10
Plurals and Modals

10.1 Introduction

Just as the interaction between first-order quantification and modalities raises a number of interesting and difficult questions, so does the interaction between plural quantification and modalities.* In this chapter, we discuss the central aspects of the problem of how plurals and modalities should be combined. We have two main goals. The first is to provide a useful map of the current literature. The second is to argue for the metaphysical claim that pluralities are modally rigid. What does this claim mean?

Consider some things, and choose any one of them. Is the chosen thing *necessarily* one of the things from which it was chosen? It is usually assumed that the answer is positive—so long as the things in question still exist.[1] If some things did not include our chosen thing, then these things would simply not be the things with which we started, that is, the things from which we made a choice. If the things from which we chose exist at all, then *necessarily*, whenever they exist, they include the chosen thing. Likewise, if some other thing is *not* one of the things from which we chose, then this too is a matter of (conditional) necessity. With the help of an existence predicate E, these two modal constraints on plural membership can be formalized as follows:[2]

(Rgd$^+$) $\Box\forall x\forall yy(x \prec yy \rightarrow \Box(Eyy \rightarrow x \prec yy))$

(Rgd$^-$) $\Box\forall x\forall yy(x \nprec yy \rightarrow \Box(x \nprec yy))$

The claim that pluralities are *rigid* is the conjunction of these constraints, which we abbreviate as (Rgd).

* This chapter is a revised and expanded version of Linnebo 2016.

[1] See Williamson 2003, 456–7; Rumfitt 2005, section VIII; Williamson 2010, 699–700; Uzquiano 2011; Williamson 2013, 246–8. The view is challenged in Hewitt 2012a.

[2] One might have thought that more existential presuppositions were needed. We will later see that that isn't so.

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The rigidity principles can be regarded as a form of extensionality, stating that pluralities are tracked extensionally across possible worlds. The ordinary principle of extensionality states that if every one of *these things* is one of *those things* and vice versa, then *these things* and *those things* are the very same things. (We will return to the question of how best to formalize this principle.) This is widely assumed to provide a criterion of identity for pluralities; and like criteria of identity in general, the principle is widely thought to hold of necessity. Even so, the principle provides no information about how pluralities are tracked across possible worlds. The rigidity principles fill this gap. They tell us that necessarily any given things have their members by necessity. A plurality is therefore not allowed to vary in its membership across possible worlds. Any variation in membership would result in our talking about some other things, not the things in question.

To appreciate how the rigidity claims for pluralities have real bite, it is useful to contrast pluralities with *groups*, such as teams, clubs, committees, and the like.[3] These entities do not have their members necessarily. Consider the Department of Philosophy of the University of Athens. It might have had other members than it in fact has: Sophus might have been hired instead of Sophia. Someone who *is* a member of this department might not have been so; and someone who *is not* a member might nevertheless have been one. Thus, a group such as a philosophy department does not have its members necessarily. The same is true of other typical groups.

If pluralities are rigid but groups are not, what explains the difference? One might try to appeal to the distinction between many and one. A plurality is many, while a group is one. But this distinction cannot explain the relevant modal difference between pluralities and groups. For a set is one and yet has its members necessarily. So rigidity is compatible with being one.

A far more promising response arises from the following basic thought: a plurality is nothing over and above its members and is thus fully specified when we have circumscribed its members. Tracking a plurality across possible worlds is therefore trivial: it is simply a matter of tracking its members. Unlike a plurality, a group *is* something over and above its members: it is not fully specified when we have circumscribed its members. For example, we additionally need to specify its membership criterion. A group such as a department of philosophy will be associated with a membership criterion that is sanctioned by the statutes of the university. So tracking a group across

[3] For useful discussions of groups, see, e.g., Landman 1989a, Landman 1989b, and Uzquiano 2004b.

possible worlds is not trivial; it goes beyond tracking each of its members.[4] By contrast, since a plurality is nothing over and above its members, there is no material available that might underwrite a non-trivial tracking across possible worlds. All we have to go on are the members. So the only way to track a plurality is the trivial one, which ensures plural rigidity.[5]

Sets—understood according to the iterative conception—resemble pluralities in this respect, with the additional and complicating factor that their members are "bound together" into a single object (see Section 4.4).

In what follows, we attempt to clarify and develop the basic thought that a plurality is fully specified when we have circumscribed its members. The result will be a disentangling and clarification of several aspects of the basic thought. We will find that plural rigidity figures at the heart of a network of ideas having to do with what we will call *the extensional definiteness* of pluralities.

10.2 Why plural rigidity matters

The question whether pluralities are rigid has emerged as the central question about the interaction between plural quantification and modalities. The reason for this has to do with the important ramifications of the question in philosophical logic, metaphysics, and the philosophy of mathematics.

One example is the debate about the relation between plural logic and second-order logic discussed in Chapter 6. Can plural logic be replaced by monadic second-order logic or even reduced to it? Or is some reduction in the opposite direction possible? If pluralities are rigid, then the two forms of logic have different modal profiles. For the modal behavior of predication is clearly non-rigid, as the following sentences illustrate.

(10.1) Timothy Williamson is a philosopher, but he might not have been one.

(10.2) Hillary Clinton is not a philosopher, but she might have been one.

[4] Uzquiano 2018 provides a systematic development of the idea that, since tracking them is trivial, pluralities can be seen as a limiting case of generally non-rigid groups.

[5] Roberts (forthcoming) provides a systematic investigation of this basic thought, resulting in a defense not only of (RGD) (which is our main concern in this chapter) but also some further modal principles.

As we have seen, however, the difference between the modal profile of predication and that of plural membership makes at least one kind of reduction problematic (see Section 6.4).

A related example concerns the semantics of predication. In Section 7.5, we showed that if we have ordered pairs at our disposal, it is technically possible to use plurals to give a semantic analysis of predication. Specifically, we can take the semantic value of a predicate to be the plurality of tuples of which the predicate is true. In addition to the lack of homophonicity even on the intended interpretation and the need for *ad hoc* tricks to handle predicates that are true of nothing, there is a violation of the constraint that semantic values should have the same modal profile as the expressions of which they are semantic values. So considerations pertaining to modal rigidity can help us decide among competing semantics.

Next, the rigidity of pluralities plays a central role in one of Williamson's main arguments for *necessitism*, the metaphysical view that, necessarily, everything necessarily exists.[6] The denial of this view is *contingentism*. When we go on to consider arguments for the rigidity of plurals, it will be important to keep in mind whether the argument is intended to be given in a necessitist setting (which is always easier) or in a contingentist setting (which requires greater care).

Finally, the question of the rigidity of pluralities plays an essential role in an approach to mathematics and to the phenomenon of indefinite extensibility developed in recent work by one of us (Linnebo 2010 and Linnebo 2013).[7]

10.3 Challenges to plural rigidity

We aim to survey a number of arguments for the claim that pluralities are rigid. Before doing that, however, we should address some alleged counterexamples to plural rigidity.

The first one involves plural descriptions. Assume that Sophia is one of *the philosophers*. Does it follow that she is *necessarily* one of the philosophers? (For simplicity, we leave implicit the assumption that the entities in

[6] See Williamson 2010, 2013.

[7] While this approach to mathematics and indefinite extensibility draws inspiration from Parsons 1983b and to some extent also Putnam 1967 and Hellman 1989, these earlier views do not rely in the same way on the rigidity of pluralities.

question still exist.) If so, we would have to accept the implausible claim that, necessarily, Sophia is a philosopher. Thus, we must deny that Sophia is necessarily one of the philosophers. Does the case of Sophia provide us with a counterexample to plural rigidity?

Long ago Kripke taught us how to respond. Let *pp* be the things such that anything is one of them if and only it is in fact a philosopher. What is necessarily the case is that Sophia is one of *pp*. But it is not necessary that *pp* are all and only the philosophers. Sophia might have become a psychologist, not a philosopher. Then she would not have been included in the ranks of the philosophers, although she would still have been one of *pp*. It is important not to misunderstand the rigidity claim.

Other apparent counterexamples involve pronouns rather than plural descriptions. A nice example is the following ad we once saw for a gym:

(10.3) Join, and become one of us!

The plural pronoun 'us' is naturally taken to stand for a plurality. But when so interpreted, the message presupposes that it is possible to become a member of a plurality of which one is not already a member.[8] If there were such a possibility, we would have a failure of rigidity.

How are we to respond to these apparent counterexamples to the rigidity of pluralities? Interesting though they are, these examples are inconclusive. Consider, for instance, a bohemian parent who upon seeing some particularly smug business school students tells her daughter:

(10.4) I'm glad you're not one of them.

It is natural to understand the parent as expressing joy that her offspring is not (in some salient respect) *like* the students in question rather than pleasure with a fact about plural non-membership. Thus, (10.4) poses no more of a challenge to the rigidity of pluralities than the following sentence poses to the necessity of identity:

(10.5) I'm glad you're not him.

In particular, the apparent counterexamples can be explained away if we allow that a plural pronoun can sometimes function as a covert description

[8] A similar example is attributed to Dorothy Edgington in Rumfitt 2005. Further examples are found in Hewitt 2012b.

or refer to a group.[9] Either way, the behavior of the pronoun would be consistent with plural rigidity: as observed above, plural descriptions can function non-rigidly, and groups need not be rigid. Of course, more work would be needed to dispel an apparent counterexample such as (10.3) and establish an alternative explanation consistent with plural rigidity. What our discussion does show, however, is that it is advantageous to base our assessment of plural rigidity on more systematic and theoretical considerations.

As mentioned, we will see that plural rigidity figures at the heart of a network of ideas having to do with the extensionality of pluralities. Since the ideas in this network are true of a *core use* of our plural resources in ordinary language and thought, we commend them as an explication of these resources. We do, however, accept the existence of non-rigid groups. So we have no trouble admitting that there may be uses of plural resources (including plural variables) to stand for groups. Nor do we have any trouble admitting that there are plural expressions (e.g. some plural descriptions) that fail to satisfy rigidity.

10.4 An argument for the rigidity of sets

It will be useful to begin our investigation of the rigidity of pluralities by reminding ourselves of an argument for the necessity of identity and distinctness made famous by Saul Kripke (1980, Lecture III) and often attributed to Ruth Barcan Marcus (1947).[10] As we will see, this argument has striking consequences for the metaphysics of sets. Throughout this chapter, we assume the modal system T as our background modal logic. When stronger modal axioms are used, this will be noted explicitly.

The argument turns on Leibniz's law:

(Leibniz) $$\Box\forall x\forall y\big(x = y \rightarrow (\varphi(x) \leftrightarrow \varphi(y))\big)$$

where, as usual, the relevant argument place of φ occurs in a transparent context, namely outside the scope of quotations and propositional attitudes.

[9] For arguments in support of the first strategy, see e.g. Heim 1990 and Neale 1990, Chapter 5. The possibility of the kind of reference assumed in the second strategy is shown by examples such as 'Yesterday, the committee/club/team met. They decided to issue a press release.' Sentences such as these are natural and contain a plural pronoun, 'they', that appears to be anaphoric on a group.

[10] See Burgess 2014 for a recent discussion of the origin of the argument.

Given the assumption $\Box(x = x)$, Leibniz's law entails $\Box\forall x\forall y(x = y \to \Box x = y)$. Moreover, given the Brouwerian axiom (or just B for short)

$$\varphi \to \Box\Diamond\varphi$$

we can also derive the necessity of distinctness: $\Box\forall x\forall y(x \neq y \to \Box\, x \neq y)$.[11] (Proof sketches of these arguments are provided in Appendix 10.B.)

A contingentist may object to the assumption of $\Box(x = x)$. After all, in a negative free logic, '$x = x$' can be used as an existence predicate, in which case what is assumed is the necessary existence of x. The problem is easily circumvented. The contingentist will have no problem with the assumption that x satisfies the following open formula whose sole argument is represented by '. . .':

$$\Box(x = x \to x = \ldots)$$

Applying (Leibniz), this enables us to derive formulations of the necessity of identity and distinctness that are acceptable to the contingentist:[12]

(□ =) $\qquad \Box\forall x\forall y(x = y \to \Box(x = x \to x = y))$

(□ ≠) $\qquad \Box\forall x\forall y(x \neq y \to \Box\, x \neq y)$

The derivation of the latter from the former relies, as before, on B.

As Kripke realized, Leibniz's law has important metaphysical consequences. The case of sets provides a nice illustration. Consider the set-theoretic principle of extensionality:

(Set-Ext) $\qquad \forall x\forall y(\forall u(u \in x \leftrightarrow u \in y) \leftrightarrow x = y)$

Leibniz's law reveals a respect in which this is quite a strong principle. Let x and y be coextensive sets. By (Set-Ext), x and y are identical. Observe now that x satisfies the open formula

$$\Box\forall u(u \in \ldots \leftrightarrow u \in x)$$

[11] In fact, as Williamson (1996) has pointed out, (□ ≠) can also be derived without use of the Brouwerian axiom by invoking suitable principles of actuality.

[12] Since the necessitation of (Leibniz) ensures $\Box(x = y \to x = x)$, the existential presupposition '$x = x$', present in (□ =), would be redundant in (□ ≠). For were $x \neq y$ to fail, the mentioned presupposition would anyway be satisfied. (Thanks to Tim Williamson for this observation.)

So by Leibniz's law, y too satisfies this formula. We conclude that two coextensive sets are subject to necessary *covariation*:

(Set-Cov) $$\forall x \forall y(\forall u(u \in x \leftrightarrow u \in y) \rightarrow \Box \forall u(u \in x \leftrightarrow u \in y))$$

Thus, the set-theoretic principle of extensionality logically entails, via (Leibniz), that two coextensive sets are necessarily coextensive.[13] In fact, since (Set-Ext) holds of necessity, (Set-Cov) does too.

This consequence of Leibniz's law is important. It brings out a fundamental difference between sets and other kinds of collections, such as groups, which are tracked across possible worlds in some intensional way. Consider the covariation claim concerning groups: "If two groups in fact have the same members, they are identical and thus necessarily have the same members." This claim is wildly implausible. Membership in a group is contingent and thus subject to "drift", in the sense that the group may have different members at different possible worlds. But once membership drift is permitted, there is no guarantee that two groups that in fact coincide will necessarily coincide. In the case of sets, by contrast, the principle of extensionality and Leibniz's law entail that there can be no such drifting apart.

The observed difference between sets and groups derives from the fact that sets, unlike groups, are subject to the principle of extensionality. Having the same members suffices for two sets to be identical, but not for two groups. Let us dig deeper. *Why* does having the same members suffice for two sets to be identical, but not for two groups? The only explanation, we contend, is that sets, unlike groups, are constituted by their members. A set is fully characterized by specifying its members. So when two sets have the same members, they are identical. By contrast, a group has additional features that go beyond its members, which means that having the same members need not suffice for identity.

It is these additional features of groups that allow them to be tracked from possible world to possible world in non-trivial ways, permitting, for example, a philosophy department to have different members at different possible worlds. Sets are fundamentally different. Since a set is constituted by its members, there is nothing other than its members to go on when tracking it from world to world. Sets are therefore tracked rigidly. Just as in the case

[13] Here and in the remainder of this section, we leave implicit the proviso that the sets still exist.

of pluralities, the claim that sets are rigid, abbreviated as (Set-Rgd), can be formalized as the conjunction of two statements:[14]

(Set-Rgd^{+}) $\Box\forall x\forall y(x \in y \rightarrow \Box(Ey \rightarrow x \in y))$

(Set-Rgd^{-}) $\Box\forall x\forall y(x \notin y \rightarrow \Box(x \notin y))$

Let us take stock. Although our argument for (Set-Rgd) does not rest on purely logical premises, it is hard to resist. The principle of extensionality holds for sets, unlike groups, because sets, unlike groups, are constituted by their members. Thus, when tracking a set across possible worlds, there is nothing other than its members to go on. This ensures that the tracking is rigid. Not so for groups, which have additional features that go beyond their members and thus permit non-trivial tracking.

These considerations give rise to a dilemma that applies not only to sets but to any other notion of collection (including groups). *Either* we have to give up the principle of extensionality (as in the case of groups), *or else* we have to accept rigidity as well (as in the case of sets).[15] There is no stable middle ground. The only explanation for the principle of extensionality also supports rigidity.[16]

We will now consider two objections to our argument for the dilemma. The first objection is based on a mereological analogue of the argument for the rigidity of sets. Let $\leq$ indicate the relation of parthood. Assume that x and y share all their parts; that is, $\forall z(z \leq x \leftrightarrow z \leq y)$. Provided that parthood is reflexive and anti-symmetric, it follows that x and y are identical. Furthermore, since necessarily x shares all of its parts with itself, it follows by (Leibniz) that necessarily x and y share all of their parts. Yet these conclusions seem compatible with parthood being non-rigid! This calls into question our claim that analogous conclusions concerning sets support the rigidity of set membership.[17]

[14] Even for a contingentist, no further existential presuppositions are required, for reasons analogous to those that apply in the case of plurals (see Section 10.6).

[15] For sets, the former option is unattractive. As Boolos (1971, 229–30) reminds us, if ever there was an example of an analytic truth, then the extensionality of sets is one.

[16] These considerations pose a challenge to Fregean and neo-Fregean approaches to collections (or extensions, or *Wertverläufe*). On the one hand, such approaches adopt the principle of extensionality as a criterion of identity. On the other hand, they view a collection as somehow "derived from" its defining (Fregean) concept, which is plausibly regarded as non-rigid. So they are potentially on a collision course with the rigidity thesis. See Parsons 1977b for a discussion of Frege's concept of extension.

[17] Thanks to Jeremy Goodman for articulating this objection.

Our response is to deny that the two cases are analogous. The crux of our argument is the claim that any reason to accept the set-theoretic principle of extensionality is also a reason to accept the rigidity of set membership. By contrast, there is a reason to accept that sameness of parts suffices for identity that is *not* also a reason to accept the rigidity of parthood. Here is the intuitive idea. To make sense of contingent parthood, it is useful to think of objects as involving both matter and form.[18] For instance, a molecule that is part of you might not have been so because tracking you across possible worlds involves more than merely tracking your matter. On this hylomorphic conception, it is natural to take parthood to be sensitive to both matter and form. Mutual parthood would then ensures identity not only of matter but also of form—and hence also ensures the identity of the objects in question. But this explanation is perfectly compatible with objects involving form, not only matter, and thus being tracked non-trivially from world to world. In short, the principle that sameness of parts ensures identity admits of an explanation that does not support the rigidity of parthood.[19]

The second objection takes its departure from the well-known fact that Leibniz's law needs to be restricted. Assume that Nikita is the shortest spy. Of course, necessarily the shortest spy is the shortest spy. But it does not follow that necessarily Nikita is the shortest spy. It is often proposed that Leibniz's law be restricted to *rigid designators*—defined as terms that refer to the same object at every world at which they refer at all—thus excluding terms like 'the shortest spy'. Ordinarily, this restriction works well. But when reasoning about sets or other kinds of collection, the restriction threatens to undermine our dilemma between denying the principle of extensionality and accepting the rigidity principles.

To understand this threat, we need to distinguish between two completely different notions of rigidity. Until the previous paragraph, we have been concerned exclusively with a metaphysical notion of rigidity. Sets and other kinds of collection are said to be rigid if their membership is a matter of necessity, in the precise sense laid down by the kind of rigidity claims stated above. But as we have just seen, there is also the semantic notion of a rigid designator.

[18] Abstract objects would be a limiting case where the material contribution is nil.

[19] A better analogue of the set-theoretic principle of extensionality is the principle that sameness of *material* parts ensures identity. Now the analogy with our argument is restored. Any reason to accept the mentioned mereological principle is also a reason to accept the rigidity of material parthood. Of course, anyone attracted to non-rigid parts should respond to this observation by denying that sameness of material parts ensures identity.

The problem is that it can be hard to disentangle the two kinds of rigidity. Assume that a term t refers at w_1 to a collection comprising a and b, where a and b are all and only the Fs at w_1. Assume that t refers at w_2 to the singleton collection of a, where a is the one and only F at w_2. Is t a rigid designator? The question cannot be answered until we have been told how to track the relevant kind of collection from world to world. If the collections are tracked extensionally, we are considering different collections, with the result that t is not a rigid designator. But if the collections are tracked intensionally in terms of their membership criterion, we may well be considering one and the same object, namely the collection of Fs, in which case t is a rigid designator after all.

The threat to our dilemma is now apparent. To show that our use of Leibniz's law is permissible, we must first show that the terms in question are rigid designators. This involves showing that they refer to the same set across possible worlds. But this presupposes that we already know how to track sets across possible worlds! As we have seen, this is a matter of answering the question of metaphysical rigidity. Our argument therefore appears powerless to answer the question of metaphysical rigidity. The permissibility of its appeal to Leibniz's law presupposes that the question has already received an affirmative answer.

Fortunately, the threat can be avoided by reformulating the restriction on Leibniz's law. Say that a term is *purely referential* if its semantic contribution to linguistic contexts in which it occurs is exhausted by its referent, or, as Quine put it, if the term "is used purely to specify its object, for the rest of the sentence to say something about" (1960, 177). Instead of restricting Leibniz's law to rigid designators, we can restrict the law to purely referential terms. After all, the semantic contribution of such terms is exhausted by supplying their referents. Assume that $t_1 = t_2$ is a true identity involving two purely referential terms, and that the relevant argument place of a formula φ occurs in an transparent context. Then of course $\varphi(t_1) \leftrightarrow \varphi(t_2)$ is true as well, as this merely says of the common referent of t_1 and t_2 that it is φ if and only if it is φ. By restricting Leibniz's law to purely referential terms rather than to rigid designators, our problem dissipates. The only terms involved in our argument are variables. And a variable is purely referential because its semantic contribution is nothing but its value. Thus, our argument for metaphysical rigidity goes through.

There is a more general lesson here as well. The problem of disentangling metaphysical rigidity from semantic rigidity points to an unfortunate feature of the notion of a rigid designator: it runs together two kinds of considerations that are best kept apart. First, there is the semantic question of whether a term is purely referential. Then, there is the metaphysical question of how its referent is to be tracked from one possible world to another. It is true that every purely referential term is a rigid designator. But our discussion shows that we get a cleaner separation of the metaphysical and semantic questions by focusing on the notion of pure reference rather than rigid designation.[20] Thus, in what follows, the default notion of rigidity will once again be the metaphysical one.

Let us sum up. As observed, Leibniz's law entails the necessity of identity. We have examined whether an analogous argument can be given for the rigidity of sets. Given (Set-Ext), we found that Leibniz's law entails (Set-Cov), which states that two coextensive sets are necessarily coextensive. This falls short of the rigidity of sets, though it is a step in that direction. To establish the desired rigidity claim, we argued as follows. Any reason to accept (Set-Ext), we argued, is also a reason to accept the rigidity of sets. For (Set-Ext) holds because sets are constituted by their members, and this insight about the nature of sets also ensures that there is nothing other than the members in terms of which a set can be tracked.

10.5 An argument for plural rigidity

We will now extend the argument from the previous section to the case of pluralities. Previously, we started with Leibniz's law. Now, we propose to start with the principle that any coextensive pluralities are indiscernible. As before, we use $xx \approx yy$ to abbreviate the claim that xx and yy are coextensive (see Section 2.3). Thus, our proposed starting point is the principle:[21]

(INDISC) $$\Box\forall xx\forall yy\big(xx \approx yy \rightarrow (\varphi(xx) \leftrightarrow \varphi(yy))\big)$$

[20] See Stalnaker 1997 and essays 1–3 of Fine 2005b for some closely related considerations.

[21] This starting point allows us to remain neutral on the question of whether there is a notion of identity between pluralities. Clearly, (INDISC) requires no such notion. If such a notion is nevertheless available—denote it with the ordinary identity sign—then (INDISC) is merely the result of contracting into a single principle the law of extensionality for pluralities

(Ext) $$\Box\forall xx\forall yy(\forall u(u \prec xx \leftrightarrow u \prec yy) \leftrightarrow xx = yy)$$

and the plural analogue of Leibniz's law:

(LEIBNIZ*) $$\Box\forall xx\forall yy(xx = yy \rightarrow (\varphi(xx) \leftrightarrow \varphi(yy)))$$

Two concerns arise. First, as we have seen, the ordinary singular version of Leibniz's law needs to be restricted. Analogous considerations apply in the plural case. Fortunately, it is easy to see that (INDISC) is suitably restricted. Since plural variables are purely referential just as much as singular ones are (only in a plural way), (INDISC) is entirely legitimate. In particular, it presupposes no prior answer to the question of the rigidity of pluralities and can thus safely be employed in an argument for this rigidity thesis.

Second, is (INDISC) acceptable from a contingentist point of view? To assess this issue, we need to be more explicit about what semantics we adopt. It is natural to use an extension of the plurality-based semantics on which '$x \prec xx$' is true at a world w relative to an assignment ss if and only if the objects assigned to 'xx' by ss exist at w and the object assigned to 'x' is one of them. This semantics makes it natural to adopt a negative free logic.[22] The inference rules for the quantifiers must then be formulated so as to make existential assumptions explicit; for instance, from $\forall x\, \varphi(x)$ we can infer $Et \rightarrow \varphi(t)$, and likewise for the plural universal quantifier. (We will shortly have more to say about the plural existence predicate.) Given these choices, it is easy to verify that (INDISC) remains a valid principle even in a contingentist setting.

We are ready to develop our argument for the rigidity of pluralities. The next step is to derive from (INDISC), an analogue of the necessity of identity. As in the set-theoretic case, we call this analogue *covariation*:

(COV) $$\Box\forall xx\forall yy(xx \approx yy \rightarrow \Box(xx \approx yy))$$

It asserts that, as matter of necessity, two coextensive pluralities are necessarily coextensive. Given the Brouwerian axiom B, we can derive the necessity of non-coextensiveness as well.

We now come to the heart of the argument. Recall the case of sets, where Leibniz's law and (Set-Ext) entail (Set-Cov). While (Set-Cov) is formally compatible with the non-rigidity of sets, it is far more plausible with rigidity. In particular, any reason to accept (Set-Ext) is also a reason to accept the rigidity of sets. Precisely the same goes for pluralities. That is, (INDISC) entails (COV). While (COV) is formally compatible with the non-rigidity of pluralities, it is far more plausible with rigidity.[23] In particular, any reason to

[22] Notice that this enables us to drop the existential assumptions Ex and Eyy from (RGD$^-$) on p. 205.

[23] If a notion of plural identity is available (see footnote 21), the case of pluralities is perfectly parallel to that of sets. For the plural analogue of Leibniz's law and (Ext) entail (COV).

accept (Cov) is also a reason to accept the rigidity of plural membership. For (Cov) holds because a plurality is nothing over and above its members, and this insight about the nature of pluralities also ensures that there is nothing other than the members in terms of which a plurality can be tracked.

We therefore face a dilemma, as in the case of sets. Either we have to give up (Indisc), which implies (Cov), or else we have to accept plural rigidity. In other words, either we need to give up the ordinary principle of extensionality encapsulated in (Indisc), or else we have to accept the full transworld extensionality associated with plural rigidity. Just as in the case of sets, the former horn is deeply unattractive, as it comes close to just changing the subject. So we conclude that plural rigidity holds.

It is worth noting that (Cov) is logically weaker than (Indisc). The covariation principle gives us precisely what its name suggests, namely that two overlapping pluralities necessarily covary. By contrast, (Indisc) states that all properties of pluralities supervene on membership. To see that the latter principle goes beyond the former, consider a department whose statutes decree that all and only tenured faculty are to be members of the Hiring Committee and of the Graduate Admissions Committee.[24] Then the two committees necessarily covary in membership. Nevertheless, the two committees have different powers, namely to hire new faculty and to admit graduate students, respectively.

To be even more specific about the relation between (Indisc) and (Cov), one can observe that the former "factorizes" into the latter and the claim that the properties of a plurality *supervene* on what we may call its *modal membership profile*:

$$\text{(Sup)} \qquad \Box\forall xx\forall yy\big(\Box(xx \approx yy) \rightarrow (\varphi(xx) \leftrightarrow \varphi(yy))\big)$$

We see this as follows. Clearly, (Cov) and (Sup) entail (Indisc), which in turn entails each of the former two principles. Moreover, (Cov) and (Sup) are logically independent and encapsulate different philosophical ideas, namely covariation in membership and supervenience of properties on modal membership profile, respectively. A more comprehensive factorization of a cluster of ideas associated with the extensionality of pluralities will be offered in Section 10.10.

[24] We assume that the statutes are partially constitutive of the committees, in the sense that, were one to change the statutes, the original committees would cease to exist and be replaced by new ones. If necessary, this persistence condition for the committees can be written into the statutes.

We wish to end this section by briefly commenting on another argument for plural rigidity based on the covariation principle (Cov). This argument, due to Williamson (2013, 245–51), begins by unpacking the two biconditionals in (Cov) to obtain:

$$\Box\forall xx\forall yy(xx \preccurlyeq yy \wedge yy \preccurlyeq xx \ \rightarrow \ \Box xx \preccurlyeq yy \wedge \Box yy \preccurlyeq xx)$$

Williamson then makes the following, interesting observation: $xx \preccurlyeq yy$ gives no support to $yy \preccurlyeq xx$, and $yy \preccurlyeq xx$ gives no support to $\Box xx \preccurlyeq yy$. Thus, he contends, $yy \preccurlyeq xx$ should imply $\Box yy \preccurlyeq xx$, and $xx \preccurlyeq yy$ should imply $\Box xx \preccurlyeq yy$. Williamson therefore concludes that (Cov) "stands or falls" with the following, inferentially stronger principle:

$$\Box\forall xx\forall yy(xx \preccurlyeq yy \ \rightarrow \ \Box xx \preccurlyeq yy)$$

It is hard not to agree.

We are now only a small step away from (RGD$^+$). All it takes to make this step is a principle asserting the existence of singleton pluralities, namely:

(Single≺) $$\Box\forall x\exists xx\Box\forall y(y \prec xx \leftrightarrow x = y)$$

As Williamson shows, this natural principle, combined with (10.5), entails (RGD$^+$).

We find this argument rather convincing. However, we believe the argument developed above is more explanatory. This argument, we recall, is an explication of the basic thought that a plurality is nothing over and above its members. Since a plurality is nothing over and above its members, the only basis for tracking it across different possible worlds is in terms of these members. We thus obtain an explanation of why pluralities are tracked rigidly, with the result that the rigidity principles are true.

10.6 Towards formal arguments for plural rigidity

We have developed an argument for plural rigidity. But, as it stands, the argument is not formally valid. Starting from (INDISC), our best formal result so far is (Cov), which states that coextensive pluralities are necessarily coextensive. Rigidity, our target, states that a plurality has the same members at any world at which it exists. We now investigate some ways to formally bridge this gap.

Since we are now aiming for formal rigor, the time has come to be entirely precise about the existential assumptions involved in our arguments. This requires a plural existence predicate that we can use to say of some things xx that they exist. As we have seen, the existence of a single object x can be expressed simply as $x = x$ (sometimes written Ex). But what about the plural existence predicate?

One may try to define plural existence distributively in terms of singular existence; that is, to define Exx as $\forall x(x \prec xx \rightarrow x = x)$. But this is unsuccessful. For a contingentist, the initial quantifier ranges only over objects that exist at the relevant world, which renders the quantified claim trivially true for any plurality xx whatsoever. Another natural but unsuccessful idea is to define Exx as $xx \approx xx$ in an attempt to imitate the definition of its singular analogue Ex as $x = x$. This too is easily seen to trivialize, for exactly the same reason as the previous attempt.

One safe option is simply to adopt a primitive collective plural existence predicate Exx, which we stipulate to be satisfied by some things at a world just in case all these things exist at the world. Another option is available as well, given the axiom that every plurality is non-empty: $\forall xx \exists y(y \prec xx)$. We can then define Exx as $\exists y(y \prec xx)$. To confirm that this works, suppose that xx don't exist at some world w. The semantics we are assuming (p. 217) ensures that Exx is false at w. Suppose instead that xx do exist at w. Then the axiom ensures that Exx is true at w. We adopt this option, rather than the first, as it is more economical.

Recall that we have assumed a negative free logic as our background for contingentist reasoning. As we already noted, this requires some restrictions on the axioms for the quantifiers.[25]

Next, we adopt the following "being constraint":

(BC) $$\Box \forall x \forall yy \Box(x \prec yy \rightarrow Ex)$$

That is, necessarily, if x is one of yy, then x exists. Clearly, this is valid on our semantics. Notice also that, given our definitions of the singular and plural existence predicates, (BC) entails:

(10.6) $$\Box(x \prec yy \rightarrow Ex \wedge Eyy)$$

[25] See Hughes and Cresswell 1996, Chapter 16, for a system of free logic in the context of modal logic.

Finally, we adopt an axiom stating that any plurality is ontologically dependent on each of its members:

(DEP) $$\Box\forall x\forall yy(x \prec yy \to \Box(Eyy \to Ex))$$

We contend that this axiom is plausible and observe that it is compatible with x being one of yy at some worlds but not at others.[26] The axiom is useful because it enables us to formulate (RGD$^+$) as we have done, rather than adopt the following, more cautious formulation with an additional existential presupposition Ex:

$$\Box\forall x\forall yy(x \prec yy \to \Box(Ex \wedge Eyy \to x \prec yy))$$

Moreover, by means of (DEP) and (BC), we are able to formulate (RGD$^-$) as we have done, rather than adopt the following, more guarded formulation:[27]

$$\Box\forall x\forall yy(x \not\prec yy \to \Box(Ex \wedge Eyy \to x \not\prec yy))$$

In the necessitist setting, of course, no existential presuppositions are needed.

With these preliminary questions clarified, we will now consider some formal arguments for plural rigidity. Each argument will first be developed from a necessitist point of view, as this is simpler. We will then use our plural existence predicate to reformulate the argument so as to work in a contingentist setting.

10.7 The argument from uniform adjunction

The first formal argument relies on an operation + of adjoining one object to a plurality. It is reasonable to assume that, necessarily, to be one of these things and that thing is to be one of these things or to be identical with that thing. We call this principle uniform adjunction:

[26] As observed by Roberts (forthcoming), (DEP) does not entail the principle that a plurality is ontologically dependent on each of its *subpluralities*:

$$\Box\forall xx\forall yy(xx \preccurlyeq yy \to \Box(Eyy \to Exx))$$

While this principle is not needed in the present context, Roberts points out that it is useful elsewhere.

[27] To verify this claim, observe first that $\Box\psi$ and $\Box(\varphi \to \psi)$ are equivalent schemes of modal propositional logic when $\Box(\neg\psi \to \varphi)$. Now let φ and ψ be $Ex \wedge Eyy$ and $x \not\prec yy$, respectively. Then $\Box(\neg\psi \to \varphi)$ is just (10.6), which was established above.

(UniAdj) $$\Box\forall xx\forall y\Box\forall x(x \prec xx+y \leftrightarrow x \prec xx \vee x = y)$$

where '$xx + y$' figures as a complex plural term. We now argue as follows. Assume $y \prec xx$. Then, by (UniAdj), we have $xx \approx xx + y$. So by applying (Cov) to xx and $xx + y$, we obtain:

(10.7) $$\Box(xx \approx xx + y)$$

Next, we observe that (UniAdj) also entails

(10.8) $$\Box(y \prec xx + y)$$

From (10.7) and (10.8), some simple modal logic ensures our desired conclusion that $\Box(y \prec xx)$.

Gabriel Uzquiano has raised a legitimate concern about the argument.[28] Recall that (10.7) is a consequence of Leibniz's law and must be restricted to purely referential terms. Is it permissible to assume that '$xx + y$' is purely referential? This can be disputed. Fortunately, we can sidestep the problematic assumption by reformulating (UniAdj). The above argument proceeds from the assumption that uniform adjunctions exist. We can express this assumption as the closure of the following plural comprehension principle:

(UniAdj*) $$\Box\forall xx\forall y\exists zz\Box\forall u(u \prec zz \leftrightarrow u \prec xx \vee u = y)$$

This principle is very weak. Indeed, it is something that even an opponent of plural rigidity should assent to, as the principle retains its plausibility even when the plural variables are allowed to range over groups.[29]

We now give our improved and official version of the argument from uniform adjunction. As before, assume $y \prec xx$. By (UniAdj*), let zz be the uniform adjunction of y to xx. From this point onward the argument proceeds exactly as before, only with zz in the role previously played by $xx+y$. (The argument is spelled out in detail in Appendix 10.B.) Notice that this argument makes no appeal to (Indisc) other than its single instance, (Cov). In this respect, the argument from uniform adjunction is like the argument from Section 10.5. This establishes (Rgd$^+$).

[28] For a mereological analogue of this concern, see Uzquiano 2014, 42.

[29] Here, as elsewhere, it is interesting to inquire whether an analogous argument can be given in mereology, to the effect that parthood is rigid. We believe the answer is negative, but this isn't the place for a proper investigation.

What does it take to obtain (Rgd$^-$)? It turns out that, in the system S5, a necessitated version of (Rgd$^+$) entails (Rgd$^-$). We prove this useful fact in Appendix 10.B and invoke it repeatedly in what follows.

Finally, let us adapt the argument to a contingentist setting. Then, uniform adjunction requires the following, more guarded formulation:

(UniAdj-c) $\Box\forall xx\forall y\Box\big(Exx \wedge Ey \rightarrow \forall x(x \prec xx + y \leftrightarrow x \prec xx \vee x = y)\big)$

As before, this principle can be formulated in a way that sidesteps worries one may have about the complex term '$xx + y$', namely by means of the following comprehension principle:

(UniAdj*-c) $\Box\forall xx\forall y\exists zz\Box\big(Exx \wedge Ey \rightarrow \forall u(u \prec yy \leftrightarrow u \prec xx \vee u = y)\big)$

Thankfully, it can be verified that (Rgd$^+$) follows and that, using axiom B, so does (Rgd$^-$). In sum, we find the argument from uniform adjunction convincing, both in a necessitist and in a contingentist setting. None of the arguments we will proceed to consider does any better, or so we will argue.

10.8 The argument from partial rigidification

Another formal argument is proposed in Williamson 2010 (699–700). The argument requires that, for any objects xx, there be some objects yy that are a partial rigidification of xx in the sense that $xx \approx yy$ but it is impossible for yy to lose any of their members. To be precise, we assume the following plural comprehension axiom:

(PartRig) $\Box\forall xx\exists yy(xx \approx yy \wedge \forall x(x \prec yy \rightarrow \Box x \prec yy))$

We can now argue as follows. Assume $y \prec xx$. Let yy be the partial rigidification of xx. Thus, we have $\Box(y \prec yy)$. By (Cov), we also have $\Box(xx \approx yy)$. The latter two claims entail $\Box(y \prec xx)$, as desired. Using (Rgd$^+$), we can, as before, obtain (Rgd$^-$).

Let us now try to develop the argument from a contingentist point of view. As usual, the comprehension axiom needs to be formulated with greater care:

(PartRig-c) $\Box\forall xx\exists yy(xx \approx yy \wedge \forall x(x \prec yy \rightarrow \Box(Eyy \rightarrow x \prec yy)))$

Assume $y \prec xx$, and let yy be the partial rigidification of xx. Applying the same strategy as in the case of necessitism, we derive $\Box(Eyy \rightarrow z \prec yy)$. By (Cov), we also have $\Box(xx \approx yy)$. This establishes $\Box(Eyy \rightarrow z \prec xx)$. The final step to our desired target, namely $\Box(Exx \rightarrow z \prec xx)$, follows by using (Cov) to show that the existence of xx necessitates the existence of yy. (The argument is spelled out in detail in Appendix 10.B.)

How does Williamson's argument compare with the argument from uniform adjunction? Let us begin by addressing the question in the necessitist setting. Both arguments rely on a single instance of (Indisc), namely (Cov). The arguments differ only with respect to the plural comprehension axioms that they invoke. We are thus left with the task of comparing the two comprehension axioms, namely (UniAdj*) and (PartRig). A careful formal investigation reveals that, against the background of (Cov) and S5, the two axioms are equivalent with each other and also with the rigidity claim. An analogous claim holds in the contingentist setting.[30] Thus, at least in the context of S5, the choice between the argument from partial rigidification and uniform adjunction is merely a matter of taste and which heuristics one prefers.[31]

10.9 The argument from uniform traversability

The last formal argument for plural rigidity that we will consider is inspired by an observation made by Ian Rumfitt (2005, 117–18). Like above, we first give a simple version of the argument that is acceptable from a necessitist point of view, and then consider how the argument can be adapted to suit the contingentist.

A finite plurality can be *traversed*, in the sense that its members can be exhaustively listed. Assume for instance that aa is the plurality whose members are a, b, and c, and that these members have names $\bar{a}$, $\bar{b}$ and $\bar{c}$, respectively. Then aa can be traversed:

[30] Does this three-way equivalence mean that the arguments for rigidity are begging the question? To think so would be to conflate deductive validity with begging the question. It is particularly important to notice that any one member of the pairs of assumptions sufficient to prove rigidity (i.e. (Cov) plus a comprehension axiom) is compatible with the failure of rigidity.

[31] Linnebo 2016 claims that, in the contingentist setting, (PartRig-c) is less plausible than (UniAdj*-c). While this may be true when the axioms are considered in isolation, our present point is that the axioms are equivalent *modulo* the mentioned assumptions. In the context of weaker modal logics, the argument from uniform adjunction can be shown to require weaker modal assumptions than the argument from partial rigidification.

$$\forall x(x \prec aa \leftrightarrow x = \bar{a} \vee x = \bar{b} \vee x = \bar{c})$$

In fact, this traversability is uniform, in the sense that it holds by necessity:[32]

$$\Box\forall x(x \prec aa \leftrightarrow x = \bar{a} \vee x = \bar{b} \vee x = \bar{c})$$

What about infinite pluralities? A straightforward generalization is available if we allow infinitary disjunctions and assume that every object a has a name $\bar{a}$:[33]

(UniTrav) $$\Box\forall x(x \prec aa \leftrightarrow \bigvee_{a \prec aa} x = \bar{a})$$

We now argue as follows. Assume $y \prec aa$. Then we can find a such that $y = \bar{a}$. By the necessity of identity, we have $\Box(y = \bar{a})$. This entails the necessitation of $\bigvee_{a \prec aa} y = \bar{a}$. Some simple modal logic ensures our target $\Box(y \prec aa)$. See Appendix 10.B for details.

Let us now consider matters from a contingentist point of view. Equation (UniTrav) must be reformulated so as to make all existential presuppositions explicit. Given any objects *aa*, we can name all of its members and use this to state that, provided *aa* still exist, to be one of *aa* is just to be identical with one of the aforementioned members. In symbols:

(UniTrav-c) $$\Box(Eaa \rightarrow \forall x(x \prec aa \leftrightarrow \bigvee_{a \prec aa} x = \bar{a}))$$

As far as we can see, this modified principle is just as plausible, given contingentism, as the original principle is, given necessitism. It is therefore satisfying to be able to verify that the original argument for rigidity goes through much as before.

We find the argument from uniform traversability less explanatory than the previous two formal arguments for plural rigidity. One problem is the lack of "conceptual distance" between the premise and the conclusion. Uniform traversability is little more than an infinitary restatement of our

[32] In fact, as Jeremy Goodman observed, if a singleton plurality is uniformly traversed by its sole member, then Uniform Adjunction allows us to *prove* that any finite plurality is uniformly traversed by its members.

[33] Of course, the choice of names depends on the particular plurality *aa*. This means that '*aa*', in the subscript to the disjunction sign, can only be understood as a plural constant, not a variable.

target claim that a plurality is fixed in its membership as we shift attention from one possible world to another (see Hewitt 2012b, 860–2, for a similar objection).[34] Moreover, this premise is needlessly strong—concerning both *what* it says and *how* it says it.

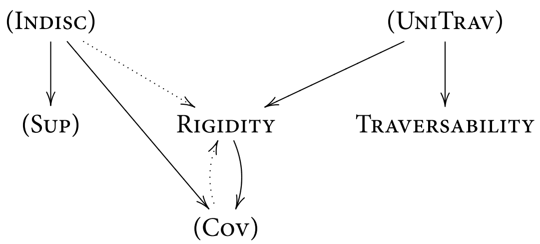
The clearest way to appreciate the strength of uniform traversability is by observing that it entails all the premises of the previous two arguments. One of these premises is (Cov). This is entailed by uniform traversability, as can be seen by a simple transitivity argument available in S5. Indeed, uniform traversability entails plural rigidity, which in turn entails (Cov), as observed at the end of Section 10.5. Moreover, it can be verified that uniform traversability entails the comprehension axioms employed in the arguments from uniform adjunction and partial rigidification.

Next, the infinitary resources employed by the argument from uniform traversability are very strong. To see this, it is useful to separate these resources from the modal claim that they are used to express. We can do so by considering what we may call *traversability*, which is (UniTrav) without the initial necessity operator. In fact, even this non-modal version of traversability has some strong consequences. As we explain in Appendix 10.A, this principle legitimizes what Paul Bernays (1935) calls "quasi-combinatorial" reasoning, that is, reasoning about infinite totalities as if they were finite.

10.10 Pluralities as extensionally definite

We have surveyed various formal and informal arguments for plural rigidity. It may be useful to summarize the most important principle that we have discussed and their inferential relations to one another.

For a necessitist, the picture looks as follows:



[34] By contrast, as observed in footnote 30, the previous two argument both relied on *two* assumptions—(Cov) and a comprehension principle—each one of which is compatible with the failure of plural rigidity.

Solid arrows represent one-way implications. Dotted arrows represent non-deductive support, but can be transformed into implications by adding suitable comprehension axioms, as discussed in Sections 10.7 and 10.8. Formal theses are in parentheses, as usual. RIGIDITY abbreviates the conjunction of the rigidity claims (RGD^{+}) and (RGD^{-}), and, in a contingentist setting, also the dependence claim (DEP).

A contingentist can use the same diagram, with two exceptions. First, (UNITRAV) must be replaced with (UNITRAV-C), whose left diagonal implication then only yields the two rigidity claims, not (DEP).[35] Second, RIGIDITY entails only a restricted version of (COV) (see p. 235).

Where does this leave us? We raised the question of whether plural rigidity, just like the rigidity of identity, can be established from purely logical premises. We have given at least a conditional answer. If (INDISC) and the "suitable comprehension axioms" identified in Sections 10.7 and 10.8 count as purely logical, then so does plurality rigidity. But is the antecedent true? As it stands, we find the notion of pure logic insufficiently clear to give a definitive answer.

We find it more productive to recall the basic thought that has animated much of our discussion in this chapter, namely that a plurality is nothing over and above some circumscribed lot of objects. *Every plurality thus exhibits extensionality in its purest form. All we have are some objects, properly circumscribed.* Pluralities are, as we will put it, *extensionally definite*. This basic thought motivates some of our central principles, especially (INDISC) and (UNITRAV), which explicate different aspects of the extensional definiteness of pluralities. The picture that emerges is thus that plural rigidity figures at the heart of a tightly interwoven network of principles that all have to do with the extensional definiteness of pluralities. These principles mutually support each other. In particular, it would be difficult and unmotivated to excise plural rigidity from this network. Plural rigidity is an integral part of our analysis of pluralities as extensionally definite.

It is particularly interesting to examine the "second floor" of the above diagram. For the principles that figure on this floor provide a factorization of all the aspects of the extensional definiteness of pluralities that are represented in the diagram. To explain what we mean, let us begin by observing that each principle on this floor represents a simple and natural idea.

[35] If desired, one can tweak (UNITRAV-C) so as to ensure that (DEP) too follows, namely by adding the following as a third (and perfectly sensible) conjunct: ($Eaa \leftrightarrow \bigwedge_{a \prec aa} E\bar{a}$).

(a) The properties of any plurality supervene on its modal membership profile, as expressed by (Sup).
(b) A plurality has a rigid membership profile: it has the very same members at any possible world at which it exists.
(c) A plurality is traversable, thus ensuring the permissibility of quasi-combinatorial reasoning applied to the plurality.

Next, we observe that the three factors entail each of the aspects of the extensional definiteness of pluralities. It suffices to verify that the items on the top floor of the diagram are entailed by those on the second. As observed on p. 218, (Indisc) can be factorized into (Sup) and (Cov). And it can be verified that (UniTrav) (or its contingentist cousin) factorizes into Rigidity and Traversability.

What remains is to verify that the three factors are logically independent of one another. To see that property supervenience, (Sup), does not follow from the other two aspects of extensional definiteness, consider again the case of committees. Imagine an oligarchic department where three senior academics *a*, *b*, and *c* have written into the department statutes that they, and they alone, are to be on the Hiring Committee and the Graduate Admissions Committee. Both committees have a rigid membership profile and are clearly traversable. Yet the two committees are not subject to property supervenience as different powers of decision are vested in them.

Next, to show that a rigid membership profile does not follow from the other two aspects, consider the case of properties, understood as objects that are individuated by the necessary coextensionality of their defining concept or condition, and tracked across possible worlds in terms of this concept or condition. Thus understood, properties exemplify the second aspect of extensional definiteness: all the characteristics of any given property are shared by any necessarily coextensive property. However, a property can be subject to contingent membership (or, perhaps better, contingent application), including when its instances are traversable. And as we have seen, the traversability of a domain ensures the traversability of any property on this domain.

Finally, we observe that traversability is not a formal consequence of the other two aspects of extensional definiteness. The principles that explicate these other two aspects do not ensure the availability of the infinitary resources needed for traversability. As Bernays observed, traversability is based on an extrapolation from the finite into the infinite. How far are we willing to extrapolate? The first two aspects of the extensional definiteness of pluralities do not, by themselves, provide any answer to this question.

10.11 The status of plural comprehension

We wish to end the chapter with some remarks about the status of the plural comprehension axioms. Many philosophers regard such axioms as utterly trivial and insubstantial.[36] Provided that a condition is well defined and has at least one instance, *of course* the condition can be used to define a plurality of all and only its instances.

This view seems to us misguided. Indeed, we suspect the view is the result of an excessive focus on ontology at the expense of other important concerns. Because plural logic is thought to incur no ontological commitments over and above those already incurred by the singular quantifiers, the plural comprehension axioms are assumed to be ontologically innocent. And because of the intense focus on ontology, one therefore concludes that these axioms are trivial and insubstantial. One of the main upshots of this chapter is that, irrespective of the question of ontological commitment, plurals are governed by strong extensionality principles whose satisfaction is a non-trivial matter. Since the plural comprehension axioms make claims about possible assignments to the plural variables, which would accordingly be governed by the non-trivial extensionality principles, these axioms too should be regarded as non-trivial (see also Williamson 2016).

To elaborate, let us consider the three factors of the extensional definiteness of pluralities. First, it is not hard to see that *traversability* is a non-trivial assumption. To say that plural comprehension is permissible on a condition φ is to say that we may reason quasi-combinatorially about all the φs. A number of disputes in the foundations of mathematics testify to the non-triviality of this assumption.[37]

Second, *property supervenience* too is a non-trivial matter. Consider the following:

(10.9) The Hiring Committee met yesterday. They decided to make an offer to Sophia.

[36] In Section 2.5, we mentioned the claims that plural comprehension axioms as "genuine logical truths" found in Boolos 1985b (342) and Hossack 2000 (422).

[37] See Feferman 2005 for a survey of debates concerning the legitimacy of impredicative reasoning in mathematics.

Is it permissible to apply the rule of plural existential generalization to 'they'? The answer must be 'no'. Generalizing in this way would ascribe the property of making a job offer to the members of the committee *considered as a mere plurality*, where in reality the property can only be ascribed to *the committee as such*. It is only the committee, not the plurality of its members, that has the power to make job offers. Indeed, the property ascribed in the second sentence of (10.9) fails to supervene on the modal membership profile. As our earlier examples show, two committees can share the same modal membership profile while differing in the powers that are vested in them.[38]

The final factor of the extensionality of pluralities is their *rigid membership profile*. This rules out, for example, the existence of a plurality of all actual and merely possible objects, that is, a plurality aa such that $\Box\forall x(x \prec aa \leftrightarrow x = x)$. For such a plurality would vary in membership from world to world. (See Linnebo 2010.)

Summing up, we have argued that pluralities are rigid and that this is in fact just one of several extensionality principles that govern pluralities. These principles explicate our basic thought that pluralities are extensionally definite. Although the principles can be split into three independent factors (of which plural rigidity is one), they go naturally together as a package. Since the extensionality principles are non-trivial, so are the plural comprehension axioms, which assert the existence of pluralities governed by these principles. This non-triviality plays an essential role in our development of a critical plural logic in Chapter 12, which restricts the plural comprehension scheme.

[38] Our example from p. 218 of the Hiring Committee and the Graduate Admissions Committee will do.

Appendices

10.A Traversability and quasi-combinatorial reasoning

We claimed in Section 10.9 that the non-modal traversability principle (UniTrav) licences what Bernays (1935) calls "quasi-combinatorial" reasoning, that is, reasoning with infinite totalities as if they were finite. Let us now spell out and defend this claim.

First, we claim that (UniTrav) ensures the permissibility of impredicative plural separation axioms of the following form:

$$\exists x(\varphi(x) \wedge x \prec xx) \rightarrow \exists yy \forall u\big(u \prec yy \leftrightarrow \varphi(u) \wedge u \prec xx\big) \tag{10.10}$$

That is, given any xx that include a φ, there are some objects that are all of the φs among xx. We show this as follows. We begin by finding a bunch of names $\bar{a}$ that provide a traversal of xx. We would like another bunch of names $\bar{b}$ that provide a traversal of just those members of xx that satisfy φ. This is easily achieved by going through the former bunch, deleting every item that names a non-φ. The resulting sub-traversal yields a quantifier free—and thus fully predicative—definition of the desired sub-plurality of xx. The upshot is that traversability functions like an axiom of reducibility, in Russell and Whitehead's famous sense, that is, as an axiom stating that every higher-order entity has a predicative definition. The reducibility afforded by (UniTrav) becomes particularly far-reaching if there is an all-encompassing or universal plurality, as is standardly assumed. We would then obtain a justification for the full impredicative comprehension scheme.

Second, when we work in the context of an intuitionistic theory, traversability ensures that quantification restricted to any plurality behaves classically. Assume that a formula $\psi(x)$—which may have further free variables—is decidable on any given argument:

$$\forall x(\psi(x) \vee \neg\psi(x))$$

In effect, this means that the property defined by $\psi(x)$ behaves classically on any given argument. Then tranversability ensures that quantification restricted to xx behaves classically as well, in the precise sense that we have the following decidability property:

$$(\forall x \prec xx)\psi(x) \vee (\exists x \prec xx)\neg\psi(x)$$

To see this, observe that by traversability this restricted quantification reduces to a conjunction of its instances, each of which has been assumed to behave classically.

10.B Proofs

We now provide proof sketches of various arguments referred to in Chapter 10. We work within modal extensions of first-order logic (FOL) and of PFO. Sentential and quantificational reasoning will apply to expressions in the extended language. When dealing with arguments in a contingentist setting, we rely on a standard negative free logic. This means that the rules for singular and plural quantifiers are restricted so as to ensure that quantifiers range over existing objects or pluralities. For instance, from $\forall x\varphi(x)$ we can infer $Ey \rightarrow \varphi(y)$. Similarly, we can infer $\exists x\varphi(x)$ from $Ey \wedge \varphi(y)$, but not from $\varphi(y)$ alone. Moreover, there are rules guaranteeing that atomic predications are false if at least one of the terms involved is empty.

The necessity of identity and distinctness. Both arguments rely on Lebniz's law. We develop them in a necessitist setting.

(1)	$x = x$	FOL
(2)	$\Box(x = x)$	1, Necessitation
(3)	$\Box\forall x\Box(x = x)$	2, FOL, Necessitation
(4)	$\Box\forall x\forall y(x = y \rightarrow (\Box(x = x) \leftrightarrow \Box(x = y)))$	Leibniz
(5)	$\Box\forall x\forall y(x = y \rightarrow \Box(x = y))$	3, 4, K, FOL

The argument for the necessity of distinctness appeals to the Brouwerian axiom:

$$\varphi \rightarrow \Box\Diamond\varphi$$

The argument goes as follows:

(0)	$\Box\forall x\forall y(x = y \rightarrow \Box(x = y))$	as shown in the previous proof
(1)	$x = y \rightarrow \Box(x = y)$	0, T, FOL
(2)	$\neg\Box(x = y) \rightarrow x \neq y$	1, FOL
(3)	$\Box(\neg\Box(x = y) \rightarrow x \neq y)$	2, Necessitation
(4)	$\Box\neg\Box(x = y) \rightarrow \Box(x \neq y)$	3, K
(5)	$\Box\Diamond(x \neq y) \rightarrow \Box(x \neq y)$	4, Definition of $\Diamond$
(6)	$x \neq y \rightarrow \Box(x \neq y)$	5, B
(7)	$\Box\forall x\forall y(x \neq y \rightarrow \Box(x \neq y))$	6, FOL, Necessitation

The proofs of the necessity of identity and distinctness in the contingentist setting are simple adaptations of the proofs just given. ⊣

Necessary set covariation. We observed in Section 10.4 that the set-theoretic principle of extensionality (Set-Ext) implies that two coextensive sets are necessarily coextensive:

(Set-Cov) $\forall x \forall y(\forall u(u \in x \leftrightarrow u \in y) \rightarrow \Box \forall u(u \in x \leftrightarrow u \in y))$

The proof is similar to that of the necessity of identity:

(1)	$\forall x \forall y(\forall u(u \in x \leftrightarrow u \in y) \leftrightarrow x = y)$	(Set-Ext)
(2)	$\forall x \forall y\big(x = y \rightarrow (\Box \forall u(u \in x \leftrightarrow u \in x) \leftrightarrow \Box \forall u(u \in x \leftrightarrow u \in y))\big)$	Leibniz
(3)	$\forall u(u \in x \leftrightarrow u \in x)$	FOL
(4)	$\Box \forall u(u \in x \leftrightarrow u \in x)$	3, Necessitation
(5)	$\forall x \forall y(\forall u(u \in x \leftrightarrow u \in y) \rightarrow \Box \forall u(u \in x \leftrightarrow u \in y))$	1, 2, 4, FOL

Note that this reasoning is available to necessitists and contingentists alike. The necessitation of (Set-Cov) can be easily obtained from the necessitation of (Set-Ext). ⊣

The remaining proofs concern modal properties of pluralities. Thus we work in a modal extension of PFO. Because of the complexity of the reasoning involved, a fully formal presentation of some arguments will not be particularly illuminating. In those cases, we prefer to reason informally about the system rather than formally within the system. All the argumentative strategies employed are meant to be essentially available, *mutatis mutandis*, to both necessitists and contingentists.

Necessary plural covariation. First, we show that (Indisc) entails that two coextensive pluralities are necessarily coextensive:

(Cov) $\Box \forall xx \forall yy(xx \approx yy \rightarrow \Box(xx \approx yy))$

(1)	$\Box \forall xx \forall yy(xx \approx yy \rightarrow (\Box(xx \approx xx) \leftrightarrow \Box(xx \approx yy))$	(Indisc)
(2)	$xx \approx xx$	PFO
(3)	$\Box(xx \approx xx)$	2, Necessitation
(4)	$\Box \forall xx \Box(xx \approx xx)$	3, PFO, Necessitation
(5)	$\Box \forall xx \forall yy(xx \approx yy \rightarrow \Box(xx \approx yy))$	1, 3, PFO

There is a perfect analogy with the case of identity. In the presence of axiom B, (INDISC) also entails that two distinct pluralities are necessarily distinct:

$$(\text{Cov}^-) \qquad \Box\forall xx\forall yy(xx \not\approx yy \rightarrow \Box(xx \not\approx yy))$$

⊣

The Barcan formula for bounded quantifiers. In the context of S5, a restricted version of the Barcan formula is derivable:

$$(\text{BFR}) \qquad \forall xx\big(\forall x(x \prec xx \rightarrow \Box\varphi(x)) \rightarrow \Box\forall x(x \prec xx \rightarrow \varphi(x)\big)$$

Here is the contingentist argument for it (the necessitist argument can be easily read off from it). Let xx be an existing plurality. Assume $\forall x(x \prec xx \rightarrow \Box\,\varphi(x))$. We conjoin Exx and apply B to the resulting conjunction to obtain a claim that will be used shortly:

$$(*) \qquad \Box\Diamond\big(Exx \wedge \forall x(x \prec xx \rightarrow \Box\varphi(x))\big)$$

Assume for reductio that the consequent of (BFR) is false, that is, $\Diamond\exists x(x \prec xx \wedge \neg\,\varphi(x))$. By the necessitation of (RGD^+) and S4, we can add a conjunct to this formula so as to obtain:

$$\Diamond\exists x(x \prec xx \wedge \neg\varphi(x) \wedge \Box(Exx \rightarrow x \prec xx))$$

We now use (*) to add a conjunct, namely the result of removing the outermost $\Box$ from (*). This yields:

$$\Diamond\exists x\big(x \prec xx \wedge \neg\varphi(x) \wedge \Box(Exx \rightarrow x \prec xx) \wedge \Diamond(Exx \wedge \forall x(x \prec xx \rightarrow \Box\varphi(x))\big)$$

A bit of modal logic on the last two conjuncts of this formula yields:

$$\Diamond\exists x\big(x \prec xx \wedge \neg\varphi(x) \wedge \Diamond\Box\varphi(x)\big)$$

Reasoning in S5, we can derive the possible existence of some x which is both φ and $\neg\varphi$. We turn this possible contradiction into an actual contradiction. So we deny that there could be something among xx that is $\neg\varphi$ and conclude $\Box\forall x(x \prec xx \rightarrow \varphi(x))$, which completes our proof. Since the proof relies on no extra-logical assumption, (BFR) may be necessitated. ⊣

Plural rigidity entails a restricted version of (Cov). The proof appeals to (BFR). Let xx and yy be two pluralities and suppose that rigidity holds. Assume $xx \approx yy$. We want to show $\Box(xx \approx yy)$. By the assumption, if $x \prec xx$, then $x \prec yy$. It follows from ($\textsc{Rgd}^+$) that $\Box(Eyy \rightarrow x \prec yy)$; in the case of necessitism, the antecedent can be dropped. Thus, in the case of necessitism, we have:

$$\forall x(x \prec xx \rightarrow \Box(x \prec yy))$$

By (BFR), we obtain:

$$\Box\forall x(x \prec xx \rightarrow x \prec yy)$$

By symmetrical reasoning, we obtain:

$$\Box\forall x(x \prec yy \rightarrow x \prec xx)$$

The last two displayed formulas entail our target claim: $\Box(xx \approx yy)$. In the case of contingentism, parallel reasoning can be carried out, though contingent on the continued existence of the pluralities in question, thus yielding a restricted version of the target claim: $\Box(Exx \wedge Eyy \rightarrow xx \approx yy)$. ⊣

($\textsc{Rgd}^-$) from ($\textsc{Rgd}^+$). In the presence of axiom B, a necessitated version of ($\textsc{Rgd}^+$) entails ($\textsc{Rgd}^-$). Assume

$$(\Box\textsc{Rgd}^+) \qquad \Box\Box\forall x\forall yy(x \prec yy \rightarrow \Box(x \prec yy))$$

Suppose for reductio that ($\textsc{Rgd}^-$) is false, that is:

$$\Diamond\exists x\exists yy(x \nprec yy \;\wedge\; \Diamond(x \prec yy))$$

The two displayed formulas entail:

$$\Diamond\exists x\exists yy(x \nprec yy \;\wedge\; \Diamond(x \prec yy \;\wedge\; \Box(x \prec yy)))$$

By B, we can add a third conjunct:

$$\Diamond\exists x\exists yy(x \nprec yy \;\wedge\; \Diamond(x \prec yy \;\wedge\; \Box(x \prec yy)) \;\wedge\; \Box\Diamond(x \nprec yy))$$

A bit of modal reasoning yields:

$$\Diamond\exists x\exists yy\Diamond\Diamond(x \prec yy \;\wedge\; x \nprec yy)$$

That is,

$$\neg\Box\forall x\forall yy\Box\Box\neg(x \prec yy \ \wedge \ x \not\prec yy)$$

But this is inconsistent in K. From this reductio, we conclude that (Rgd⁻) holds. Note that, since axiom 4 enables us to necessitate (Rgd⁺), we have also shown that (Rgd⁻) follows from (Rgd⁺) in S5. ⊣

A central concern in Chapter 10 was to provide formal arguments in support of plural rigidity. We focused on three principles that can yield such arguments: uniform adjunction, partial rigidification, and uniform traversability.

(UniAdj*) $\Box\forall xx\forall z\exists yy\Box\forall u(u \prec yy \leftrightarrow u \prec xx \vee u = z)$

(PartRig) $\Box\forall xx\exists yy(xx \approx yy \wedge \forall x(x \prec yy \rightarrow \Box x \prec yy))$

(UniTrav) $\Box\forall x(x \prec xx \leftrightarrow \bigvee_{a \prec xx} x = \bar{a})$

We now prove the various claims made in the main text.

(UniAdj*) entails (Rgd). The proof relies on axiom B. We first derive (Rgd⁺). Assume that $z \prec xx$. It follows from (UniAdj*) and axiom T that are yy such that:

$$\forall u(u \prec yy \leftrightarrow u \prec xx \vee u = z) \ \wedge \ \Box\forall u(u \prec yy \leftrightarrow u \prec xx \vee u = z)$$

An obvious consequence of this fact is that $\Box(z \prec yy)$. Since $z \prec xx$, so we also have that $xx \approx yy$. By (Cov), $\Box(xx \approx yy)$. But $\Box(z \prec yy)$ and $\Box(xx \approx yy)$ entail that $\Box(z \prec xx)$. Thus:

$$z \prec xx \rightarrow \Box(z \prec xx)$$

The variables are arbitrary and the reasoning relies only on a necessary non-logical premise, so we conclude:

$$\Box\forall z\forall xx(z \prec xx \rightarrow \Box(z \prec xx))$$

The other component of the rigidity claim, (Rgd⁻), can be proved similarly by appealing to (Cov⁻), which was proved to follow from (Indisc) and B. Alternatively, we can obtain (Rgd⁻) from (Rgd⁺) in S5, as shown above. ⊣

(PartRig) entails (Rgd). The axioms of S5 are used. Assume that $z \prec xx$. By (PartRig) and T, there are *yy* such that:

$$xx \approx yy \land \forall x(x \prec yy \to \Box x \prec yy)$$

So $\Box(z \prec yy)$. By (Cov), $\Box(xx \approx yy)$. Therefore, $\Box(z \prec xx)$. So we have shown that:

$$\forall z \forall xx(z \prec xx \to \Box(z \prec xx))$$

The modal status of (PartRig) guarantees that this holds of necessity. We can then derive (Rgd$^-$) from (Rgd$^+$) in S5. ⊣

So far we have two arguments for plural rigidity. The first relies on (UniAdj*) and makes use of axiom B. The second relies on (PartRig) and can be carried out in S5. We claimed that, against the background of (Cov) and S5, the following three statements are equivalent: (UniAdj*), (PartRig), and (Rgd). An analogous claim holds in the contingentist setting. We have already proved that each of the former statements entails the third. So it suffices to establish the two converse entailments.

(Rgd) entails (UniAdj*). Let *xx* and *z* be arbitrary. By plural comprehension, there are *yy* such that:

$$(\dagger) \qquad \forall u(u \prec yy \leftrightarrow u \prec xx \lor u = z)$$

We want to show that this generalization holds by necessity. Let us first prove that the left-to-right direction of (†) holds by necessity. Suppose that $u \prec yy$. Then either $u \prec xx$ or $u = z$. If $u \prec xx$, then (Rgd) implies $\Box(u \prec xx)$ and thus $\Box(u \prec xx \lor u = z)$. If $u = z$, then $\Box(u = z)$ and thus $\Box(u \prec xx \lor u = z)$. So, in either case, $\Box(u \prec xx \lor u = z)$. Since *u* is arbitrary, we have established that:

$$\forall u(u \prec yy \to \Box(u \prec xx \lor u = z))$$

By (BFR), we obtain:

$$(*) \qquad \Box \forall u(u \prec yy \to u \prec xx \lor u = z)$$

Now we prove the necessity of the other direction. We proceed by *reductio* and suppose the opposite, which can be written as:

$$\Diamond\exists u((u \prec xx \vee u = z) \wedge u \nprec yy)$$

A bit of logical manipulation yields the following disjunction:

$$\Diamond\exists u(u \prec xx \wedge u \nprec yy) \ \vee \ \Diamond\exists u(u = z \wedge u \nprec yy)$$

But each disjunct leads to contradiction. Consider the first disjunct. By the converse of (BFR), that is:

$$\Diamond\exists u(u \prec xx \wedge \varphi(u)) \ \rightarrow \ \exists u(u \prec xx \wedge \Diamond\varphi(u))$$

we obtain:

$$\exists u(u \prec xx \ \wedge \ \Diamond(u \nprec yy))$$

But given how yy were introduced, if $u \prec xx$, then $u \prec yy$. By (Rgd), $\Box(u \prec yy)$, which contradicts $\Diamond(u \nprec yy)$. Now consider the second disjunct. It entails that $\Diamond(z \nprec yy)$. But this contradicts $\Box(z \prec yy)$, which follows from $z \prec yy$ by (Rgd). We conclude from the reductio that:

(**) $$\Box\forall u(u \prec xx \vee u = z \rightarrow u \prec yy)$$

Our target claim, (UniAdj*), is an immediate consequence of the conjunction of (*) and (**). ⊣

(Rgd) entails (PartRig). This is straightforward and requires no special modal assumption.

(1)	$\Box\forall xx\forall x(x \prec xx \rightarrow \Box(x \prec xx))$	(Rgd)
(2)	$\Box\forall xx\big(xx \approx xx \ \wedge \ \forall x(x \prec xx \rightarrow \Box(x \prec xx))\big)$	1, PFO, K
(3)	$\Box\forall xx\exists yy\big(xx \approx yy \ \wedge \ \forall x(x \prec yy \rightarrow \Box(x \prec yy))\big)$	2, PFO, K ⊣

The last formal argument for plural rigidity considered in Chapter 10 relies on the principle of uniform traversability:

(UniTrav) $$\Box\forall x(x \prec xx \leftrightarrow \bigvee_{a \prec xx} x = \bar{a})$$

The formulation of this principle (and of the resulting argument) requires an infinitary extension of PFO.

(UniTrav) entails each instance of (Rgd). Let *xx* be any plurality. By (UniTrav), we can find a traversal:

$$(*) \qquad \Box\forall x(x \prec xx \leftrightarrow \bigvee_{a \prec xx} x = \bar{a})$$

By T and Universal Instantiation, we obtain $z \prec xx \leftrightarrow \bigvee_{a \prec xx} z = \bar{a}$. Now, the necessity of identity entails $\bigvee_{a \prec xx} z = \bar{a} \rightarrow \bigvee_{a \prec xx} \Box(z = \bar{a})$. And basic modal logic ensures $\bigvee_{a \prec xx} \Box(z = \bar{a}) \rightarrow \Box \bigvee_{a \prec xx} z = \bar{a}$. Combining the three preceding formulas, we obtain:

$$z \prec xx \rightarrow \Box \bigvee_{a \prec xx} z = \bar{a}$$

From this and (*) we derive:

$$z \prec xx \rightarrow \Box(z \prec xx)$$

Since z is arbitrary, we can universally generalize to establish our desired conclusion:[39]

$$\forall x(x \prec xx \rightarrow \Box(x \prec xx))$$

The modal profile of (*) ensures that this conclusion holds of necessity. ⊣

[39] Can we proceed to universally generalize on '*xx*' as well? In fact, this move is unavailable because (UniTrav) is a axiom *scheme*, which for each *xx* states that there is a traversal, but which provides no uniform way of specifying such a traversal.