12
Critical Plural Logic

12.1 Introduction

An inconsistent triad figured centrally in the previous chapter. We cannot simultaneously accept universal singularizations, unrestricted plural comprehension, and absolute generality. So at least one of these *prima facie* attractive assumptions has to be abandoned. Which one?

We began by rejecting generality relativism, which abandons absolute generality. This left us with a choice between rejecting universal singularizations and rejecting unrestricted plural comprehension. We proceeded to take a closer look at the former option, which has received far more attention than the latter. Our discussion revealed some serious problems with this popular option. One problem is that there remains pressure to accept universal singularizations, in particular when we examine how plural logic can be used to illuminate set theory. Another problem is that this version of absolute generality faces difficulties akin to those of generality relativism. Overall, the previous chapter thus motivates taking a closer look at the last remaining option, namely to restrict the plural comprehension scheme.

At the outset, this option seems unpromising. How could there *not* be some things that are all and only the φs? Provided there is at least one φ, the mentioned claim seems obviously true, as observed for example by Boolos and Hossack (see Section 2.5). Clearly, an explanation would be needed of why the plural comprehension scheme must be restricted. Our proposed explanation is simple, at least in essence. True, all we need to do to define a plurality is circumscribe the objects in question; in particular, there is no postulation of a set or any other "plural entity" over and above these objects. *But the objects in question do need to be circumscribed.* And as we shall see, on some metaphysical views, reality as a whole resists proper circumscription. If a view of this sort is right, there can be no universal plurality, as this

The Many and the One: A Philosophical Study of Plural Logic. Salvatore Florio and Øystein Linnebo,
Oxford University Press. © Salvatore Florio and Øystein Linnebo 2021.
DOI: 10.1093/oso/9780198791522.003.0012

would require circumscribing something uncircumscribable. It follows that the plural comprehension scheme must be restricted.[1]

The aim of this chapter is to explore and develop these considerations. This leads to a third way, we argue, between generality relativism and the traditional form of generality absolutism.

12.2 The extendability argument

Let us reconsider the kind of extendability argument that is typically used to motivate generality relativism. In a nutshell, the argument takes the following form.[2]

> Assume that we quantify over absolutely everything. For every condition $\varphi(x)$, we can define a set $\{x : \varphi(x)\}$ of all objects satisfying the condition:
>
> $$(*) \qquad \forall x(x \in \{x : \varphi(x)\} \leftrightarrow \varphi(x))$$
>
> Consider the condition '$x \notin x$' and the resulting set $R = \{x : x \notin x\}$. If R is in the range of the quantifier '$\forall x$' in the associated instance of $(*)$, a contradiction follows by familiar Russellian reasoning. Therefore, R is outside the range of this quantifier, and we weren't quantifying over absolutely everything after all.

Clearly, the crux of the argument is the use of an arbitrary condition to define a set subject to the requirement $(*)$.

Arguments of this form can be frustrating, however. Why is it permissible to define the mentioned sets? Generality relativists take an extremely liberal view of what constitutes a permissible mathematical definition. They claim that this liberalism supports the mentioned crux and thus also their relativist conclusion. From the point of view of a generality absolutist, however, this extreme liberalism is unacceptable—indeed provably so, as can be seen by considering the following simple truth of first-order logic:

[1] Some alternative strategies for defending this thesis can be found in Spencer 2012 and Hossack 2014. Essentially the same view is also defended in Linnebo 2010, although the quantifiers used here correspond to his "modalized quantifiers" '$\Box\forall$' and '$\Diamond\exists$'.

[2] See Dummett 1991 (especially Chapter 24), Parsons 1974b, Glanzberg 2004, and Fine 2006.

$$\neg\exists y \forall x(Rxy \leftrightarrow \neg Rxx)$$

Our inability to define the offending Russellian set is just an instance of this logical truth, obtained by replacing '*R*' with the relativists' desired notion of membership in a set.

Thus, in its standard form, the extendability argument fails to resolve the debate about the possibility of absolute generality. The argument turns on the permissibility of certain definitions, which absolutists have good reason to reject. We believe progress can be made by means of a more nuanced formulation of the argument. To explain what we have in mind, it is useful to start with an analogy.

Suppose you detest web pages that link to themselves.[3] So you wish to create a web page that links to all web pages that are innocent of this bad habit. In other words, you wish to create a web page that links to all and only the web pages that do not link to themselves. Can your wish be fulfilled? The answer depends on how your wish is analyzed. Should the scope of the crucial plural description—'the web pages that do not link to themselves'—be narrow or wide? Depending on the scope of the description, your wish can be analyzed in either of the following two ways:

(N) You wish to design a web page y such that, for every web page x, y links to x if and only if x does not link to itself.

(W) There are some web pages xx such that, for every web page x, x is one of xx just in case x does not link to itself, and you wish to design a web page y that links to all and only xx.

On the narrow scope reading (N), your wish is flatly incoherent. The desired web page would have to link to itself just in case it does not link to itself. On this reading, your wish is no better than the wish to bring about the existence of a Russellian barber:

(B) You wish there to be a barber y such that, for all x, y shaves x if and only if x does not shave himself.

On the wide scope reading (W), by contrast, there is no conceptual or mathematical obstacle to the fulfillment of your wish. First, you identify all

[3] See Linnebo 2016, Section 7, and Linnebo 2018, Section 3.3. This example has been independently used by Brian Rabern in teaching and on social media.

the web pages xx that refrain from the bad habit of self-linking. Then, you create a new web page that links to all and only xx.

What explains this stark difference between the two readings? The heart of the matter is how one specifies the target collection, that is, the web pages of which you wish to create a comprehensive inventory. (As before, we use the word 'collection' in an informal way for anything that has a membership structure, such as a set, class, plurality, or indeed even a Fregean concept—where the relation between instance and concept is regarded as a membership structure.) On (N), the target is specified intensionally by means of the condition 'x does not link to itself'. This intensional specification means that the target shifts with the circumstances. First, you find that there is no web page of the sort you wish for. So you attempt to fulfill your wish by changing the circumstances, that is, by creating a web page of the desired sort. But since the target is specified intensionally, this new web page must itself be taken into account when assessing whether your wish has been satisfied—which of course it has not, as logic alone informs us.

By contrast, on the wide scope reading (W), the target is specified extensionally by means of the plurality xx. This extensional specification ensures that the target stays fixed when you change the circumstances. (Here we invoke the modal rigidity of pluralities, which was defended in Chapter 10.) You can thus fulfill your wish by creating a new web page that links to all and only xx. Although xx are described, in the original circumstances, by means of a condition that is prone to paradox, there is no requirement that xx should remain so described in alternative circumstances. Like any other plurality, xx are tracked rigidly across alternative circumstances, not in terms of any description that these objects happen to satisfy.

With this analogy in mind, let us return to the question of what is a reasonable liberalism about mathematical definitions. Suppose you care about sets, not web pages. You wish to define a set by specifying its elements. As our web page analogy reveals, it is essential to distinguish between two different ways in which the elements of the would-be set might be specified. You might specify the elements *intensionally*, by means of a condition $\varphi(x)$:

(I) You wish to define a set y such that, for every object x, x is an element of y if and only if $\varphi(x)$.

Alternatively, you might specify the elements of the would-be set *extensionally*, by means of a plurality xx:

(E) You wish to define a set y such that the elements of y are precisely xx.

Can either wish be fulfilled?

This is a question about what it takes for a mathematical definition to be permissible. We claim that the proposed definition is often problematic when the target is specified intensionally, but always permissible when the target is specified extensionally. Our defense of these claims will be informed by our web page analogy.

Let us begin with the negative claim that (I) is often problematic. The reason is simple. We can hardly be more liberal about mathematical definitions than we are about objects that we literally (and easily) construct, such as web pages. This means we need to be extremely cautious about which definitions of sets we deem permissible when the target is specified intensionally. To illustrate how such definitions can be problematic, observe that one instance of the intensionally specified wish (I) is an analogue of the problematic narrow-scope wish (N) concerning web pages:

(N′) You wish to define a set y such that, for every object x, x is an element of y if and only if x is not an element of itself.

Just as (N) is flatly incoherent, so, we contend, is (N′). This takes care of the negative claim, showing also that the standard extendability argument is too quick. In the next section, we defend the positive claim that (E) is always permissible.

12.3 Our liberal view of definitions

Suppose that the target set is specified extensionally by means of a plurality xx. Then this specification ensures that the target won't shift with the circumstances. We therefore have no difficulty making sense of circumstances in which xx define a set, much as we have no difficulty making sense of circumstances in which some given web pages yy are precisely the ones to which some new web page links.

We can be far more specific, though. Consider a dispute between a proponent and an opponent of the proposed definition. Suppose both parties accept a domain dd. The proponent now wishes to define one or more sets of the form $\{xx\}$, where xx are drawn from dd. She does not insist that the sets to be defined be among dd; in this sense, the sets may be "new". To shore up the proposed definition, she provides the following account of what it takes for a "new" set to be identical with another set or have a certain element:[4]

(i) $\{xx\} = \{yy\}$ if and only if $xx \approx yy$
(ii) $y \in \{xx\}$ if and only if $y \prec xx$

These clauses achieve something remarkable. They provide answers to all atomic questions about the "new" sets of the form $\{xx\}$ in terms that are concerned solely with the "old" objects in *dd*, objects that were available before the definition. That is, all atomic questions about the "new" objects receive answers in terms of the "old" objects that both parties to the dispute accept.

In fact, this is merely an instance of the more general liberal view of definitions encountered in Sections 4.4 and 5.8. According to this view, it suffices for a mathematical object to exist that an adequate definition of it can be provided, where the adequacy is understood as follows. Consider a domain *dd* of objects standing in certain relations. We would like to define one or more additional objects. Suppose our definition provides truth conditions for every atomic predication concerned with the desired "new" objects in the form of some statement concerned solely with the "old" objects with which we began. Thus, every atomic question about the "new" objects receives an answer in terms that are solely about the "old" objects. Then, according to our liberal view, the definition is permissible. This idea can be applied not only to sets but also to mereological sums (as we saw in Section 5.8), cardinal numbers, and so on. For example, the cardinal numbers of two pluralities are identical if and only if the pluralities are equinumerous.[5]

Our liberal view of definitions is an explication of a theme that one often encounters in mathematicians' own reflections on their practice. A striking example is the following passage by Cantor.

> Mathematics is in its development entirely free and only bound in the self-evident respect that its concepts must both be consistent with each other and also stand in exact relationships, ordered by definitions, to those concepts which have previously been introduced and are already at hand and established. [...] [T]he *essence* of *mathematics* lies precisely in its *freedom*. (Cantor 1883, 896)

[4] In fact, the right-to-left direction of (i) follows from the plural indiscernibility principle (Indisc) introduced in Section 2.4.

[5] See discussion in Linnebo 2018, Section 3.3.

A similar sentiment is expressed by other mathematicians, such as David Hilbert and Henri Poincaré. Hilbert writes:

> As long as I have been thinking, writing and lecturing on these things, I have been saying [...]: if the arbitrarily given axioms do not contradict each other with all their consequences, then they are true and the things defined by them exist. This is for me the criterion of truth and existence.
>
> (Letter to Frege of 29 December 1899, in Frege 1980, 39–40)

According to Poincaré, "[m]athematics is independent of the existence of material objects; in mathematics the word 'exist' can have only one meaning; it means free from contradiction" (1905, 1026).

Let us apply our liberal view of definitions to the case of sets. It is instructive to compare with the situation where the desired set is specified intensionally, by means of a membership condition. Again, we start with some objects *dd* accepted by both parties. A more extreme proponent of liberal definitions may wish to define sets of the form $\{x : \varphi(x)\}$, where any parameters in the membership condition $\varphi(x)$ are drawn from *dd*. As before, she does not insist that these sets be among *dd*; they may be "new". The opponent will rightly challenge her to provide an account of what it takes for "new" sets to be identical or to have certain elements. Given the intensional specification of the desired sets, her answers will be as follows:

(i′) $\{x : \varphi(x)\} = \{x : \psi(x)\}$ if and only if $\forall x(\varphi(x) \leftrightarrow \psi(x))$
(ii′) $y \in \{x : \varphi(x)\}$ if and only if $\varphi(y)$

These answers are potentially problematic in a way that their extensional analogues, (i) and (ii), are not. An interesting example is the attempt to define a set $a = \{x : x \in x\}$. If this definition is to succeed, there must be an answer to the question of whether a is an element of itself. But the only answer we receive from clause (ii′) is that $a \in a$ if and only if $a \in a$. Of course, this is useless.[6] More tellingly, the answer is not stated in terms of the objects accepted by both parties to the dispute. An atomic question about the "new" object a receives an answer that essentially involves *this very object*; there is no reduction to the "old" objects among *dd*.

[6] It could be worse. When we ask whether the Russell set $b = \{x : x \notin x\}$ is an element of itself, we receive an inconsistent answer.

Notice that it is of no avail for the extreme liberal to allow a to lie outside of dd, that is, in our parlance, to be "new". The set a is specified intensionally, by means of the membership condition '$x \in x$', and we cannot "outrun" this specification. Even in a domain that strictly extends dd, a is, by definition, the set of all and only the objects that satisfy the condition '$x \in x$'. By contrast, when a set is specified extensionally by means of a plurality xx, it does help to consider a domain that strictly extends dd. Even if xx are, say, all the sets among dd that are not elements of themselves, xx need not satisfy this plural description in an extended domain. For xx are tracked rigidly into the extended domain, not by means of the description. This makes the world safe for the desired set $\{xx\}$, provided that the set is located outside of dd. Notice also the striking parallelism with the case of web page design. Suppose you want a web page to link to all and only the members of some collection of web pages, for example, the collection of web pages that do not link to themselves. If the target collection is specified intensionally, it is of no avail to create a *new* web page: you cannot "outrun" this problematic specification. By contrast, if the collection is specified extensionally, there is no obstacle to the creation of the desired web page.

The picture that emerges is that there is a fundamental difference between the proposed definitions of sets depending on whether the target is specified extensionally or intensionally. In the former case, every atomic question about the "new" objects receives an answer expressed solely in terms of the "old" objects, whereas in the latter case, this kind of reduction is often unavailable. The proposed definitions are therefore often unacceptable when the target is specified intensionally. In the case of an extensional specification, on the other hand, a proponent of liberal definitions is in a much stronger position. She has laid out certain definitions, which are mathematically fruitful and have the desirable property that all atomic questions about the "new" objects receive answers in terms that are acceptable to her opponent. Granted, she cannot force her opponent to accept the proposed definitions: he does not contradict himself when he rejects them. But she can justifiably accuse her opponent of dogmatism that stifles scientific progress. He dogmatically clings to certain beliefs that stand in the way of fruitful mathematics. By insisting that dd are all-encompassing—and thus that there can be no "new" objects outside of dd—he privileges certain metaphysical or logical dogmas over good mathematics.

12.4 Why plural comprehension has to be restricted

In the previous chapter, we defended the permissibility of absolute generality. And we have just argued that any given objects can be used to define a set. Thus, we have defended two of the three assumptions that we know to form an inconsistent triad. We therefore have no alternative but to reject the third assumption, that is, to restrict the plural comprehension scheme.

Here is an intuitive and more direct version of our argument for that conclusion. To define a plurality, we need to circumscribe some objects. But when we circumscribe some objects, we can use these objects to define yet another object, namely their set, in a way that would not be possible were the objects in question not circumscribed. And since yet another object can be defined, it follows that the circumscribed objects cannot have included *all* objects. Thus, reality as a whole cannot be circumscribed: there is no universal plurality. Consequently, the plural comprehension scheme needs to be restricted.

It might be objected that traditional plural logic is so compelling that the correct response to our findings is not to reject it but to reconsider our defenses of absolute generality and the view that every plurality can be used to define a set. This response deserves a hearing. So let us explain why we believe it is appropriate to reject traditional plural logic. First, we have, as already mentioned, offered positive arguments for the two other assumptions that make up the inconsistent triad.

Second, we have identified major difficulties with each of the two other responses to the inconsistent triad. Both relativism and traditional absolutism suffer from a serious expressibility deficit—the former because of its relativism, the latter because it is pushed up through the type-theoretic hierarchy. Our alternative solution avoids these expressibility problems. Without a universal plurality, Plural Cantor no longer entails Plural Profusion. And without Plural Profusion, we can no longer prove the Ascent Theorem, which appeared to show that the need for a generalized semantics forces us higher and higher up in the type-theoretic hierarchy. In fact, even if one accepts a type-theoretic hierarchy despite not being *forced* up it, there is a way to restore full expressibility, as we explained in Section 11.7.

Ultimately, though, we believe the debate must be decided by theoretical considerations. Which of the three horns of the trilemma is theoretically most satisfying? We hope this book as a whole will show that the widely ignored third horn has major theoretical attractions, which very likely exceed those of its two rivals.

Let us compare our third alternative—*critical (generality) absolutism*, as we shall call it—with its two rivals. We begin by reminding ourselves of how the three types of view respond to the inconsistent triad.

type of view	universal singularization	absolute generality	unrestricted plural comprehension
traditional absolutism	✗	✓	✓
relativism	✓	✗	✓
critical absolutism	✓	✓	✗

Just like the traditional absolutist, our view accepts that our quantifiers can achieve a form of absolute generality. We differ from the traditional absolutist only in our insistence that absolute generality is generality over an extensionally *in*definite domain, which consequently does not sustain unrestricted plural comprehension.

More nuanced comparisons are possible as well. Although we deny that reality as a whole can be circumscribed, there are also restricted domains of quantification that are extensionally definite and can thus be specified as a plurality. Let *dd* be one such domain. Consider a condition $\varphi(x)$ that has an instance among *dd*. We can then use a plural separation principle to define some objects *xx* that are all and only the objects among *dd* that satisfy $\varphi(x)$.[7] This shows that the unrestricted plural comprehension scheme is valid whenever the domain of quantification is extensionally definite. Instead, an extensionally definite domain does not permit a universal singularization within this domain. For recall that our argument for the permissibility of singularizing pluralities as sets forces us out of any given extensionally definite domain. So universal singularization fails when all our quantifiers are relativized to such a domain. The following table provides a summary of these observations:

type of domain	universal singularization	unrestricted plural comprehension
extensionally definite	✗	✓
extensionally indefinite	✓	✗

[7] See Appendix 10.A for a justification.

This table enables some illuminating comparisons of our view with its two main rivals. First, when a domain is extensionally definite, the correct view on plural comprehension and universal singularization is precisely that of traditional absolutism. This traditional view is entirely correct—when relativized to any given extensionally definite domain. Its only error is the assumption that reality as a whole can be circumscribed, that is, that there is a universal plurality. *Modulo* this single—though important—error, we are in agreement with the traditional absolutist.

Second, consider relativism. Of course, we disagree with the relativists on the important question of the possibility of absolute generality. But there is something more interesting to be said. There are, as noted, many extensionally definite domains, namely those that can be specified as a plurality. Suppose we restrict our attention to generality over such domains. Thus restricted, the relativist's claim is right: given any extensionally definite domain, there is indeed an even larger such domain. But the relativists fail to appreciate the important distinction between domains that are extensionally definite and those that are not. They appear tacitly to assume that all domains are extensionally definite. It is only restricted to such domains that their extendability claim has force, as argued in Sections 12.2 and 12.3. Moreover, by ignoring the possibility of extensionally *in*definite domains, they inflict upon themselves a gratuitous expressibility deficit, which we avoid by accepting generality over an extensionally indefinite domain of absolutely everything.[8]

12.5 The principles of critical plural logic

By advocating a restriction of the plural comprehension scheme, we depart from the traditional formulation of plural logic. To emphasize this departure, let us call our approach *critical plural logic*.[9]

How, exactly, does our critical plural logic differ from the traditional version? We accept standard classical first-order logic. Furthermore, we allow the plural quantifiers to be governed by axioms and rules analogous to

[8] To be fair, many relativists achieve a form of absolute generality through schematic generality, as discussed in Section 11.4. But this form of generality is either subject to its own expressibility deficit (by permitting only Π_1-generalizations) or will be transformed into a version of our preferred form of absolute, but extensionally indefinite, generality.

[9] This label is inspired by Charles Parsons's 'Infinity and a Critical View of Logic' (2015). Some examples of this approach to logic are discussed in a forthcoming special issue of *Inquiry* edited by Mirja Hartimo, Frode Kjosavik, and Øystein Linnebo.

those governing the first-order quantifiers.[10] We also retain the axiom stating that pluralities are non-empty and the axiom scheme stating that coextensive pluralities are indiscernible (see Section 2.4). Our quarrel with traditional plural logic concerns only the question of what pluralities there are, or, in other words, the question of which plural comprehension axioms to accept. It is therefore incumbent on us to clarify what pluralities we take there to be. It is insufficient merely to observe that the plural comprehension scheme needs to be restricted in some way or other to avoid a universal plurality. We need some "successor principles" to the unrestricted plural comprehension scheme to tell us what pluralities there in fact are.

How should these successor principles be chosen and motivated? When discussing this question, it is useful to recall the intuitive version of our argument for restricting the plural comprehension scheme. To define a plurality, we need to circumscribe some objects. But when we circumscribe some objects, we can use these objects to define yet another object, namely their set. It follows that the circumscribed objects cannot have included *all* objects and thus, in particular, that reality as a whole cannot be circumscribed.

Clearly, this argument hinges on the idea that every plurality is circumscribed, or, as we also put it, *extensionally definite*. Can this notion of extensional definiteness guide our search for successor principles and help us justify, or at least motivate, the resulting principles? Here we face a fork in the road, depending on whether or not we attempt to provide an analysis of extensional definiteness in more basic terms, and on this basis, try to provide the requisite guidance and justification.

There have been several attempts to provide such an analysis. Linnebo 2013 proposes a modal analysis inspired by Cantor's famous distinction between "consistent" and "inconsistent" multiplicities. Here is how Cantor explains the distinction in a famous letter to Dedekind of 1899:

> [I]t is necessary... to distinguish two kinds of multiplicities (by this I always mean definite multiplicities). For a multiplicity can be such that the assumption that all of its elements 'are together' leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as 'one finished thing'. Such multiplicities I call *absolutely infinite* or *inconsistent multiplicities*... If on the other hand the totality of the elements of a multiplicity can be thought of without contradiction as 'being together', so that they can be gathered together into 'one thing', I call it a *consistent multiplicity* or a 'set'. (In Ewald 1996, 931–2)

[10] But, of course, we should insist that the formulation of logical rules be neutral with respect to which comprehension axioms are validated.

Using the resources of modal logic, it is relatively straightforward to formalize Cantor's notion of a multiplicity being "one finished thing", namely, that all possible members of the multiplicity can exist or "be together". Or, changing the idiom slightly, there is no possibility of the multiplicity gaining yet more members at more populous possible worlds.[11] Based on this analysis, Linnebo 2013 proves various principles of extensional definiteness, which in the present context amount to principles concerning the existence of pluralities.

Another analysis of extensional definiteness is inspired by Michael Dummett's suggestion that a domain is definite just in case quantification over this domain obeys the laws of classical logic, not just intuitionistic.[12] Intriguingly, it turns out that a fairly natural development of this Dummettian suggestion validates almost the same principles of extensional definiteness as the modal analysis.[13] Yet other analyses may be possible as well. We invite the readers to explore.

Here we wish to pursue the other fork in the road, namely to leave the notion of extensional definiteness unanalyzed and instead to use our intuitive conception of the notion, coupled with abductive considerations, to motivate principles of extensional definiteness. This strategy has both advantages and disadvantages: it is more general, as it avoids specific theoretical commitments; but it also provides less leverage and thus less of an independent check on the proposed principles of definiteness. In any case, we believe this is an option worth exploring. We thus ask what it is for a collection to be circumscribed or extensionally definite.

First, since every single object can be circumscribed, there are singleton pluralities:

$$\forall x \exists yy \forall z (z \prec yy \leftrightarrow z = x)$$

Second, because the result of adding one object to a circumscribed plurality is also circumscribed, we accept a principle of adjunction. Given any plurality xx and any object y, we can adjoin y to xx to form the plurality $xx + y$ defined by:

$$\forall u (u \prec xx + y \leftrightarrow u \prec xx \vee u = y)$$

[11] See pp. 248–9 for an explication of this idea in a modal language.

[12] A closely related idea is found in Solomon Feferman's widely circulated and discussed manuscript, "The Continuum Hypothesis is neither a definite mathematical problem nor a definite logical problem" (Feferman unpublished).

[13] See Linnebo 2018.

Moreover, we already argued that a plural separation principle is well motivated (see Appendix 10.A). Suppose you have circumscribed a collection and have formulated a sharp distinction between two ways that members of the collection can be. Then the subcollection whose members are all and only the objects that lie on one side of this distinction is in turn circumscribed. More formally, given any plurality *xx* and any condition $\varphi(x)$ that has an instance among *xx*, there is a plurality *yy* of those members of *xx* that satisfy the condition:

$$\exists x(\varphi(x) \land x \prec xx) \rightarrow \exists yy \forall u(u \prec yy \leftrightarrow u \prec xx \land \varphi(u))$$

Next, there are some plausible union principles. Let us begin with a simple case. Since two circumscribed collections can be conjoined to make a single such collection, a principle of pairwise union is plausible. Given any plurality *xx* and any objects *yy*, there is a union plurality *zz* defined by:

$$\forall xx \forall yy \exists zz \forall u(u \prec zz \; \leftrightarrow \; u \prec xx \lor u \prec yy)$$

A generalized union principle can also be motivated. Consider some circumscribed collections, each with its own unique tag. Suppose that the collection of tags is also circumscribed. Then the "union collection" comprising all the items that figure in at least one of the tagged collections is circumscribed. This motivates a generalized union principle to the effect that the union of an extensionally definite collection of extensionally definite collections is itself extensionally definite. We can formulate this as the following schema. Suppose there are *xx* such that:

$$\forall x(x \prec xx \rightarrow \exists yy \forall z(z \prec yy \leftrightarrow \psi(x,z)))$$

Then there is *zz* such that:

$$\forall y(y \prec zz \leftrightarrow \exists x\,(x \prec xx \;\land\; \psi(x,y)))$$

Although the generalized union principle does not, on its own, entail the pairwise one, this entailment does go through in the presence of the singleton and adjunction principles.[14] It therefore suffices to adopt the generalized union principle.

[14] *Proof sketch*. Consider two pluralities *xx* and *yy*. Assume there are two distinct objects, say *a* and *b*, to tag these pluralities. (If there is only a single object, the pairwise union of *xx* and *yy* is a singleton plurality.) Now apply the generalized union principle to the formula '$(x = a \land y \prec xx) \lor (x = b \land y \prec yy)$', observing that *a* and *b* form a plurality. This yields the pairwise union of *xx* and *yy*.

The principles accepted so far do not entail the existence of any infinite pluralities; indeed, they have a model where every plurality is finite. Is it possible for an infinite collection to be circumscribed and thus to correspond to a plurality? This question calls to mind the ancient debate about the existence of completed infinities. Aristotle famously argued that only finite collections can be circumscribed, and that a collection can be infinite only in the potential sense that there is no finite bound on how many members the collection might have. This remained the dominant view until Cantor boldly defended the actual infinite and the existence of completed infinite collections. The natural numbers provide an example. Aristotle denied, whereas Cantor affirmed, the existence of a completed collection of all natural numbers.

We are interested in an analogous question concerning pluralities. Let '$P(x, y)$' mean that x immediately precedes y. Following first-order arithmetic, we accept that every natural number immediately precedes another:[15]

$$\forall x \exists y \, P(x, y) \tag{12.1}$$

We would like to know whether there is a circumscribed collection, or plurality, of all natural numbers. More precisely, we would like to know whether there are some objects xx containing 0 and closed under P, in the following sense:

$$\exists xx(0 \prec xx \land \forall x \forall y(x \prec xx \land P(x, y) \rightarrow y \prec xx)) \tag{12.2}$$

Although asserting the existence of such a plurality is a substantial step, it has also been a tremendous theoretical success, as mathematics since Cantor has clearly demonstrated. On abductive grounds, we therefore recommend accepting (12.2), conditional on (12.1), as a plural analogue of the set-theoretic axiom of Infinity.

It will be objected that this conditional principle is concerned specifically with the natural numbers and thus lacks the topic neutrality of a logical law. The objection is entirely reasonable and points to the need for a more general principle that justifies transitions such as the one from (12.1) to (12.2). There is nothing special about 0 and the functional relation P. So, for *any* plurality xx and functional relation, there should be a plurality yy

[15] Aristotle would only accept a weaker, modal analogue of this principle, namely $\Box \forall x \Diamond \exists y \, P(x, y)$, where the modal operators represent metaphysical modalities.

containing xx and closed under that function. We therefore claim that the desired generalization is the schematic principle that every plurality can be closed under function application:

$$(12.3) \qquad \forall x \exists! y\, \psi(x,y) \rightarrow \forall xx \exists yy (xx \preccurlyeq yy \wedge \forall x \forall y (x \prec yy \wedge \psi(x,y) \rightarrow y \prec yy))$$

We adopt this as the official plural principle of infinity. In practice, however, it doesn't much matter whether we accept this more general schematic principle or merely (12.2), conditional on (12.1). For in the presence of first-order arithmetic, ordered pairs, and the other principles concerning pluralities, these two principles of infinity are provably equivalent.[16]

A plural analogue of the axiom of Replacement is plausible as well. Consider a plurality of objects. Now you may replace any member of this plurality with any other object, or, if you prefer, leave the original object unchanged. Then the resulting collection is also circumscribed and thus defines a plurality of objects. We formalize this as follows.

$$\forall xx [\forall x (x \prec xx \rightarrow \exists! y\, \psi(x,y)) \rightarrow \exists yy \forall y (y \prec yy \leftrightarrow \exists x (x \prec xx \wedge \psi(x,y)))]$$

It is pleasing to observe that this plural version of Replacement follows from the generalized union principle and the singleton principle. And, as in the case of sets, the plural principle of replacement entails that of separation.[17]

To sum up, we started with some core assumptions shared with traditional plural logic: first-order logic, axioms and rules governing the plural quantifiers, and the principles (Non-empty) and (Indisc). Next, our intuitive

[16] *Proof sketch*. The only hard direction is to show that the specific conditional entails the general one. Consider any xx, and assume that ψ is functional. For every member $a \prec xx$, we contend that there is a plurality zz_a containing a and closed under ψ. Given this contention, the generalized union principle enables us to define the desired plurality yy as the union of all the pluralities zz_a. To prove the contention, we observe that, using ordered pairs and plural quantification, we can produce a formula $\theta(n,y)$ which expresses that n is a natural number and that y is the nth successor of a in the series generated by ψ. We do this by letting $\theta(n,y)$ state that $\langle n,y \rangle$ is a member of every plurality containing $\langle 0,a \rangle$ and closed under the operation $\langle m,u \rangle \mapsto \langle m+1,v \rangle$, where v is the unique object such that $\psi(u,v)$. Now we apply the generalized union principle to the plurality of all natural numbers and the formula θ to obtain the desired plurality zz_a.

[17] *Proof sketch*. Consider xx and a condition $\varphi(x)$. Assume $\varphi(a)$ for some member a of xx. Now apply the principle of replacement to the condition $\psi(x,y)$ defined as $(\neg\varphi(x) \wedge y = a) \vee (\varphi(x) \wedge y = x)$. This yields the subplurality of those members of xx that satisfy $\varphi(x)$.

conception of extensional definiteness motivates the following three principles concerning pluralities:

- singleton
- adjunction
- generalized union

An additional principle receives a more theoretical justification:

- infinity

These four principles, in addition to the core assumptions just mentioned, constitute the system we call *critical plural logic*.

As observed, the first three of these principles entail some other plausible principles:

- separation
- pairwise union
- replacement

Moreover, it is straightforward to verify that each principle of critical plural logic can be derived from traditional plural logic. In essence, each of the pluralities we licence is a subplurality of the universal plurality licenced by traditional plural logic. Critical plural logic is therefore strictly weaker than the traditional system. This relative weakness is for a good cause, as will emerge clearly in Section 12.7, where we explore the connection between critical plural logic and set theory. This connection is far simpler and, we believe, more natural than in the case of traditional plural logic.

12.6 Extensions of critical plural logic

When stronger expressive resources are accepted, various extensions of critical plural logic can be formulated and justified. The addition of superplural resources provides an obvious example. This addition enables us to express analogues of the principles of critical plural logic. Here we will focus on two novel and more interesting principles.

First, we can formulate a principle of extensional definiteness that corresponds to the familiar set-theoretic axiom of Powerset. We can do this entirely without mention of sets by using superplurals. For any plurality *xx*, there is a superplurality *yyy* of all subpluralities of *xx*:

$$\forall xx \exists yyy \forall zz (zz \prec yyy \;\leftrightarrow\; zz \preccurlyeq xx)$$

The justification for this "powerplurality" principle is less straightforward than in the case of the earlier principles. It relies on what Bernays (1935) calls "quasi-combinatorial" reasoning: a combinatorial principle that is compelling for finite domains is extrapolated to infinite domains. The powerplurality principle is certainly reasonable when the plurality *xx* is finite: we can then list all of its subpluralities. The general principle is a big and admittedly daring extrapolation of the finitary principle into the infinite. Its justification is thus partially abductive: the big and daring extrapolation has proved to be a theoretical success. Just like its set-theoretic analogue, the principle fits into a coherent and fruitful body of theory, as will be explained shortly. The principle also provides important information about which superpluralities there are.

Second, superplurals make it possible to formulate plural choice principles. For example, given a superplurality *xxx* of non-overlapping pluralities, there is a "choice plurality" whose members include one member of each plurality of *xxx*. That is, for each such *xxx* we have:

$$\exists yy \forall zz (zz \prec xxx \;\rightarrow\; \exists! y (y \prec zz \;\wedge\; y \prec yy))$$

As in the case of the powerplurality principle, plural choice principles are extrapolations from the finite into the infinite, and their is justification is partially abductive.[18]

In sum, the addition of superplural resources enables us to formulate and justify an extended critical plural logic. Two distinctive principles are:

- powerplurality
- choice

[18] See Pollard 1988 for a defense of the Axiom of Choice on the basis of a plural choice principle. If ordered pairs are available, there is less of a need for superplurals to express choice principles. For example, we can assert that for any relation coded by means of a plurality of ordered pairs, there is a functional subrelation with the same domain, again coded by means of a plurality of ordered pairs.

Of course, yet stronger principles can be countenanced as ever greater expressive resources are considered.

12.7 Critical plural logic and set theory

The various plural principles we have discussed provide valuable information about sets. To see this, recall the correspondence we have advocated between pluralities and sets:

(i) $\{xx\} = \{yy\}$ if and only if $xx \approx yy$
(ii) $y \in \{xx\}$ if and only if $y \prec xx$

Using this correspondence, the plural principles entail analogous set-theoretic axioms.

However, there are two reasons to worry that the plural principles will not lead to Zermelo-Fraenkel set theory. First, since we do not ordinarily admit an empty plurality, there is a threat of losing the empty set. Some ways to address this threat were discussed in Section 4.4. One solution is to allow an empty plurality. Another is to allow the "set of" operation $xx \mapsto \{xx\}$ to be what Oliver and Smiley (2016, 88) call a "co-partial" function, which can thus take the value $\varnothing$ on an undefined argument. Either way, we can prove the existence of an empty set.

Second, since plural logic is applied to all sorts of objects, the mentioned correspondence introduces impure sets, that is, sets of non-sets. The relevant comparison is therefore not ZFC, but ZFCU—the modified system which accommodates urelements (see Section 4.7). Recall that this system is obtained by making explicit the quantification over sets in the axioms of ZFC. Whenever a quantifier of an axiom of ZFC is intended to range over sets even when urelements are introduced, we explicitly restrict this quantifier to sets by means of a predicate 'S' intended to be true of all and only sets.

Our aim, then, is to use critical plural logic and the correspondence principles (i) and (ii) to justify axioms of ZFCU. We define '$S(x)$' as '$\exists xx(x = \{xx\})$'. This enables us, it turns out, to derive the axioms of Empty Set, Pairing, Separation, Union, Infinity, and Replacement. (The proofs are relatively straightforward.) Moreover, the axiom of Extensionality follows immediately from the correspondence between pluralities and sets, and

Foundation can be seen as explicating how sets are successively formed from pluralities of elements, and as justified on that basis.[19]

To derive the axioms of Powerset and Choice, we need to go beyond critical plural logic. Choice follows naturally from the superplural choice principle discussed in the previous section. Deriving Powerset is less straightforward. Given any set *a*, we want to prove the existence of its powerset. To do so, we need to show that there is a plurality comprising all of *a*'s subsets. How might this be done? One option, inspired by the iterative conception of set, is to postulate the existence of such a plurality, on the grounds that when *a* was formed, all its elements were available, thus giving us the ability also to form all of *a*'s subsets. We prefer to utilize the powerplurality principle of the previous section, reasoning as follows. Let *aa* be the elements of *a*, and consider their superplurality *bbb*. For every subset *x* of *a*, if $x = \{xx\}$ for some *xx*, then $xx \prec bbb$. That is, *bbb* circumscribe all the subpluralities of *aa*. But if some pluralities are jointly circumscribed, so are the unique sets formed from precisely these pluralities. This gives us the desired plurality of subsets of *a*. (This reasoning assumes that the extended, superplural logic contains a replacement principle that allows us to replace each plurality of a superplurality with a unique object and thus arrive at a plurality.)

Our discussion shows that critical plural logic, and the plausible superplural extensions thereof, have great explanatory power, especially in connection with the correspondence principles (i) and (ii). Still, one might worry that things are too good to be true. Do we even know that our assumptions—the mentioned plural logics and the correspondence principles—are jointly consistent? This worry can be put to rest by proving that these assumptions are consistent relative to ZFC. For critical plural logic and the correspondence principles, we do this by translating plural quantifiers as first-order quantifiers restricted to non-empty sets. An analogous relative consistency result can be given for the described extension of critical plural logic. In that case, superplural quantifiers are translated as first-order quantifiers restricted to non-empty sets of non-empty sets.

[19] Relative to the other axioms of ZFC, Foundation is equivalent to the following induction scheme:

> Suppose that every urelement is φ and that, for every *xx* each of which is φ, $\{xx\}$ too is φ. Then everything is φ.

This induction scheme explicates the idea that every set is generated by means of the "set of" operation.

Let us end with some more general observations. First, on the view we have defended, plural logic lacks one of the features commonly ascribed to pure logic, namely epistemic primacy vis-à-vis all other sciences (see Section 2.5). To see this, we need only recall the extent to which our defense of critical plural logic relies on abductive considerations, in particular, on considerations about what constitutes a permissible mathematical definition. Moreover, some of the principles of critical plural logic—infinity, powerplurality, and choice—specifically received an abductive justification.

Second, our view forges a close connection between the principles of critical plural logic and the axioms of set theory, which suggests that critical plural logic and its extensions have non-trivial mathematical content. Let us explain. We have provided a factorization of set theory into two components: the correspondence principles, which link pluralities and their corresponding sets, and critical plural logic, which provide information about what pluralities there are and how these behave. Clearly, the strong mathematical content of set theory derives from these two components. It is the correspondence principles that introduce sets as mathematical objects by characterizing what Gödel called the the "set of" operation (see Section 4.6). What sets there are, however, will depend on what pluralities are "fed into" this operation and is determined in large part by the plural logic that is brought to bear. We can study this dependence by keeping the correspondence principles fixed, while varying the plural logic to which they are applied. As just observed, our extended critical plural gives rise to full Zermelo-Fraenkel set theory. If we remove the plural principle of infinity, the result is a comparatively weak theory of hereditarily finite set. Alternatively, suppose we retain that plural principle of infinity but impose a predicativity requirement on the generalized union principle (and thus also plural replacement and separation).[20] Then a broadly predicative set theory ensues. In short, when we keep the correspondence principles fixed but vary the plural logic, we obtain set theories with wildly different mathematical content. This observation strongly suggests that some of the mathematical content of the resulting set theory derives from the plural logic to which the correspondence principles are applied, not solely from these principles. If this is correct, it follows that a theory can have substantial mathematical content without any commitment to mathematical objects.

[20] Specifically, we require that the formula $\psi(x, y)$ be predicative, in the sense that it contain no bound plural variables.

To come to terms with the possibility of mathematical content even in the absence of mathematical objects, it is useful to recall Bernays's notion of quasi-combinatorial reasoning, whereby principles that are compelling in finite domains are extrapolated to infinite ones. Bernays and others regard such reasoning as distinctively mathematical and a major watershed in the foundations of mathematics, marking the onset of serious infinitary reasoning. Since critical plural logic and its extensions embody, and are motivated by, such reasoning, Bernays would regard both the notion of a plurality and the principles of critical plural logic as distinctively mathematical in character. This is particularly clear for the plural principles of infinity, powerplurality, and choice, whose justification explicitly relied on quasi-combinatorial reasoning.

Is the mathematical content of plural logic compatible with our view that pluralities can be used to explain sets? We believe it is. The explanation in question is a broadly metaphysical one: we make sense of a set $\{xx\}$ as "formed" from its elements xx. There is no conflict between this explanation and the view that plural logic has non-trivial mathematical content. Indeed, on this view, the indisputable mathematical content of set theory is in part inherited from that of plural logic.[21]

Finally, the view that logic can have mathematical content has important consequences concerning how we choose a "correct" logic. Some starkly different views are found in the literature. At one extreme we find Frege, who claims that logic codifies "the basic laws" of all rational thought, and the laws of logic must therefore be presupposed by all other sciences. He writes:

> I take it to be a sure sign of error should logic have to rely on metaphysics and psychology, sciences which themselves require logical principles. (Frege 1893/1903, xix)

This "logic first" view has been very influential. Following Frege, logic is often regarded as epistemologically and methodologically fundamental. All disciplines, including mathematics, are answerable to logic rather than vice versa.

At the opposite extreme we find Quine, whose radical holism leads him to assimilate logic and mathematics to the theoretical parts of empirical science. Logic and mathematics, he claims, are not essentially different

[21] Thanks to Hans Robin Solberg for raising this concern.

from theoretical physics: although they go beyond what can be observed by means of our unaided senses, they are justified by their contribution to the prediction and explanation of states of affairs that *can* be observed.

These extremes are not the only views, however. In particular, one need not be a radical holist to reject the Fregean logic-first view. What are sometimes called "critical views of logic" represent a less dramatic departure from Frege.[22] These views hold that the logical principles governing some subject matter may depend on features of this subject matter or of our discourse about it. The views thus stop short of Quine's radical holism and emphasize instead a more local entanglement of logic with some particular discipline, such as mathematics, semantics, or some part of metaphysics. As a result of this entanglement, logic is answerable to one's views in this other discipline.

The revision of plural logic that we have defended provides a good example of such a critical view of logic. Avoiding any commitment to Quinean holism, we have argued that the principles of plural logic are entangled with our theory of correct mathematical definitions. Specifically, we have defended a liberal theory of mathematical definitions, and on the basis of this theory, we have argued that plural comprehension needs to be restricted more than has traditionally been assumed.

12.8 Generalized semantics without a universal plurality?

We wish to address an open question that, despite not being directly about plurals, is nevertheless relevant to the view we have defended in this chapter and the previous one.

The question concerns how semantics should be done if we adopt critical plural logic. When a language quantifies over an extensionally definite domain, the answer is straightforward: we may as well employ the usual set-based model theory, since all of the relevant constructions, relative to this domain, can be done in set theory. But what about languages that quantify over an extensionally *in*definite domain, such as the important domain of absolutely everything? For such languages, plural logic is no better off than set theory for the purposes of developing a generalized semantics; after all, we have argued that there is no plurality corresponding to the domain of absolutely everything. What to do?

[22] See footnote 9 on p. 278.

Our recommendation is to use Fregean concepts to do the job previously done by pluralities. Instead of giving a plurality-based semantics, we should give a second-order semantics based on Fregean concepts, as explained in Section 7.5. Of course, this means that we must accept enough Fregean concepts to serve the needs of semantics; in particular, we need a universal concept to serve as our absolutely unrestricted domain of quantification.

While promising, this strategy raises some hard follow-up questions, two of which are particularly pressing. First, we argued that every plurality defines a set, which forced us to restrict plural comprehension (see Sections 12.3 and 12.4). Can an analogous argument be given that every Fregean concept defines some sort of object, thus forcing us to restrict second-order comprehension? If so, this might imperil our strategy of using higher-order logic to develop a generalized semantics. Second, we argued that traditional plural logic, when combined with absolute generality and a desire for a generalized semantics, forces us to ascent to higher and higher levels of the plural hierarchy (see Section 11.5). This gives rise to an expressibility deficit akin to that which afflicts generality relativism. Does our use of second-order logic, combined with the same assumptions, force an analogous ascent in the conceptual hierarchy, resulting in an analogous expressibility deficit?

Let us take the questions in order. Concerning the first, recall that our argument that every plurality defines a set relies essentially on the extensional definiteness of pluralities. Without the assumption of extensional definiteness, our liberal view of definitions would not licence the relevant definitions. Suppose we wish to use some sort of collection to define a set whose elements are precisely the members of this collection. We showed that when the collection is extensionally definite—as any plurality is—the assumptions of our liberal view are satisfied and the attempted definition succeeds. But when the collection fails to be extensionally definite—as is often the case with Fregean concepts—the assumptions are not satisfied and the definition can be dismissed as illegitimate. Thus, our argument does not extend from pluralities to Fregean concepts. As far as the views defended in this book are concerned, it is wide open which Fregean concepts, if any, define corresponding objects and what the resulting restriction on second-order comprehension, if any, would have to be.

The second question, concerning an ascent into the conceptual hierarchy and a resulting expressibility deficit, is harder. As discussed, we wish to retain absolute generality and to develop a generalized semantics. Suppose that we also accepted traditional second-order logic, with its unrestricted second-order comprehension scheme. Then all the assumptions of the argument developed in Section 11.5 would be satisfied, and we would thus be forced to ascend higher and higher in the conceptual hierarchy. Wouldn't this be unacceptable? Let us consider three increasingly ambitious responses.

The least ambitious response is to bite the bullet and admit the forced ascent into the conceptual hierarchy as well as the ensuing expressibility deficit. The resulting view would be on a par with generality relativism, which suffers from an analogous expressibility deficit. But the resulting view would at least have one advantage vis-à-vis traditional generality absolutism, namely that it retains full expressibility with respect to the plural hierarchy, accommodating the universal singularization provided by the transition from some things to their set. Even so, we do not find this option very appealing: it is much too close to the generality relativism that we sought to avoid.

A more ambitious response, if forced to ascend into the conceptual hierarchy, is to lift the veil of type distinctions along the lines explained in Section 11.7, so as to return to an untyped language that ensures full expressibility. We regard this strategy as promising but are mindful of the fact that it will be significantly harder to implement it in the case of the conceptual hierarchy than in the case of the plural one. The intuitive barrier to lifting the veil appears greater for the conceptual hierarchy than for the plural one. Perhaps this is because we are familiar with cumulativity in the latter case but not the former.

Moreover, in the conceptual case, we face the question of how to handle predication in the one-sorted language used to lift the veil. When the veil is lifted, all the predicates of the original typed language are subsumed under a single category. (Of course, predicates would still be distinguished according to their adicity.) But the one-sorted language will include predicates of its own, such as a generalized identity predicate and the application predicate 'η', both of which figured crucially in the translation into the one-sorted language described in Section 11.7. Presumably, these predicates too must have semantic values. But if so, we face a treacherous dilemma. Can these semantic values figure as values of the single sort of variables? If they can, then paradox will threaten, for example, by considering the interpretation of '*x* does not apply to itself'. If they cannot, we will have failed to restore full expressibility.[23]

The most ambitious response is to reject traditional second-order logic in favor of a more critical one that restricts the second-order comprehension scheme. This would mean that the assumptions on which the ascent phe-

[23] See Hale and Linnebo forthcoming for discussion of whether, and if so how, this dilemma can be resolved.

nomenon relies are no longer granted, with the result that the pressure to ascend ceases. What kind of restriction might be appropriate? The desired critical higher-order logic would have to balance two potentially opposing needs. It would have to be weak enough to block the ascent, while simultaneously being strong enough to serve the needs of generalized semantics. Although much remains to be investigated, there are indications that this balancing act might be doable.[24]

12.9 What we have learnt

This book has centered around the three overarching questions outlined in Chapter 1. We would like to end our discussion by revisiting those questions and recapitulating the answers we have developed throughout the book.

First there was:

> THE LEGITIMACY OF PRIMITIVE PLURALS
>
> Should the plural resources of English and other natural languages be taken at face value or be eliminated in favor of the singular?

As emphasized already in Chapter 1, different considerations pull in different directions. On the one hand, the success of set theory suggests that we may be able to dispense with the many: the "ones" provided by set theory suffice for our theoretical needs. On the other hand, considerations from natural language and the paradoxes appear to show that it is impossible consistently to eliminate every "many" in favor of corresponding "ones".

While natural language does indeed provide some evidence in favor of primitive plurals, we have argued that this evidence is not entirely conclusive (Chapter 2). We have even stronger reservations about the arguments from paradox (Chapter 3). The stronger of these arguments depend on both absolute generality and traditional plural logic. While we accept absolute generality, we have argued that this form of generality makes traditional plural logic problematic. However, this does not mean that we give up on primitive plurals. There is a significant though underappreciated reason in their favor: plural resources are of great value for the explanation of sets (Chapter 4). In short, while some of the well-known arguments for primitive

[24] Promising approaches are developed in Fine 2005a, Linnebo 2006, Schindler 2019, and work in progress by Sam Roberts.

plurals are less compelling than many philosophers think, a less well-known argument is quite powerful.

Once primitive plurals have been recognized, our second overarching question arises, namely:

> HOW PRIMITIVE PLURALS RELATE TO THE SINGULAR
>
> What is the relation between the plural and the singular? We have been particularly interested in the circumstances under which many objects correspond to a single, complex "one" and whether any such correspondence can shed light on the complex "ones".

Attempts to answer these questions are constrained by a key fact:

> PLURAL CANTOR
>
> For any plurality *xx* with two or more members, the subpluralities of *xx* are strictly more numerous than the members of *xx*.

Plural Cantor leads to a tricky trilemma (Chapter 11): we cannot simultaneously accept the possibility of absolute generality, the existence of universal singularizations, and traditional plural logic. Part IV explored the theoretical consequences of choosing among its three horns, which amounts to accepting one of the following three "package deals":

(i) generality relativism;
(ii) generality absolutism with traditional plural logic but no universal singularizations;
(iii) generality absolutism with critical plural logic and universal singularizations.

We have defended the third package based on three sets of considerations. To begin with, the first two options are afflicted with expressibility deficits. Moreover, if we restore full expressibility by lifting the veil of type distinctions, there is a compelling argument for abandoning traditional plural logic, as is done by the third option. Finally, this option is also supported by a plausible account of permissible definitions, according to which a mathematical object exists whenever an adequate definition of it can be provided.

Our third and final overarching question addresses the significance of plural logic as a tool:

THE SIGNIFICANCE OF PRIMITIVE PLURALS

What are the philosophical and (more broadly) scientific consequences of taking plurals at face value?

The fact that primitive plural resources are available in thought and language is itself highly significant. Philosophers often make strong further claims on behalf of primitive plurals, in particular, that they:

(i) help us eschew problematic ontological commitments, thus greatly aiding metaphysics and the philosophy of mathematics;
(ii) ensure the determinacy of higher-order quantification;
(iii) require us to redo semantics in a way that uses primitive plurals not only in the object language but also in the metalanguage.

We have argued that these three claims are severely exaggerated. Plural quantification is not ontological innocent, at least not in the most interesting sense of this expression (Chapter 8). While it may not carry ontological commitments to *objects*, it does carry substantive commitments that can be precisely measured by means of a Henkin semantics. The same semantics reveals that plural logic, even on its plurality-based model theory, is not immune from non-standard interpretations and fails to secure a gain in expressive power.

Finally, employing primitive plurals in the semantics is not always required. When the domain is extensionally definite—which domains of natural language typically are—the domain can be represented either as a plurality or as its corresponding set. So wherever we can use plurals, we can also use sets (or, for that matter, individual sums). Linguists can therefore largely be acquitted of the charge of error. The only exception concerns certain uses of language of particular interest to philosophers, namely those where the domain is all-encompassing and thus, on our view, extensionally *in*definite. In such cases, we must look beyond the extensional resources of plural logic, set theory, and individual mereology.

Let us finally return to the question of whether plural logic is really "pure logic". Three alleged features of pure logicality were identified in Section 2.5: topic-neutrality, formality, and epistemic primacy. We have no quarrel with the idea that plural logic is topic neutral, that is, that it is applicable to reasoning about any domain. However, we have argued that some domains are extensionally *in*definite and for that reason require critical rather than traditional plural logic. We focused on two components of formality. We grant that plural logic does not discriminate between objects, but we deny that it is ontologically innocent, at least in the most interesting sense of that term. Finally, does plural logic permit a special kind of epistemic primacy? A negative answer was defended in Section 12.5, where we argued that both the concepts of plural logic and the justification of its principles are entangled with set theory.

All in all, we hope to have shown that plural logical is a tool of great value and theoretical interest, both in its own right and for the hard questions it raises concerning the relation between the many and the one—in semantics, metaphysics, and the philosophy of mathematics. The value of this tool does not require that it qualify as "pure logic" in any robust sense.