way over the ordinary domain.[1]

The second alleged virtue of plural logic is expressive power. To see this point, consider first the case of second-order logic with its two kinds of

* Most of this chapter derives from Florio and Linnebo 2016.

[1] Defenses of the innocence of plural logic in the sense just defined are put forth, among others, by Boolos 1984b, and Boolos 1985a, Yi 1999, Yi 2002, Yi 2005, Yi 2006; Hossack 2000, Oliver and Smiley 2001 and 2016; Rayo 2002; McKay 2006.

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traditional set-based model theory. In standard semantics, the second-order quantifiers range over the full powerset of the first-order domain, whereas in Henkin semantics the second-order quantifiers may range over a subset of this powerset. This gives rise to an interesting debate about semantic determinacy. That is, does our linguistic practice single out, relative to a given domain, the interpretation given by the standard semantics as the correct one?[2] An important aspect of this question is that it is only on the standard semantics that second-order logic can truly be said to offer more expressive power than first-order logic. For second-order logic on the Henkin semantics may be regarded as a version of first-order logic, namely a first-order system with two sorts of quantifiers. As such, it has all the main metalogical features of first-order logic: it is complete, compact, and has the Löwenheim-Skolem property. But, for the same reason, it fails with respect to the main accomplishments of second-order logic with the standard semantics. Chiefly, it does not discriminate between importantly different classes of structures, such as countable and uncountable ones, and it fails to ensure the categoricity of arithmetic and analysis, and the quasi-categoricity of set theory.

In this respect, plural logic on the plurality-based model theory, as well as higher-order logic on a parallel higher-order model theory, is thought to provide a significant improvement over second-order logic on the set-based model theory. Indeed, one finds many claims to the effect that plural logic, on the plurality-based model theory, is immune to the threat of non-standard (Henkin) interpretations that confronts higher-order logics on their more traditional, set-based model theory. Nearly all writers who have embraced plural logic on the plurality-based model theory ascribe to this system metalogical properties which presuppose that the semantics is standard rather than Henkin, but without flagging this as a substantive presupposition as one would do as a matter of routine in the case of systems with a set-based model theory.[3] The failure to make this presupposition explicit strongly suggests that the only plurality-based interpretation is the standard one. So it is naturally interpreted as a commitment to the standard semantics rather than the Henkin alternative. Why else claim that plural logic—not plural logic *with standard semantics*—lacks a complete axiomatization and

[2] See Shapiro 1991, Chapter 8. A notable consequence of the view that second-order quantification is determinate is the thesis famously held by Kreisel and others that the Continuum Hypothesis is either true or false (for discussion, see Weston 1976).

[3] See, for instance, Rayo and Uzquiano 1999, 315–18; Yi 1999, 180–1; Hossack 2000, 439–41; Yi 2006, 256–7; McKay 2006, 139–43; Rayo 2007, 215; Oliver and Smiley 2016, 246–9.

compactness, yielding the categoricity of arithmetic and analysis, and the quasi-categoricity of set theory?

In any case, a striking feature of the literature on this novel kind of model theory for plural logic is the near-absence of debate about the semantic determinacy of plural quantification thus interpreted.[4] Indeed, on the plurality-based approach, the only interpretation of the plural quantifiers that has been articulated is the standard one. No analogue of Henkin semantics has been developed. The following diagram sums up the kinds of semantics currently available:

	standard semantics	Henkin semantics
set-based semantics	A. Tarski	L. Henkin
plurality-based semantics	G. Boolos	—

The apparent absence of a plurality-based Henkin semantics has no doubt influenced the ensuing debate. It has encouraged the thought that plural logic on the plurality-based model theory is immune from non-standard interpretation, and thus the thought that plural logic does better than higher-order logic on the set-based model theory in securing a gain in expressive power.

As appealing as this common picture of plural logic may be, we believe that it is far too optimistic. Our aim in this chapter is to develop an alternative picture, one in which both alleged virtues of plural logic—ontological innocence and expressive power—are much less significant than they are made out to be. We argue that set-based and plurality-based model theory are on a par with respect to worries about indeterminacy. So no progress is made by switching from the former to the latter. We do not take a stand on which side of the debate prevails; though in the absence of a compelling argument, we urge caution about the determinacy claims. Moreover, we articulate a generalized notion of ontological commitment according to which plural logic is not, after all, innocent. This provides, for the first time, a precise development of some ideas by Parsons 1990 (section 6), Hazen 1993, Shapiro 1993, and Linnebo 2003. Our focus is on plural logic, though much of what we say would apply, *mutatis mutandis*, to second- and higher-order logics that quantify into predicate position.

[4] The same is true for a higher-order model theory for higher-order logic, though Rayo and Yablo 2001 provides a rare exception.

Our pursuit of the mentioned aims uses as its main tool a semantics for plural logic that fills the gap in the above diagram. Accordingly, the first part of the chapter is devoted to the development and defense of a plurality-based Henkin semantics. (Technical details can be found in this chapter's appendices.) In the second part of the chapter, we reconsider the alleged virtues of plural logic in light of the new semantics. The resulting picture is one in which the role of plural logic as a philosophical tool is substantially diminished.

8.2 A plurality-based Henkin semantics

As announced, our first step is to construct a plurality-based Henkin semantics for plural logic and thus populate the empty quadrant in the above diagram. Although from a technical standpoint this is largely a straightforward adaptation of the familiar set-based Henkin semantics, arguing for its philosophical legitimacy is not straightforward. Once the resources needed to develop a plurality-based Henkin semantics are identified, they must be shown to be in good standing vis-à-vis the resources used to develop the plurality-based standard semantics.

Our object language will be $\mathcal{L}_{\text{PFO}}$. As with the set-based model theory, the plurality-based Henkin models consist of a domain for the first-order quantifiers, a representation of the range of the plural quantifiers, and an interpretation function that specifies the semantic values of the non-logical terminology of the language. The crucial difference is that, in our case, the first-order domain, the range of the plural quantifiers, and the interpretation functions will not be set-theoretic objects.

A domain *dd* for the first-order quantifiers will consist of *some things*—any things in the domain of the metatheory. Next, to represent the range of the plural quantifiers, we need a "collection" *D* of pluralities. We will think of *D* as a plural concept, but an alternative interpretation is available: *D* may be taken to be a superplurality. We remain neutral between these interpretations.

The pluralities 'in' *D* will be exactly those that instantiate *D*. We require that the two domains be connected in the following way: for every *xx* such that *D*(*xx*) (that is, *xx* instantiate *D*), *xx* are among *dd*. In symbols:

$$\forall xx(D(xx) \rightarrow xx \preccurlyeq dd)$$

Finally, we continue to assume the model-theoretic framework presented in Section 7.3. In particular, we assume that the metatheory is equipped with a pairing operation so that an interpretation function can be defined as some ordered pairs *ii* specifying the semantic value or values of each non-logical item in the vocabulary of the object language. (A more precise formulation of the semantics is provided in Appendix 8.A.) As is well known, the standard deductive system for second-order logic is sound and complete with respect to set-based Henkin semantics. As one would expect, this result carries over to the case of plurality-based Henkin semantics for plural logic. A completeness proof is given in Appendix 8.B.

Two aspects of our semantics deserve to be highlighted. First, as in the plurality-based standard semantics, plural quantifiers in our plurality-based Henkin semantics do not range over any special kind of set-like objects. Rather, they range plurally over things in the domain of the first-order quantifiers. Second, the formulation of the semantics requires expressive resources that go beyond those of plural logic. The variable D, used to represent the non-standard interpretations for the plural quantifiers, introduces a form of third-order quantification. As interpreted above, D stands for a plural concept. The alternative is to give D a superplural interpretation. (See Chapter 9 for a discussion of superplurals.) Either interpretation of D might raise worries about the legitimacy of the additional expressive resources required by our semantics. So let us address this issue next.

8.3 The legitimacy of ascending one order

As shown in Chapter 7, a standard version of the plurality-based model theory for PFO does not require expressive resources beyond those of PFO+. So, when describing standard interpretations of $\mathcal{L}_{\mathrm{PFO}}$, there is no need to invoke a variable D. This is only needed if we wish to "select" a non-standard range for the plural quantifiers. In the plurality-based standard semantics, a sentence of the form $\exists vv\ \varphi$ is true in a model of the language just in case *some things* among those in the first-order domain satisfy the formula φ. The formulation of this clause relies only on plural quantification. In our Henkin semantics, we want to impose the additional requirement that the things satisfying the formula also be among the pluralities represented by D.

The expressive economy of the plurality-based standard semantics may be thought to constitute an important advantage of that semantics over our Henkin alternative, especially when coupled with some skepticism about the legitimacy of expressive resources going beyond PFO+. However, we believe that this advantage of the plurality-based standard semantics over our Henkin alternative is not significant. For, as we will now argue, the additional expressive resources required by our semantics are available, and they are needed anyway for independent semantic reasons.

As observed in Section 7.5, it is relatively straightforward to develop a formal system of third-order quantification suitable to develop the plurality-based Henkin semantics (see Rayo 2006). Thus the expressive resources under discussion are available at least in the sense of belonging to the inventory of possible semantic mechanisms. Moreover, there is evidence from natural language that such resources are available also in the stronger sense of being actually in use. On the one hand, familiar arguments for the presence in natural language of quantification into predicate position extend from singular to plural predicates. In Section 6.1, we observed that examples such as 'John is everything we wanted him to be' are naturally regimented using bound variables in predicate position (Higginbotham 1998, 251, but see also Rayo and Yablo 2001). The same conclusion vis-à-vis plural predicates is suggested by analogous examples involving plural predication, such as 'John and Mary are everything we wanted them to be'. This vindicates the interpretation of *D* in terms of plural concepts. On the other hand, it has been argued that natural languages such as English contain superplural expressions (see Oliver and Smiley 2004, Oliver and Smiley 2005, and Oliver and Smiley 2016, Section 8.4; Linnebo and Nicolas 2008), which provides at least *prima facie* support for the superplural interpretation of *D*. (Again, see Chapter 9 for details.)

An important reason why the expressive resources required by our semantics are needed anyway has to do with absolute generality. This emerged in Chapter 7, where a plurality-based standard semantics for PFO was carried out in PFO+ and the same kind of semantics for PFO+ was carried out in a richer metalanguage including either superplural quantification or quantification over concepts. Let us recapitulate the main idea.[5]

An attractive feature of the plurality-based standard semantics is that it allows us to capture models whose first-order domain of quantification contains absolutely everything. By means of the plural resources available in the metalanguage, one can define models in which the first-order quantifiers range over *all objects*. But, if quantification over absolutely everything is

[5] As we noted in Chapter 7, the appeal to resources going beyond PFO+ can also be motivated by parity constraints other than absolute generality.

possible, developing a model theory for plural logic requires the introduction of a new non-logical predicate. Specifically, it requires the introduction of a plural predicate functioning as a satisfaction predicate (see Rayo and Uzquiano 1999). However, once the original language of plural logic has been expanded to include plural predicates, ascending one order higher becomes unavoidable. For it is now known that a model theory for the language expanded to include plural predicates will require a language that is one order higher than plural logic (see Chapter 11). So, if one wants to do justice to the possibility of quantifying over absolutely everything, semantic considerations push the expressive resources up one order.

Whether one interprets this higher-order quantification as quantification over plural concepts or as superplural quantification, semantic reflection will eventually lead the proponent of the plurality-based standard semantics to embrace the expressive resources needed to formulate the plurality-based Henkin semantics. Since the additional resources needed to formulate our Henkin semantics are available and needed anyway for independent semantic reasons, we conclude that the expressive economy of plurality-based standard semantics does not constitute a significant advantage over our plurality-based Henkin semantics.

8.4 Does ontological innocence ensure determinacy?

The previous two sections establish that there exist plurality-based yet non-standard interpretations of a plural language. This is significant. For it is commonplace to maintain that plural logic on the plurality-based model theory is determinate. The view goes back at least to Boolos's famous argument that plural logic is non-firstorderizable. The argument is based on plural logic's alleged ability to distinguish standard from non-standard models of arithmetic (Boolos 1984a, Boolos 1984b, and Boolos 1985a). But of course, if our plurality-based non-standard interpretations are admitted, then plural logic is no better equipped to make such distinctions than, say, a first-order set theory. This contrasts with the widespread view that, when formulated with the help of plural quantification, arithmetic and analysis are categorical, and set theory is quasi-categorical; and relatedly, that plural logic is not axiomatizable (see footnote 3). To be perfectly clear: we are not claiming that all proponents of this view deny or fail to recognize the existence of plurality-based non-standard interpretations. Our claim is that their remarks are potentially misleading because they suggest that the only plurality-based interpretation is the standard one.

It might be responded that, while we have shown that plurality-based non-standard interpretations exist, they can safely be set aside as unintended or illegitimate. Doing so would restore the determinacy of plural logic, which the views just referenced all presuppose. The key question, it seems to us, is whether this response is any better than the analogous response for traditional set-based interpretations. That is, does plural logic on a plurality-based model theory have a *better* claim to determinacy than plural logic on a set-based model theory? Let *Plural Robustness* be the view that the plurality-based model theory is superior in this regard. A defense of Plural Robustness would have to show that the plurality-based standard interpretations are in better standing vis-à-vis their (plurality-based) Henkin rivals than the set-based standard interpretations are vis-à-vis their (set-based) Henkin rivals. Our aim in this section is to articulate and reject a natural defense of Plural Robustness. In the next section, we argue that the two forms of standard semantics are equally well (or poorly) placed against their respective Henkin rivals and that Plural Robustness should therefore be rejected.

Plural Robustness has considerable initial plausibility. An explicit defense is due to Hossack, who nicely lays out the argument as follows:

> The singularist [a proponent of a set-based model theory] cannot solve the problem of indeterminacy, but the pluralist [a proponent of a plurality-based model theory] can. [...] Plural set theory has no non-standard models, so the indeterminacy problem does not arise for pluralism. [...] [P]lural variables range plurally over the very same particulars that the singular variables range over individually. *Therefore the pluralist does not confront an independent problem of identifying what the plural variables range over.* [...] Plural sentences therefore provide the missing additional constraint we were seeking on admissible interpretations. This is why the pluralist [a proponent of a plurality-based model theory] is able to solve the indeterminacy problem, though the singularist cannot do so.
>
> (Hossack 2000, 440–1, our emphasis)

As we understand it, the argument has as its point of departure the other virtue that plural logic is widely believed to enjoy, namely ontological innocence. According to this view—which we call *Plural Innocence*—plural quantification does not incur ontological commitments to entities beyond those in the first-order domain. In particular, plural quantification is not reducible to singular quantification over sets or mereological sums, nor does it involve reference to such entities. Rather, plural variables range plurally over objects in the ordinary, singular domain. And the use of such variables incurs ontological commitments only to objects in this ordinary domain, not to any sets or sums of such objects.

Of course, Plural Innocence is not uncontroversial (see Resnik 1988, Parsons 1990, Hazen 1993, and Linnebo 2003); we too take issue with it below. But if the thesis is false, so is an essential premise of the argument we wish to reject, and we are done. In the remainder of this section we therefore proceed on the assumption that the thesis is true.

It would be very natural to think that Plural Innocence supports Plural Robustness. Since the plural quantifiers do not range over any kind of "plural objects", such as the subsets of the first-order domain, we do not—as Hossack observes—"confront an independent problem of identifying what the plural variables range over." Plural quantifiers just range plurally over the very same domain that the singular quantifiers range over. This contrasts with the set-based model theory for second-order logic, where the standard interpretation requires one to single out a range for the second-order quantifiers that contains *all* the subsets of the first-order domain. The possibility of failing to single out such a range gives rise to the possibility of non-standard interpretations in the set-based model theory. Since Plural Innocence ensures that no new range of entities needs to be singled out for the plural quantifiers, this thesis renders plural logic on the plurality-based model theory immune to non-standard interpretations, or at least more immune than plural logic on the set-based model theory.

However, we contend that our plurality-based Henkin semantics is just as innocent as the plurality-based standard semantics. On both semantics, plural variables range plurally over objects in the ordinary, first-order domain. The only difference is that, on our semantics, the range of the plural variables can be so restricted as to make room for general interpretations in addition to the standard one.

In fact, this notion of ontological innocence can be understood in a less and in a more demanding way. The less demanding way requires the ontological innocence of the plural quantifiers. Then our claim that plural quantification is innocent on the plurality-based Henkin semantics is incontrovertible. Since the semantics is plurality-based, the plural quantifiers do not range over special kinds of objects. They range plurally over the objects in the first-order domain. This is the sense of ontological innocence operative in the argument from Plural Innocence to Plural Robustness spelled out above.

One might also want innocence in a more demanding form that includes the resources employed by the model theory itself. (For instance, the plurality-based model theory uses a pairing operation which is not ontologically innocent.) Our semantics may possess a high degree of innocence even in this more demanding sense. For there are arguments, akin to the one developed by Boolos himself, for the ontological innocence of the third-order quantification that binds the variable *D*. This is fairly straightforward in the case of the "superplural" interpretation of *D*. As for the official interpretation of *D* as a plural concept, one may argue for its innocence along the lines of Rayo and Yablo 2001 (see also Wright 2007). Moreover, in the more demanding sense of innocence the two semantics appear to be on equal footing. As argued above, an appeal to higher-order resources is unavoidable when the defender of the plurality-based standard semantics attempts to articulate a model theory for a language containing plural predicates (as she will have to do when formulating the model theory for her own metalanguage). So, when seen from this perspective, the semantic machinery of the plurality-based standard semantics is no more innocent than that of its Henkin competitor.

We conclude that, no matter which understanding of Plural Innocence is assumed, the plurality-based Henkin semantics has as good a claim to innocence as the standard semantics. This shows that Plural Innocence does not support Plural Robustness. For there is an innocent semantic option, namely the plurality-based Henkin semantics, for which Plural Robustness fails. This poses a challenge for defenders of Plural Robustness. If their claim is not supported by Plural Innocence, then what, if anything, does support it?

8.5 The semantic determinacy of plural quantification

The question of semantic determinacy, we recall, is whether the unique correct interpretation of our quantificational practice is the one associated with the standard interpretations. We contend that plural logic with the traditional set-based model theory and plural logic with plurality-based model theory are on a par with regard to semantic determinacy.

Two remarks about this *parity thesis*—as we shall call it—are in order. First, our contention is that the determinacy claims concerning plurality-based model theory stand or fall with the corresponding determinacy claims concerning set-based model theory. We remain agnostic about whether they stand together or fall together; though as mentioned, in the absence of compelling arguments, we urge caution about the determinacy claim. Second, the parity thesis includes, but goes beyond, the claim that Plural Robustness is false. If Plural Robustness is false, then no *additional* assurance of determinacy is gained by switching from a set-based to a plurality-based model theory. Our parity thesis consists of this claim and its converse.

We submit that the parity thesis has a great deal of plausibility whenever the domain of quantification is set-sized, as is the case of higher-order quantification over the natural numbers or the reals. Assume that the domain is a set d, and let dd be its elements. (We indicate this relationship by writing $d = \{dd\}$.) In the case of the set-based model theory, we need to single out a special object—the standard interpretation—from a large pool of other objects—the Henkin interpretations. In the case of the plurality-based model theory, we need to single out a special way of ranging over the domain dd—the standard way—from a large pool of other ways of ranging over dd—the Henkin ways. *Why should it be any easier—or harder—to single out an object from a pool of objects than to single out a way from an isomorphic pool of ways?* Since the two tasks are isomorphic, whatever can be said in one case carries over to the other.

While these considerations capture the gist of our argument, some work remains to be done to establish the parity thesis in full generality, that is, independently of the assumption that the domains of the plurality-based model theory are set-sized.[6] Consider first the possibility that plural logic is determinate on the plurality-based model theory and *in*determinate on the set-based model theory. If plural logic is determinate on the plurality-based model theory, this means that the only plurality-based Henkin interpretation is the standard one. *A fortiori*, no non-standard plurality-based Henkin interpretation can be countenanced in which the elements dd of the domain form a set d. But this is incompatible with the idea that non-standard *set-based* Henkin interpretations are legitimate, since the legitimacy of an interpretation would then depend entirely on the way in which the interpretation is described. Non-standard Henkin interpretations with set-sized domains would be legitimate when described set-theoretically but illegitimate when described with the help of higher-order resources. So we must conclude that plural logic on the set-based model theory is determinate too, and thus Plural Robustness is false.

[6] On the critical plural logic we develop in Chapter 12, every plurality defines a set and the mentioned assumption is always satisfied. Thus, if we are right, the considerations of this paragraph and the next become redundant.

We now consider the converse. Might plural logic be determinate on the set-based model theory but not on the plurality-based model theory? We believe the answer is negative. The determinacy of plural logic on the set-based model theory rules out non-standard interpretations whenever the domain is set-sized. So, if plural logic admits non-standard interpretations on the plurality-based model theory, such interpretations could only arise when the domain is too large to form a set. As a result, the type of interpretation legitimate for the plural quantifiers would vary depending on the size of the domain. That is, the interpretation of the plural quantifiers would be standard whenever the domain forms a set but may be non-standard when the domain is too big to form a set. Why should that be so? Since plural quantifiers are treated as logical, this asymmetry would be implausible. Thus, it appears that if plural logic is determinate on the set-based model theory, it must also be determinate on the plurality-based model theory.

8.6 The metaphysical determinacy of plural quantification

We now briefly examine a different determinacy question pertaining to plural and other forms of higher-order quantification. This question is metaphysical and challenges a presupposition of the semantic determinacy question discussed above. Consider a domain $d = \{dd\}$. Is there a determinate maximal set of subsets of *d* or a determinate maximal concept of being a subplurality of *dd*? Where the semantic question asks whether our practice uniquely singles out as correct a maximal interpretation of the plural and higher-order quantifiers, the metaphysical question asks whether the sort of thing we are attempting to uniquely single out even exists.

Many philosophers and mathematicians have defended a negative answer in cases where the domain is infinite. Their skepticism is fueled in part by our inability to answer some fairly immediate questions about the set of subsets (or its analogue in the case of plurals). A well-known example is Cantor's Continuum Hypothesis, which provably resists an answer by ZFC and has so far resisted an answer from widely accepted further axioms.

The metaphysical question is interesting in part because it might provide a reason to prefer the plurality-based model theory over the set-based model theory. For metaphysical determinacy might hold in the case of pluralities but fail in the case of sets. However, we don't think that this is so. More generally, we believe there is a determinate totality of subpluralities of the things *dd* that serve as our domain if and only if there is a determinate totality of subsets of $d = \{dd\}$. To see this, consider the conditions that would define these two totalities, namely '$xx \preccurlyeq dd$' and '$x \subseteq d$'. We contend that the pluralities satisfying the former condition are in one-to-one correspondence with the sets satisfying the latter. Provided that this contention is right, it is hard to see how one of the conditions could define a determinate totality while the other fails to do so.[7]

It remains only to defend our contention. For every set satisfying the condition '$x \subseteq d$', the plurality of its elements satisfies the condition '$xx \preccurlyeq dd$'. Next, we observe that every plurality satisfying the latter condition forms a set, by the axiom of Separation and the fact that $d = \{dd\}$. Moreover, this set satisfies the former condition. Thus, we can go back and forth between sets and pluralities satisfying the two conditions. Indeed, we obtain the promised one-to-one correspondence by observing that a set is sent to the plurality of its elements, which are sent back to the original set.

What about the case where *dd* do not form a set? Our considerations leave open whether in this case there is a determinate totality of subplutalities of *dd*. But any trouble here would only serve to limit the advantage of the plurality-based model theory over its set-based rival.

8.7 A generalized notion of ontological commitment

Let us finally consider the debate about the ontological commitments of plural logic. According to Boolos and his followers, plural languages are ontologically innocent. For instance, when you say that you had a bowl of Cheerios for breakfast, you are talking exclusively about the Cheerios, not about a set of them, their sum, or any kind of "plural entity". Call this the *narrow notion* of ontological commitment. It will be made precise below. We have seen how to develop a model theory for a plural object language in a plural metalanguage in which the semantic values of a plural variable is one or more objects from the ordinary first-order domain. This model theory upholds the view that the use of plural quantifiers incurs no new commitments to sets, sums, or any other kind of plural entities (Boolos 1985a).

The opposite side responds by disputing the *prima facie* case for the ontological innocence of plural quantification. For instance, commenting on

[7] Here we rely on an analogue of Replacement for our intuitive notion of determinate totality. In Chapter 12 we develop a notion of extensional definiteness that obeys the mentioned principle.

Boolos's example 'there are some sets which are all and only the non-self-membered sets', Parsons writes:

> in a context of this kind a quantifier like 'there are some sets' is saying that there is a plurality of some kind. Cantor's notion of 'multiplicity' and Russell's of 'class as many' were more explicit versions of this intuitive notion, both attempting to allow that pluralities might fail to constitute sets. (Parsons 1990, 326)

(See also Hazen 1993, Shapiro 1993 and Linnebo 2003, as well as Resnik 1988 for a more "singularizing" version of the view.) The model theory developed in a plural metalanguage cuts both ways. Both parties to the debate can agree that if the use of the plural quantifiers in the metalanguage is innocent, then so is their use in the object language. One party will assert the antecedent, while the other will deny the consequent. Thus there are two internally coherent views on the matter, and we appear to have reached a standoff.

The best way to make progress, we believe, is by considering two competing construals of the notion of ontological commitment. If one understands this notion in the *narrow sense* (as concerned exclusively with the existence of objects) and takes an object to be the value of a singular first-order variable, then the plurality-based model theory does indeed show that plural logic is ontologically innocent. For this model theory does not use singular first-order variables to ascribe values to the plural variables of the object language; rather, this ascription is made by means of plural variables of the metalanguage.

There is, however, a *broad notion* of ontological commitment. According to this notion, ontological commitment is tied to the presence of existential quantifiers of *any logical category* in a sentence's truth conditions. If this notion is operative, then even the plurality-based model theory shows that plural locutions incur additional ontological commitments. The resulting view is an analogue of that espoused by Frege when he held that quantification into predicate position incurs its own distinctive kind of commitment, not to objects but to concepts.

Before a meaningful debate can take place about which notion of commitment is more interesting and appropriate, both notions need to be clearly articulated. We will now show that our plurality-based Henkin semantics is precisely the tool we need in order to articulate the more inclusive notion.

Let us begin with the narrow notion, which ties ontological commitment to the values of singular first-order variables. Here is one of Quine's more helpful statements of his view.

> The ontology to which an (interpreted) theory is committed comprises all and only the objects over which the bound variables of the theory have to be construed as ranging in order that the statements affirmed in the theory be true. (Quine 1951, 11)

This suggests the following precise definition. A theory T is committed to κ objects that are φ if and only if every model of T contains at least κ objects satisfying the formula φ.

In light of our work in earlier sections, it is straightforward to extend this criterion of commitment to plural variables. In both cases, the formulation of the criterion relies on the use of quantifiers that are assumed to be antecedently understood in the metatheory. A theory T is committed to κ pluralities that are φ if and only if every *plurality-based Henkin model* of T has a range D of the plural quantifiers containing at least κ pluralities satisfying the formula φ. (Of course, the proper way to talk about many pluralities is by means of plural concepts or super-pluralities, as discussed above.)[8] It is important to note that the appeal to plurality-based *Henkin* models is essential. If we had instead appealed to Boolos-style plurality-based *standard* models, then the ontological commitment of any theory involving plural quantifiers would be trivially determined by the ontological commitments of the first-order quantifiers of the theory. For any theory would incur commitments to all and only the pluralities based on the objects to which the theory is committed. By contrast, the definition of commitment to pluralities that we have proposed has the desirable feature that a theory's commitment to pluralities can add information over and above its commitment to objects.

The value of this information is most easily appreciated when it is denied that there is a single maximal interpretation of the plural quantifiers, that is, when the metaphysical determinacy of these quantifiers is denied. When this is denied, there can be no hope of determining the theory's commitments to pluralities directly on the basis of its commitments to objects. Instead, one must assess the commitments to pluralities independently, using the

[8] In a perfectly analogous way, we can define a notion of ontological commitment incurred by quantification into predicate position.

generalized Quinean criterion set out above. To illustrate this point and, more generally, the value of our notion of commitment to pluralities, let us consider a puzzle due to Hazen (1993, 135). Consider the scheme of plural comprehension:

$$\exists x\, \varphi(x) \;\rightarrow\; \exists xx\, \forall x\, (x \prec xx \leftrightarrow \varphi(x))$$

Which instances of the scheme should we accept? The traditionalist (whose position is enshrined in the standard semantics for plural logic) accepts *all instances*—with the obvious and uncontroversial proviso that $\varphi(x)$ not contain xx free. This traditional view faces various challenges. According to predicativists, for example, we should only accept plural comprehension axioms that are predicative in the sense that $\varphi(x)$ does not contain any bound plural variable. And according to the critical plural logic we develop and defend in Chapter 12, plural comprehension needs to be restricted so as to avoid commitment to a universal plurality or other pluralities that are not properly circumscribed. As Hazen observes, there is a clear and intuitive sense in which these non-traditional views are committed to fewer pluralities than the traditionalist. Thus, if a notion of commitment is to be worth its salt, it must capture this sense. And this is exactly what our broad notion of ontological commitment enables us to do. Using this notion, we can maintain that the traditionalist, unlike the predicativist, takes on commitments to impredicatively defined pluralities. By contrast, had we assumed the plurality-based standard semantics, this conclusion would not have been available.

Our notion of commitment to pluralities is also useful in cases where the metaphysical determinacy of plural quantification is granted. When this is granted, there is a notion of commitment to pluralities—namely the one associated with the maximal interpretation of the plural quantifiers—according to which these commitments supervene on the commitments to objects. Once the commitments to objects of a theory have been determined, so have the commitments to pluralities associated with the maximal interpretation. It must therefore be conceded that there is no *further* question concerning the theory's commitments to pluralities. However, the supervenience of one parameter on certain others does not mean that there is no genuine and theoretically interesting question as to the value of this parameter! In our case, even if metaphysical determinacy ensures that the commitments to pluralities of a theory are uniquely determined by its commitments to objects, we still want to know how many, and what kind of, pluralities the theory is committed to. Even if one believes in the metaphysical determinacy of plural quantification, one may have views about how strong, or mathematically rich, one's notion of subplurality is (e.g. Shapiro 1993 and Parsons 2013). The notion of commitment to pluralities that we have articulated allows such views to be expressed.

An example might be helpful. Assume that the commitments to objects of a theory involve an omega-sequence, which we may think of as the natural numbers. If metaphysical determinacy holds, then there is a sense in which the commitments to pluralities are determined by the commitments to objects. Even so, we can ask *which* pluralities the theory is committed to. Different answers are possible. For instance, a theorist who believes the axiom of constructibility, $V = L$, may answer that the only subpluralities of the "natural numbers" to which the theory is committed are the ones that are constructible (in the sense that they correspond to sets in the constructible hierarchy L). Another theorist—who rejects the axiom of constructibility—may disagree and insist that the commitments to pluralities go beyond the constructible ones.

It may be objected to the broad notion of commitment that the commitments associated with plural and higher-order quantifiers is not a form of *ontological* commitment but perhaps, following Quine, of *ideological* commitment. We see little point in quarreling over terminology. A more interesting question is whether ideological commitments in this sense give rise to fewer philosophical problems, or whether they are philosophically less substantive, than ontological commitments narrowly understood. It is far from obvious why this should be so. Indeed, it seems to us that questions involving the broad notion of commitment can be just as interesting and problematic as those involving the narrow ones. How are we to understand the values of different sorts of variables—in extensional or intensional terms? Which such values are there and which comprehension axioms should we therefore accept? How do we trace a value from one context (e.g. time or possible world) to another?

In light of these considerations, we are inclined to agree with Parsons when he writes that, on the narrow notion,

> ontological commitment may just not have the significance that both nominalists and many of their opponents attribute to it, or that Boolos seems to attribute to it in the case of proper classes. That might be a victory for the Innocence Thesis, but it would be a Pyrrhic victory.
>
> (Parsons 2013, 173)

Thus, if Parsons is right, then either Plural Innocence is false, or else it is true but not nearly as interesting as one might have thought.

Our primary goal in this section has been not so much to adjudicate this debate as to prepare the ground for a precise and well-informed debate. We have done so by using our plurality-based Henkin semantics to provide a clear articulation of a generalized notion of commitment. Still, on the picture emerging from our discussion, the role of plural logic as a philosophical tool appears substantially diminished. As we have shown, plural logic is not immune from the threat of non-standard interpretations, and the promised gain in expressive power has not been established. Although we do not take a stand on which side of the debate prevails, we have, in the absence of a compelling argument, urged caution about the determinacy claims.

Further, there is a precise and interesting sense in which plural logic may be said to be committing. Whether this commitment is ontological or ideological, it is a full-fledged form of commitment nonetheless.

8.8 Applications reconsidered

The conclusion we have just reached is in stark contrast to the common picture of plural logic canvassed in Section 2.5. According to that picture, plural logic is "pure logic" and hence also ontologically innocent, and it provides greater expressive power than first-order logic. In Section 2.6, we explained how this common picture has sustained some important applications of plural logic, thereby contributing to the view that plural logic has great philosophical significance. We focused on four such applications, which concern logicism, nominalism, semantics, and categoricity arguments in philosophy of mathematics. Let us briefly reconsider these applications in light of the preceding discussion.

By itself, our rejection of Plural Innocence and Plural Robustness does not force any logical revision. Our arguments can be accepted while retaining the traditional version of plural logic that we have used so far. This means that our arguments do not affect technical applications of plural logic, including to logicism. Logicists can employ plural logic in developing their views, provided that such views are compatible with the failure of Plural Innocence.

The case of nominalism is different. The use of plural logic in some nominalistic projects relies essentially on the alleged ontological innocence of plural quantification. Eliminating certain kinds of complex objects in favor of pluralities will be less significant if one accepts that plural quantification incurs commitments that go beyond those of first-order quantification. Nominalists can trade some commitments to objects for new commitments to pluralities. But they will still face some substantive metaphysical and epistemological questions about the nature and extent of the new commitments.

In semantics, the main application of plural logic was to develop a plurality-based model theory. This application is unaffected by our conclusions concerning the innocence and determinacy of plural quantification. Indeed, our argument for the existence of non-standard interpretations of plural logic used precisely the framework of plurality-based model theory. What about absolute generality? Since the Henkin semantics subsumes all the standard interpretations, the new semantics is just as congenial to absolute generality as the standard one.[9] (As mentioned, however, the use of plural logic to represent absolute generality faces an entirely different challenge; see Chapters 11 and 12.)

Finally, plural logic has been held up as an appealing alternative to second-order logic in order to overcome the expressive limitation of first-order logic and hence make available categorical characterizations of important mathematical structures. Given our rejection of Plural Robustness, this application of plural logic becomes highly problematic. Because Plural Robustness fails, plural logic is *not* immune to the threat of non-standard interpretations, and the desired gain in expressive power remains in doubt.

In sum, we have found that plural logic lacks some key features that pure logic has been thought to have, in particular ontologically innocence; nor is the logic immune to worries about indeterminacy.[10] This calls into question some popular applications of the system. As we have stressed, however, plural logic has other important applications, particularly in accounting for sets, which do not require those features. Plural logic is thus of great interest and theoretical value, just not in the way that many of its earlier proponents have argued.

[9] In Section 2.6, we claimed that if plural talk is not ontologically innocent, then the use of plural logic to capture absolute generality would appear to be undermined. The claim was made in the context of what we now call the narrow notion of ontological commitment and was explicitly linked to the existence of set-like objects (see p. 27).

[10] A more comprehensive summary of our view on the extent to which plural logic counts as pure logic can be found in the final section of the book.

Appendices

8.A Henkin semantics

Let us provide a more precise formulation of the plurality-based Henkin semantics for PFO. This semantics is a variant of the standard semantics illustrated in Section 7.3. The difference is that some key definitions are relativized to a plural concept D functioning, in effect, as a domain for the plural quantifiers.

We want to characterize a Henkin interpretation. We start with a plurality dd serving as the first-order domain. Then we relativize to D the previous definition of an interpretation function ff (Section 7.3) by adding this requirement: for every plural constant tt, there is at least one x such that $\langle tt, x\rangle \prec ff$, and for all xx such that

$$\forall y(y \prec xx \leftrightarrow \langle tt, y\rangle \prec ff)$$

it holds that $D(xx)$. The requirement captures the idea that, in any interpretation function, a plural constant tt denotes some things that instantiate D, specifically those appearing as second coordinates of pairs whose first coordinate is tt.

An interpretation of the object language is obtained by combining the domains dd and D with an interpretation function ff relative to dd and D. Given how these three components have been characterized, an interpretation is not an object or the value of a single higher-order variable. But such components can be 'merged' so as to be represented by a single variable I, whose value is a plural concept (or, alternatively, a superplurality) that codes the three components. Quantifying over interpretations then amounts to quantifying over plural concepts (or superpluralities). For convenience, however, we speak of an interpretation as a triple and represent it as $\langle dd, D, ff\rangle$.[11]

[11] Here is one way of doing the coding. In keeping with the notation introduced in Section 7.5, we let '$\langle y, xx\rangle$' stand for the ordered pairs obtained by pairing y with each x in xx. Then, given dd, D, and ff, there is I such that for all yy, $I(yy)$ if and only if one of the following holds:

(1) $yy \approx \langle a, dd\rangle$;
(2) there are zz such $D(zz)$ and $yy \approx zz$;
(3) $yy \approx \langle b, ff\rangle$;

where a and b are any two distinct objects. The plural concept (or superplurality) so characterized can be used as a surrogate for the triple $\langle dd, D, ff\rangle$.

We also relativize to D the previous definition of a variable assignment *ss*: we require that for every plural variable *vv*, there is at least one x such that $\langle vv, x\rangle \prec ss$, and for all *xx* such that

$$\forall y(y \prec xx \leftrightarrow \langle cc, y\rangle \prec ss)$$

it holds that $D(xx)$. This means that a plural variable *vv* is assigned some things that instantiate D, specifically those appearing in the assignment as second coordinates of pairs whose first coordinate is *vv*.

Before defining the notion of truth in an interpretation, let us introduce some additional notation, following our convention in Section 7.2. For any model $\langle dd, D, ff\rangle$, variable assinglment *ss*, and non-logical expression E, let $[\![E]\!]_{\langle dd,D,ff\rangle,ss}$—but, in fact, we will write $[\![E]\!]_{ff,ss}$ leaving the domains implicit—indicate the semantic value or values of the expression E relative to the model $\langle dd, D, ff\rangle$ and the variable assignment *ss*.

We are ready to give the inductive characterization of truth in an interpretation via satisfaction clauses. In the Henkin semantics, a formula φ is true in an interpretation $\langle dd, D, ff\rangle$ relative to a variable assignment *ss* based on D, written $\langle dd, D, ff\rangle \vDash_H \varphi\ [ss]$, just in case:

(i) if φ is $t_1 = t_2$, then $[\![t_1]\!]_{ff,ss} = [\![t_2]\!]_{ff,ss}$;
(ii) if φ is $S^n(t_1, \dots, t_n)$, then $\langle [\![t_1]\!]_{ff,ss}, \dots, [\![t_n]\!]_{ff,ss}\rangle \prec [\![S^n]\!]_{ff,ss}$;
(iii) if φ is $\exists v\, \psi$, then $\langle dd, D, ff\rangle \vDash_H \psi\ [ss(v/x)]$ for some $x \prec dd$;
(iv) if φ is $\exists vv\, \psi$, then $\langle dd, D, ff\rangle \vDash_H \psi\ [ss(vv/xx)]$ for some $xx \preccurlyeq dd$ such that $D(xx)$;
(v) the clauses for the logical connectives are the obvious ones.

As usual, the satisfaction clauses ensure that if φ is a sentence, we can ignore variable assignments.

We say that an interpretation $\langle dd, D, ff\rangle$ is *faithful* if it satisfies every instance of the plural comprehension scheme:

$$\exists v\, \varphi(v) \ \rightarrow\ \exists vv \forall v(v \prec vv \leftrightarrow\ \varphi(v))$$

Logical consequence is defined with respect to faithful models only. (Of course, when we are interested in systems with restricted plural comprehension, we modify the definition so as to consider all models that satisfy the relevant comprehension scheme.) A sentence φ is a consequence of a set of sentences Δ in the Henkin semantics (written '$\Delta \vDash_H \varphi$') if, for every faithful interpretation $\langle dd, D, ff \rangle$ satisfying every member of Δ, $\langle dd, D, ff \rangle \vDash_H \varphi$.

8.B Completeness of the Henkin semantics

Let us now prove that traditional plural logic defined in Section 2.4 is sound and complete with respect to the plurality-based Henkin semantics formulated above. We use the symbol $\vdash$ to denote the relation of provability is this system. We want to show that, for any sentence φ and set of sentences Δ, $\Delta \vDash_H \varphi$ (if and) only if $\Delta \vdash \varphi$. The shortest and most elegant way of proving this is through a squeezing argument.

First, it is a routine exercise to verify that traditional plural logic is sound with respect to the plurality-based Henkin semantics, which means that

(8.1) if $\Delta \vdash \varphi$, then $\Delta \vDash_H \varphi$.

Now consider the familiar set-based Henkin semantics for second-order logic. (See, for instance, Shapiro 1991, Section 4.3.) It is relatively straightforward to adapt this semantics to PFO. An interpretation is given by a triple $\langle d_1, d_2, f \rangle$, where d_1 is a non-empty set, d_2 (the range of the plural quantifiers) is a set of non-empty subsets of d_1, and f is interpretation function from the non-logical vocabulary of the language to elements of d_1 (for singular terms), elements of d_2 (for plural terms), and possibly empty sets of n-tuples from d_1 (for singular n-ary predicates). Plural membership ('is one of') is systematically interpreted as set-theoretic membership. Let us use the symbol $\vDash_h$ for the resulting relation of logical consequence when confined to *faithful* interpretations, namely those satisfying every instance of plural comprehension. So $\Delta \vDash_h \varphi$ means that φ is a logical consequence of Δ in the set-based Henkin semantics. In other words, for every faithful interpretation $\langle d_1, d_2, f \rangle$, if $\langle d_1, d_2, f \rangle \vDash_h \psi$ for every member ψ of Δ, then $\langle d_1, d_2, f \rangle \vDash_h \varphi$.

It is evident that every set-theoretic model just described corresponds to a plurality-based Henkin model. Take any model $\langle d_1, d_2, f \rangle$. Then its corresponding plurality-based model $\langle dd, D, ff \rangle$ is one in which dd are the elements of d_1, D is the concept of being a plurality that forms a set in d_2, and *ff* is an interpretation function that matches *f*.[12] This correspondence establishes the following:

(8.2) If $\Delta \vDash_H \varphi$, then $\Delta \vDash_h \varphi$.

Finally, we can easily adapt the standard proof that second-order logic is complete with respect to the set-based Henkin semantics (Henkin 1950) to show that traditional plural logic is complete with respect to the set-based Henkin semantics outlined in the paragraph just above.

This gives us that

(8.3) if $\Delta \vDash_h \varphi$, then $\Delta \vdash \varphi$.

Putting together the last three numbered claims, we obtain the result we wanted to prove:

(8.4) $\Delta \vDash_H \varphi$ (if and) only if $\Delta \vdash \varphi$.

So traditional plural logic is complete with respect to the plurality-based Henkin semantics. Therefore, it is also compact and axiomatizable.

[12] Specifically, if $f(t) = x$, then $\langle t, x\rangle \prec ff$. If $f(aa) = \{xx\}$, then $\forall y(\langle aa, y\rangle \prec ff \leftrightarrow y \prec xx)$. And, for any n-tuple $\langle x_1, \ldots, x_n\rangle$, $\langle S^n, \langle x_1, \ldots, x_n\rangle\rangle \prec ii$ if and only if $\langle x_1, \ldots, x_n\rangle \in f(S^n)$.