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Noether as Mathematical Structuralist

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1. Introduction

Emmy Noether's student B. L van der Waerden wrote of her that the maxim by which she always let herself be guided was that "all relations between numbers, functions, and operations become clear, generalizable, and truly fruitful only when they are separated from their particular objects and reduced to general concepts." This chapter will show how Noether's emphasis on abstraction and generalization of frameworks and results contributed to the abstract conception of structure found in contemporary mathematics. Doing so will demonstrate her contribution to structuralist methodology, though she did not herself advocate many philosophical views that we now associate with articulations of structuralism, such as the idea that structures are the real objects of mathematical study. Instead, Noether can be seen as exemplifying what Reck and Price (2000) have called *methodological structuralism*, as opposed to *philosophical structuralism*. The former approach notes that many of the entities studied in mathematics, such as various different number systems and geometrical spaces, are studied primarily in terms of their structural features, and considers this to be the proper approach to mathematical practice. Further, it contends that it is of no real mathematical concern what the intrinsic nature of such mathematical entities might be above and beyond such structural features. What distinguishes this approach from philosophical structuralism is that the methodological structuralist is only purporting to make claims about how we ought to do mathematics, namely confining the scope of the view to mathematical *practice*. Philosophical structuralism goes beyond the claims about correct practice to ask what the further implications of a structuralist methodology might be:

> The way many contemporary philosophers of mathematics (as well as philosophers of language and metaphysicians) specify it further is this: How are we supposed to think about reference and truth along these lines, e.g., in the case of arithmetic? And what follows about the existence and the nature of the natural numbers, as well as of other mathematical objects, even if the answer doesn't matter mathematically? Put more briefly, what are the semantic

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and metaphysical implications of a structuralist methodology? (Reck and Price 2000, 346–347)

We might answer these semantic and metaphysical questions in a variety of different ways, from "thin" views that reject the very question of the real nature of mathematical entities, perhaps in favor of a formalist or inferential characterization, to "thick" Platonist views that consider structures to be real, if nonphysical, entities. While a lot of focus in contemporary structuralism has to with these philosophical questions, they rest on a characterization of mathematical practice that is nevertheless underpinned by methodological structuralism. While one can articulate a methodological structuralist view without committing oneself to any particular version of philosophical structuralism, the converse would seem like a strange move. After all, endorsing philosophical structuralism without also believing it to be the correct, or at least an appropriate, way of doing mathematics would suggest that the correct answers to the philosophical questions rest on an ill-advised methodology. This, while perhaps logically consistent as a view, seems nonetheless to be self-undermining.

To return to Noether, then, the purpose of this chapter is to demonstrate how she contributes to this philosophical tradition by enabling the very mathematical developments that make it possible to be a methodological structuralist in the first place. I will do this by tracing her development as a mathematician and seeing the ways in which she came to exemplify a structuralist approach to mathematical practice and lay the technical groundwork for further work on mathematical structure. This biographical look at Noether will follow the periods into which Hermann Weyl divided her career and methodological styles when he delivered her obituary. First, in Noether's early work, she worked in an algorithmic, constructive style, having begun her career studying under Paul Gordan. But she truly grew into her own as an algebraist, having been encouraged to study abstract algebra by Ernst Fischer. In the second period Weyl identifies, Noether worked on invariant theory, some of which comprised her habilitation work, but then turned to the theory of ideals, which is arguably one of her most important mathematical contributions, and the most important for structuralism. This chapter will focus primarily on Noether's middle and later work rather than her work under Gordan, which she had a tendency to dismiss later on in life. Though in many ways, her contributions to ideal theory are generalizations of work that had already been done by others, most notably Dedekind, it is exactly her emphasis on generalization that embodies her pioneering approach to abstract algebra and contributed to the abstract conception of structure used in contemporary mathematics. For example, Noether's work on commutative rings was similar to Dedekind's *Theory of Algebraic Integers*, but proved the results for arbitrary integral domains and domains of general rings. And her work on non-commutative rings generalized work in representation theory. I also point out that it was not just in Noether's own work, but also in her influence on her students such as Mac Lane and van der Waerden, who went on to provide their own significant contributions to algebra, in which her structural approach to mathematics can be seen.

Several themes will emerge in outlining the development of Noether's methodological structuralism. One will be a commitment to abstraction and generalization—consistently finding ways to treat objects from a perspective that showcases the underlying concepts rather than relying on features of individual number systems. Another will be the use of axiomatic methods; indeed, the structural approach is often associated with the axiomatic approach in the historical literature, and in Noether's case in particular, we can see her use of axioms as exemplifying her commitment to working with structural definitions. Indeed, according to a well-known classification of axioms due to Feferman (1999), the type of axioms that Noether primarily uses are called *structural* axioms. These organize the practice of mathematics by providing the definitions of well-known and recurring types of structures. They can be contrasted with *foundational* axioms, which are taken to be universal throughout mathematics by providing definitions for fundamental concepts such as number and set. Finally, we can see what Koreuber (2015) has called "conceptual mathematics," an approach that has been described by Stein (1988) as follows: "*The role of a mathematical theory is to explore conceptual possibilities*—to open up the scientific *logos* in general, in the interest of science in general" (Stein 1988, 252). This point of view is often associated with Cantor's and Dedekind's advocating free creation in mathematics, but can be seen in Noether's methodology as well. We can see that she is not too preoccupied with the extent to which the concepts she studies are instantiated, preferring instead to focus on the relationships between them.

What follows, though, will be organized biographically rather than thematically, as we shall see how these tendencies emerge in Noether's thought as she develops as a mathematician. The next two sections will discuss the three epochs into which Weyl divided her work. The first will briefly discuss Noether's early work on invariant theory, starting with the formal and algorithmic approach influenced by Gordan, and moving on to her adoption of the Hilbert-style approach to invariants. The second section will consider her work in algebra and the development of the general theory of ideals as well as her contributions to non-commutative algebras. Throughout each of these periods we can see ways in which the themes of generality and axiomatization inform her approach. I will conclude the chapter by relating Noether's methodological structuralism to some contemporary philosophical structuralist views articulated by Schiemer (2014), Landry (2011), and Awodey (1996, 2004), and considering the extent to which they are compatible with each other.

2. Invariant Theory

Emmy Noether's doctoral dissertation was written under Paul Gordan at Erlangen, entitled "On Complete Systems of Invariants for Ternary Biquadratic Forms." Invariant theory is a branch of algebra whose early systematization can be attributed to Arthur Cayley, but which is now often associated with the work of Hilbert and Gordan. The development of Noether's structural approach to mathematics can be seen in her departure from Gordan-style work on invariants in favor of a Hilbert-style approach. As we will see, she did not start out as a methodological structuralist, having been trained in systems of complex symbolic calculations and equations by her supervisor.

Briefly, the study of *invariants* considers transformations of polynomial forms. An *invariant* of a polynomial form is an expression in its coefficients that changes only by a factor determined in a fixed manner by the transformation. This area of mathematical research arose from the work of Cayley, James Sylvester, and others, on the algebraic relationships that hold between the coefficients of higher-degree polynomial forms (Kosmann-Schwarzbach 2011, 29–30). To put this more precisely,[1] a polynomial form is a homogenous polynomial—one whose nonzero terms all have the same degree. This might be done by adding an extra variable. The *discriminant* of a polynomial is a fixed quantity determined by an equation on its coefficients. For example, the quadratic form is given by

$$F(x,y) = Ax^2 + Bxy + Cy^2$$

and its discriminant is given by $\Delta_F = B^2 - AC$. Now suppose that we transform our initial polynomial form by substituting the variables x, y with linear combinations of new variables x', y' and substitution coefficients $\alpha, \beta, \gamma, \delta$:

$$x = \alpha x' + \beta y'$$

$$y = \gamma x' + \delta y'.$$

This transformation defines a new form $F'(x', y')$ each of whose coefficients A', B', C' depends on the substitution coefficients as well as the initial coefficients A, B, C. In general, an *invariant* of $F(x,y)$ is an expression I_F in the coefficients of F such that any transformation of F into a form F', such as

$$F(x,y) = F(\alpha x' + \beta y', \gamma x' + \delta y') = F'(x', y'),$$

[1] This exposition is largely drawn from McLarty (2012).

is such that $I_{F'} = (\alpha\delta - \beta\gamma)^{n} I_F$. So the analogous expression $I_{F'}$ in the coefficients of F' is the product of I_F and some power of an expression in the substitution coefficients. As it happens, for the quadratic form, the discriminant $\Delta_F = B^2 - AC$ is an invariant—and in fact, all of its invariants are powers of the discriminant. This in some sense lets us think of the discriminant as providing a complete system of invariants for the quadratic form.

Gordan's best-known contribution to invariant theory was the solution of his eponymous problem: given any polynomial form in two variables of arbitrary degree, he was able to develop a method for calculating a finite complete system of invariants for that form. That is, he found a routine through which such a finite basis for the invariants of any binary polynomial form could be calculated. Its main drawback, however, was that actually carrying out these calculations for forms of higher degree proved to be relatively infeasible. A lot of Gordan-style mathematics involved applying symbolic transformation rules to complex equations; this frequently involved prohibitively long lists of formulas, and the development of routines that were impractical actually to carry out.

Noether did work on Gordan-style problems using these very methods for some time, but would abandon his algorithmic approach in favor of a new approach to invariant theory developed by David Hilbert. This is likely due to the influence of one of Gordan's successors, Ernst Fischer, who was a proponent of the Hilbert-style approach to invariants, and had a clear influence on Noether's development. This eventually led to her being invited to Göttingen by Hilbert and Felix Klein. The dramatic differences between Hilbert's and Gordan's respective approaches to invariant theory can be seen in Hilbert's own solution of the Gordan Problem, in which he provided a proof by contradiction of the existence of a finite basis for certain invariants. This means that he did not produce an actual finite basis, nor a procedure through which one could be determined. Instead, his proof by contradiction simply demonstrated that one must exist, whatever it may look like.

Upon reading the proof, Gordan is said to have remarked, "Das ist nicht Mathematik; das ist Theologie [This is not mathematics; this is theology]" (Kimberling 1981, 11), though it has been pointed out that the extent of Gordan's resistance to Hilbert's proof is often exaggerated (McLarty 2012). Certainly a non-constructive proof would have seemed illegitimate from the perspective of Gordan's algorithmic methodology, and he did not initially find Hilbert's proof to be clear. But our interest here lies in the fact that for his student Noether, this marked a turn toward such non-constructive approaches to invariant theory, and to mathematics more generally.

Noether's subsequent work in differential invariant theory, some of which constituted her 1919 habilitation work, proved to be extremely significant in theoretical physics— a connection she was able to develop further in Göttingen working with Hilbert, who had already discussed with Einstein the possibility of enlisting Noether's help on some open problems with general relativity. In particular, conservation laws such as the law of conservation of energy did not seem to work the same way within the framework of general relativity as they did in classical mechanics. Some connections between invariant theory and the conservation of quantities had already been made by mathematicians such as Joseph Lagrange, but, as Kosmann-Schwarzbach (2011) argues, it was with Noether that these connections were made in their full generality. Differential invariants are sought in the case of forms whose coefficients are functions; when they are not constant functions, their derivatives are found in the transformed expressions. In her approach to differential invariants, we can already see evidence of Noether's conceptual approach to mathematical problems:

> The second study, *Invariante Variationsprobleme*, which I have chosen to present for my habilitation thesis, deals with arbitrary, continuous groups, finite or infinite, in the sense of Lie, and derives the consequences of the invariance of a variational problem under such a group. These general results contain, as particular cases, the known theorems concerning first integrals in mechanics and, in addition, the conservation theorems and the identities among the field equations in relativity theory. (Noether 1919, quoted in Kosmann-Schwarzbach 2011, 49)

What this quotation illustrates is that the conservation laws in physics for which she is famous are special cases of more general theorems that she was able to prove about Lie groups. As Kosmann-Schwarzbach (2011) points out, the symbolic Gordan-style method of calculating these invariants could find solutions, but did not reveal any general connections. Instead, Noether's more conceptual view, in which the invariants in the conservation laws are seen as special cases of something more general, was the first full treatment of this problem. But beyond this important work that was crucial to modern physics, she did not continue this line of research for much longer, and turned instead to work in algebra and the theory of ideals, a domain that would further showcase her ability to think in terms of general concepts and the relationships between them, and continue the development of her methodological structuralism.

3. Rings and Ideals

The second period of Noether's mathematical work that I will explore covers her work in abstract algebra, especially her groundbreaking contributions to ideal theory in the 1920s. The important pieces here are her 1921 paper "Idealtheorie in Ringbereichen" and subsequent 1926 paper "Abstrakter Aufbau der Idealtheorie." Many of the foundations for ideal theory were laid well before then by Ernst Kummer, working on factorization problems in the cyclotomic integers, and further developed by Leopold Kronecker so that they could be extended to systems of complex numbers. However, a contrasting approach, explicitly rejecting Kummer's and Kronecker's more algorithmic methods, was taken by Richard Dedekind in several versions of his theory of algebraic integers, and it is the latter's work that is taken up and generalized by Noether, to become what we now think of as ideal theory proper. We will see through this history how Dedekind's structural approach was further refined and generalized by Noether.

In 1846, Kummer introduced ideal prime factors for the cyclotomic integers, which had turned out to be quite useful in the study of higher reciprocity laws.[2] Cyclotomic integers are integers of the form

$$a_0 + a_1\theta + \cdots + a_n\theta^n,$$

where the $a_i \in \mathbb{Z}$ and θ is a primitive p-th root of unity, a complex number $\neq 1$ such that $\theta^p = 0$. For such integers, Kummer discovered that unique factorization fails for $p = 23$, and published this result in 1844. This means that in rings of cyclotomic integers $\mathbb{Z}[\theta]$, where θ is a primitive p-th root of unity as previously described, Kummer was able to find distinct decompositions of some ring elements into irreducible factors. Kummer's development, then, of the notion of ideal prime factors was intended to restore some, albeit weakened, form of unique factorization to the rings he was studying.[3] But what he defined when he introduced them were not the ideal prime factors themselves, but rather the multiplicity by which they divided cyclotomic integers in the rings in question. The idea was that if we conjecture the existence of the divisors, we can provide rules for calculating divisibility by them. The methods for determining the calculations were also limited in their application, which sufficed for Kummer's purposes, since he was studying reciprocity laws rather than aiming to develop a theory of ideals in his own right (Edwards 1980, 1992). But for further applications, it was useful to develop a more general description of divisibility by these ideal factors, for which we turn to Kronecker. In Kummer's 1859 paper on reciprocity laws, in which the most general version of his own theory appeared, he wrote that

[2] At the time, Kummer just regarded these as a special kind of complex number, but now we have a geometrical interpretation of these kinds of integers which warrants the use of the term "cyclotomic integers," since the roots of cyclotomic polynomials lie on the unit circle in the complex plane.

[3] Though as it turned out, his work in this area was also applied to Fermat's last theorem. See Edwards (1977) for more details on Kummer's theory and its applications in that area.

> Kronecker would very soon (*nächstens*) publish a work "in which the theory of the most general complex numbers" [meaning, surely, the most general algebraic number field] "is completely developed with marvelous simplicity in its connection with the theory of decomposable forms of all degrees." (Edwards 1992, 7)

No such theory appeared until 1881, when Kronecker published his *Grundzüge einer arithmetischen Theorie der algebraischen Grössen*. In this work, Kronecker developed a theory of divisors, which did generalize Kummer's theory to some extent, applicable as it was to general algebraic number fields (though Kronecker, as a constructivist, would likely not have accepted several algebraic number fields that we do today). While Kronecker's theory, like Kummer's, was based on a divisibility test, the main difference between the two is that Kronecker's does not test for divisibility by an ideal prime factor, but for divisibility by the greatest common divisor, which may be ideal or prime (or both). Further, Kronecker does not make use of the notion of a prime because primality is relative to the particular field in question, while the idea of a greatest common divisor is not (Edwards 1980, 353). Then in Kronecker's version of divisor theory, he is able to determine, independent of the underlying field, whether or not the greatest common divisor of some numbers divides an algebraic integer, in a more general fashion than Kummer's theory can.

However, in the interim period between Kummer's announcement and the appearance of Kronecker's *Grundzüge*, Richard Dedekind went through several versions of his own theory of ideals, which would lay some important foundations for Noether's own work in the area. In contrast with Kummer and Kronecker, Dedekind's approach to ideal theory was to explicitly define the ideal divisors in terms of sets of numbers in the domain. So he does not focus, as Kummer and Kronecker do, on the multiplicity by which a given ideal divides a number. Rather, he focusses on the properties possessed by collections of numbers that are divisible by some given factor. In other words, for any algebraic integer a in our domain, we consider the collection of all multiples of a, denoted by $i(a)$. This is called the principal ideal (*Hauptideal*) generated by a. It is easy to see that these ideals satisfy certain closure properties. Namely,

(1) If b and c both belong to $i(a)$, then both $b + c$ and $b - c$ belong to $i(a)$.
(2) If b belongs to $i(a)$, then for any c in the domain, bc also belongs to $i(a)$.

But now, we realize that a did not have to be an algebraic integer in the first place. Even if it was one of Kummer's ideal prime factors, $i(a)$ would still satisfy (1) and (2). And indeed, these two conditions turn out to be both necessary and sufficient for characterizing the ideal numbers, as each algebraic integer can be identified with its unique principal ideal.

Now, Dedekind comments several times on his dissatisfaction with Kummer's theory and his reasons for developing his own theory in such a different way. In particular, he writes that

> the greatest circumspection is necessary to avoid being led to premature conclusions. In particular, the notion of *product* of arbitrary factors, actual or ideal, cannot be exactly defined without going into minute detail. Because of these difficulties, it has seemed desirable to replace the ideal number of Kummer, which is never defined in its own right, but only as a divisor of actual numbers ω in the domain $\mathfrak{o}$, by a *noun* for something which actually exists. (Dedekind 1877, 94)

So an improvement of Dedekind's theory over Kummer's is that the ideal divisors are now identified with things that actually exist and are defined in their own right. Dedekind also writes that it was this very consideration—that the mathematical objects should form the basis of the theory—that led him to develop his theory of ideals in his distinctive way. While Kummer (and Kronecker) have a divisibility test at the heart of their theory, at the heart of Dedekind's theory is the set-theoretic notion of an ideal. To obtain unique factorization, each ideal corresponds to a well-defined list of "prime ideals," each of which divides it with a particular multiplicity. The concepts of multiplication and division are also given set-theoretic interpretations.

An ideal A is a multiple of B, or is divisible by B, exactly when every number in A is also in B, or when A is a subset of B. Yet alongside that notion, we also have the definition of multiplication for ideals such that for ideals A and B, their product AB is defined to be the set of all numbers ab and their sums such that $a \in A$ and $b \in B$. Now, one way to see the central problem of the work is as the task of showing that divisibility in this sense coincides with multiplication in this sense. For Dedekind writes that we see immediately that AB is divisible by both A and B, but "establishing the complete connection between the notions of divisibility and multiplication of ideals succeeds only after we have vanquished the deep difficulties characteristic of the nature of the subject" (Dedekind 1877, 60). For certainly, the definition of multiplication suggests an alternate notion of divisibility (analogous to that in the integers) such that A is divisible by B exactly when there is another ideal R such that $A = BR$. And what Dedekind aimed at showing is that the two notions of divisibility coincide. The difference between Dedekind and Kummer's approaches to divisibility is an excellent illustration of the difference between the conceptual approach that we will see in Noether, and the algorithmic approach that Kronecker employed. As part of his criticism of Kummer, Dedekind wrote that

> Kummer did not define ideal numbers themselves, but only the divisibility of these numbers. If a number α has a certain property A, to the effect that α satisfies one or more congruences, he says that α is divisible by an ideal number corresponding to the property A. While this introduction of new numbers is entirely legitimate, it is nevertheless to be feared at first that the language which speaks of ideal numbers being determined by their products, presumably in analogy with the theory of rational numbers, may lead to hasty conclusions and incomplete proofs. And in fact this danger is not always completely avoided. On the other hand, a precise definition covering *all* the ideal numbers that may be introduced in a particular numerical domain $\mathfrak{o}$, and at the same time a general definition of their multiplication, seems all the more necessary since the ideal numbers do not actually exist in the numerical domain $\mathfrak{o}$. To satisfy these demands it will be necessary and sufficient to establish once and for all the common characteristic of the properties $A,B,C,\ldots$ that serve to introduce the ideal numbers, and to indicate, how one can derive, from properties A,B corresponding to particular ideal numbers, the property C corresponding to their product. (Dedekind 1877, 57)

Then the methodological issue that Dedekind has with the Kummer-style algorithmic approach is that it might lead to imprecise definitions or perhaps incoherent ones. Given that Dedekind is not a mathematical Platonist (though Kummer and Kronecker are no Platonists either), the importance of precise definitions in ensuring the proper, legitimate creation of mathematical objects is not to be underestimated. The underpinnings for Dedekind's structuralism are arguably based in the potential for precise logical definition (Reck 2003; Yap 2009), and this approach is continued by Noether in her own work (Yap 2017).[4]

Noether's paper "Idealtheorie in Ringbereichen" (Noether 1921) generalizes Dedekind's unique factorization results for the algebraic integers into the more abstract setting of arbitrary rings. Given the introduction of the ring axioms between Dedekind's work and Noether's, this was a natural extension of the conceptual approach that both favored. In Noether's case, the focus on finding the best definitions possible for the concepts was characteristic of her methodological structuralist approach. Now, since Noether's work greatly resembles and builds on Dedekind's, I will not go through many of the details here, though they are discussed in other places (Corry 2004; Yap 2017). She also defines ideals as

[4] Avigad (2006) and (Reck and Ferreirós, this volume) provide more in-depth treatments of Dedekind on ideal theory in particular, so we will return again to Noether.

sets, rather than focusing on ideal divisors, and defines concepts such as divisibility and decomposition in terms of set-theoretic concepts like inclusion and intersection.

The main difference between Noether's and Dedekind's contributions to ideal theory is in their generality, one of our central themes, though it is better described as an extension of Dedekind's methodological trajectory than as a change. In writing about Dedekind's own work on ideal theory, Avigad (2006) notes among the advantages of the axiomatic method that it allows for greater generality, and that it allows for a smoother transference of prior results. While other treatments of ideals, including Dedekind's, had relied on properties of algebraic integers that can be taken for granted, Noether was defining ideals in a more arbitrary setting. Rather than being able to rely on known facts about concrete mathematical entities, the decomposition theorems that Noether proved had to follow from general defining properties of sets of elements in a ring. The main thing in Noether (1921) that was taken for granted as a property of ideals was the ascending chain condition (a.c.c), which states that every chain of ideals ordered by inclusion has a maximal element. More precisely, if we have a chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$, then there is an index n after which all the ideals are equal, so $I_n = I_{n+1} = \dots$. This condition was explicitly used to prove her decomposition results, but in 1921 was simply stated without proof. In contrast, Noether (1926) made it explicit that the a.c.c. was simply a condition on rings that Noether was interested in.

Her 1926 "Abstrakter Aufbau der Idealtheorie" made the axiomatic approach (another central theme) even more explicit and laid out the structural conditions that rings might satisfy at the outset. In this case, the a.c.c. was just one of the conditions that rings might or might not satisfy. Others included a multiplicative unit element and a lack of zero divisors. But in contrast to 1921, these were not assumed, but treated as contingent. This means that throughout the work, Noether considered rings that satisfied different conditions, so that we discover what follows from each one. Frequently over the course of the work, she will specify what type of ring $\mathfrak{R}$ is intended to be: whether it need only be a ring of some kind or whether it also needs to satisfy some other properties such as the a.c.c. These rings, then, are seen simply as instances of mathematical entities that satisfy various conditions. The use of the axiomatic method, then, is to facilitate the use of merely structural definitions of objects. It also allowed Noether to generalize, in that she could abstract away from different standardly assumed properties of mathematical entities, to consider more general cases of objects.

One other example of abstracting away from a standardly assumed property is commutativity. Though it is a standard property of algebraic integers, Noether in 1926 is careful to specify when a ring under discussion needs to be commutative, and when it only needs to satisfy the basic ring axioms or other conditions.

That approach was sufficiently fruitful that she eventually worked seriously on theories of non-commutative rings, as part of her continuing research trajectory toward studying entities of greater and greater generality. But even before her published work on the subject in the 1930s, we can see the structural approach at work, in which commutativity is only presupposed when it is required and theorems are proved with as much generality as possible. For instance, the theory of integers is initially introduced in terms of a commutative ring with no zero divisors and a multiplicative unit (Noether 1926, 29), while various isomorphism theorems presuppose nothing beyond the ring axioms (Noether 1926, 39).

The move toward the non-commutative setting, however, is importantly modern. In giving up commutativity for multiplication, we take a step away from the intended interpretation of ideals as ideal divisors for the algebraic integers, to consider what else rings as structures could be used to represent, allowing for a wider domain of application. The concept of module that she used in her work on ideals turned out to be a helpful device when it came to representation theory, a branch of algebra that uses vector spaces as representations of groups. Since the vector spaces used in representation theory can be seen as special cases of modules over rings, Noether was once again able to provide a more general structure to use as a mathematical tool (Noether 1929). The work on representation theory in hypercomplex numbers was also further extended into the domain of non-commutative algebras (Noether 1933).

Noether's move to a more general setting such as the theory of rings yielded the ability to make use of tools that describe very general relationships between structures, such as homomorphisms and isomorphisms. In 1926, she explicitly assumes only ring properties (and module properties, respectively), without any other axioms, in order to prove several isomorphism theorems, and theorems relating rings to their quotients. These results are then used for calculations with relatively prime ideals and subsequent decomposition results. Such theorems, as Noether herself notes, can be seen in Dedekind as well, but only as special cases of her own results (Noether 1926, 41). We will also see in the next section that this use of morphisms is further developed by some of Noether's students who went on to lay the foundations of category theory.

And although we find little in the way of autobiographical reflection on her approach to mathematics, Noether's colleagues and students provide a fairly uniform picture.[5] With respect to Noether's use of generalization as a way of developing mathematically fruitful connections, Weyl observes in his memorial address,

[5] Dedekind as well does not do much philosophical writing, and many of the philosophical positions we now attribute to him are extrapolated from his criticisms of other approaches and general methodological comments. So it seems fair to take a similar interpretive stance with respect to Noether.

> She possessed a most vivid imagination, with the aid of which she could visualize remote connections; she constantly strove toward unification. In this she sought out the essentials in the known facts, brought them into order by means of appropriate general concepts, espied the vantage point from which the whole could best be surveyed, cleansed the object under consideration of superfluous dross, and thereby won through to so simple and distinct a form that the venture into new territory could be undertaken with the greatest prospect of success. (Weyl 1981, 147)

This quotation could be taken to apply to both ideal theory and representation theory, as branches of algebra in which Noether was able to develop this more general vantage point. In the case of ideal theory, Noether was able to connect Fraenkel's definition of a ring to Dedekind's work on algebraic integers in order to give a more general treatment of the latter's factorization theorems. And in the case of representation theory, she connected work from Frobenius and Dickson in order to develop a general treatment of non-commutative algebras. Also, the full quotation that opened this chapter can be found in another of Noether's obituaries, in which her student van der Waerden writes,

> One could formulate the maxim by which Emmy Noether always let herself be guided as follows: *All relations between numbers, functions, and operations become clear, generalizable, and truly fruitful only when they are separated from their particular objects and reduced to general concepts.* For her this guiding principle was by no means a result of her experience with the importance of scientific methods, but an a priori fundamental principle of her thoughts. She could conceive and assimilate no theorem or proof before it had been abstracted and thus made clear in her mind. She could think only in concepts, not in formulas, and this is exactly where her strength lay. In this way she was forced by her own nature to discover those concepts that were suitable to serve as bases of mathematical theories. (van der Waerden 1981, 101)

Both Weyl and van der Waerden are consistent in their assessment of Noether as fundamentally committed to what I have called methodological structuralism; she was a mathematician who, at least in her mature work, preferred to think about the relationships between concepts rather than developing formulas or doing calculations. So while the generality of her thinking and fruitful use of axiomatics is apparent, situating Noether within a taxonomy of modern structuralist views is to some extent speculative, given the lack of her own philosophical writing. Nevertheless, we can consider which philosophical structuralisms are compatible with Noether's own methodological structuralism, and the extent to which her methodological views could support one philosophical picture over another.

4. Structuralism, Categories, and Invariants

There is no such thing as a single canonical philosophical structuralist view. They tend to at least have in common a view of mathematical objects as defined or determined by the structures to which they belong and some commitment to methodological structuralism. Sometimes, this means that mathematical objects are seen as "thin" or "incomplete" in the sense that they have no distinguishing properties other than those they possess in virtue of belonging to a particular mathematical structure. While mathematical objects may have other properties, such as the fact that the number one might have the property of being the number of moons of Earth, this is an accidental property that the number has, rather than one making it the thing that it is.

Within these relatively broad constraints, there are a range of positions, as well as a variety of different classifications of such views.[6] For our purposes, it will be most useful to compare Noether's mathematical methodology to two views that are closely connected to the areas of mathematics in which she worked: category-theoretic structuralism, as articulated by Awodey (1996, 2004) and Landry (2011), and invariant-based structuralism as outlined by Schiemer (2014). Noether's connection to category theory comes directly through her students and others who worked with her. For instance, Saunders Mac Lane, credited as one of the founders of category theory, studied with Noether in Göttingen, and is also an important figure in the history of structuralism (McLarty, this volume). Invariant-based structuralism builds on much of the work done by category-theoretic structuralists, but also accounts for issues raised for structuralism by, e.g., Carter (2008). Both are explicitly based on the idea of structure as it can be captured by various branches of abstract mathematics.

The reason for bringing in categories and invariants is the fact that the very concept of structure as it is used in mathematics can be hard to pin down as a single unified concept. Category theory has sometimes been discussed as a potential foundation for mathematics generally, but as Awodey describes it, it can also be used as a way to understand what we mean when we talk about mathematics as a field that deals essentially with structures. This might not be an easy task, because of the wide variety of mathematical structures and the number of different areas in mathematics that use them. The appeal of category theory as a kind of foundation for mathematics, then, is appealing because of its generality and flexibility in characterizing different kinds of mathematical structures. Landry (2011), for instance, gives a list of different categories that can be used to

[6] See Reck and Price (2000); Parsons (1990); Hellman (2005) for various classifications of different structuralist positions.

organize the mathematical structure involved in the concepts of group, set, and topological space, among others. All that we need to do is assign different kinds of entities to be objects and morphisms.

However, in offering categories as a means for the analysis of structure, the kind of foundation that category theory offers is not a foundation in the traditional sense—what Awodey calls "bottom up." Rather,

> The "categorical-structural" [approach] we advocate is based instead on the idea of specifying, for a given theorem or theory only the required or relevant degree of information or structure, the essential features of a given situation, for the purpose at hand, without assuming some ultimate knowledge, specification, or determination of the "objects" involved. (Awodey 2004, 56)

What this means is that the categorical foundations only need to be foundations insofar as they allow us to specify what is essential about the objects that we are interested in. So categories can serve as a foundation for mathematics because of the flexible way in which they permit the characterization of a diverse range of mathematical structures. This alternative approach also results in a different interpretation of the schematic nature of mathematical theories. I can illustrate this in terms of Noether's 1926 work, in which she is very careful to specify which properties of rings she is assuming in each section, for which definitions. For example, when Noether begins her introduction of prime and primary ideals, she simply says to let $\mathfrak{R}$ be a commutative ring, and is clear that no other assumptions are required. We can read this hypothetically, as stating that the definitions and theorems apply *if* an object satisfies the properties for being a commutative ring.

But this is unlike an eliminative structuralist view, or one in which we remove reference to individual mathematical objects by reinterpreting mathematical statements as being implicitly universally quantified. For in order for them to be interpreted in terms of universal quantification, there must be a preexisting domain over which we quantify. Rather, Awodey (2004) advocates for the indeterminacy in the objects being taken seriously, rather than taking a modal approach as does, for instance, Hellman (1989). Further, rather than the focus being on the relations between objects (as a focus on the relations presupposes the relata), morphisms in categories are a perfectly good autonomous concept on which to base the analysis of structure (Awodey 2004, 61). They are also a natural extension of the isomorphisms and homomorphisms on modules that Noether uses in 1926. In this more general situation, so long as the category of rings is sufficient to model the different types of rings, commutative, Dedekindian, etc., that Noether is interested in, it can form a perfectly good basis for her definitions. In that case, these various rings would simply be objects in the category ring, while ring homomorphisms are its morphisms.[7]

So even though Noether's work on ideals and rings preceded the development of categories, the top-down approach to category-theoretic foundations that Awodey and Landry advocate are a natural philosophical overlay atop Noether's mathematical structuralism. I have already noted Noether's deft use of axioms in her work on the theory of rings, and the extent to which it matured over time to place increased emphasis on the conceptual and structural of rings. For instance, the later work, such as her 1926 paper, focused on the connections between the properties of various rings and the theorems that could be proved about them, and this meshes nicely with many characterizations of mathematical practice to which category-theoretic structuralism claims to be faithful:

> The structural perspective on mathematics codified by categorical methods might be summarized in the slogan: The subject matter of pure mathematics is invariant form, not a universe of mathematical objects consisting of logical atoms. This trivialization points to what may ultimately be an insight into the nature of mathematics. The tension between mathematical form and substance can be recognized already in the dispute between Dedekind and Frege over the nature of the natural numbers, the former determining them structurally, and the latter insisting that they be logical objects. (Awodey 1996, 235)

The connection between Noether and Dedekind was famously emphasized by Noether herself, who was said to have remarked, "*Es steht alles schon bei Dedekind* (It is all already in Dedekind)" (quoted in Corry 2004, 250). While in this case she was talking about the decomposition results that she had proved, Awodey's characterization of his structuralist position suggests applying this remark to Noether's methodological views as well. After all, not only did Noether extend Dedekind's results to a more general setting, she also arguably extended his use of structural methods by treating the concept of mathematical structure with a greater degree of abstraction (Yap 2017). In fact, this very same move was arguably employed to extend Noether's work to more general settings by her student and category theorist Mac Lane (see McLarty, this volume), which makes category-theoretic structuralism a natural philosophical view to consider alongside Noether's methodological view. It is, after all, in category theory that many of the concepts that Noether worked with so fruitfully, such as morphisms, get treated in thoroughly general terms.

There are, however, some criticisms of structuralism that we might want to consider as well, which apply to both philosophical and methodological

[7] While I have argued in this chapter for the importance of morphisms to Noether's work, precursors for such ideas are also arguably in Dedekind (Reck and Ferreirós, this volume).

structuralisms. In particular, Carter (2008) considers the methodological claim that mathematics is the study of structure, arguing that this is inaccurate when considered as an overall view of mathematical practice. While she certainly agrees that mathematics deals with structures, it is unclear that there is any single sense of "structure" that will suffice and that can accurately be captured by a structuralist account. This ambiguity about the sense of "structure" can even be traced to the Noether school, at least according to some of its members. Mac Lane (1996) notes that the word "structure" was used in various informal ways by algebraists such as Noether and her students in the 1930s, and given the extent of its ambiguity, might not be able to form the basis of a philosophy of mathematics. Carter, following Mac Lane's discussion, gives examples of two distinct uses of structure that can be found in mathematical practice.

1. Structure over sets that is used to compute invariants of this set.
2. A case where "structure" is extracted in order to change relations between objects. (Carter 2008, 123)

In the first use of "structure," we want to obtain some information about a certain kind of mathematical object. In the case of Galois theory, we might want to determine whether a given polynomial is solvable by radicals. A permutation group based on invariance of the roots can be associated with this polynomial, which is called its Galois group. If the Galois group is solvable,[8] then the polynomial is also called solvable by radicals. So this is a case in which we obtain information about an object because of a certain structure that is associated with it, which we might say is a structure that the object or set has.

In the second use of "structure," we can consider cases in which we have to discover some information about certain structures in order to situate them among more general ones. For instance, we might have to determine how to treat some structures category-theoretically, and in order to do so, need to determine which category they should be subsumed under. In doing so, however, we in effect move objects from one structure to another, which has the following consequences:

> The fact that objects or "places" are moved between structures seems to go against the dictum that "places have no distinguishing features except those determined by the structure in which they have a place" which is taken as implying the claim that "places from different structures can not be identical." Firstly, we have seen that the properties of places or objects can be determined by different structures

[8] A solvable group is one that has a normal series whose normal factors are abelian.

> that they are part of. Secondly, the properties of an object in a given structure can be used to consider the object as part of another structure. (Carter 2008, 130)

To clarify, some features of mathematical practice seem at odds with some central structuralist claims about the identity of objects. If objects technically have different properties in different structures, then it is hard to make sense of moving objects between structures. So this suggests that there is something more to being a particular object than the structure to which it belongs. Carter, however, does not deny that structures are extremely important in mathematics, or even that they are central to mathematical practice, simply that structures cannot be all there is. And this does speak to some extent against Noether's tendency to generalize existing results to more abstract domains. After all, the category-theoretic way of modeling structure represents somewhat more of a difference between algebraic integers and abstract rings than might be warranted by Noether's remarks that it was all already in Dedekind.

I will now turn to an alternative characterization of structuralism based on abstract mathematics, namely Schiemer's structuralism based on invariants. The invariants that Schiemer considers are more general than isomorphisms; rather, they determine equivalence relations on objects that have a certain common property. The purpose of introducing them can be related to some of the issues that Carter raises with structuralist views, such as determining what counts as a structural property of mathematical objects in the first place, given that we might sometimes move an object to a different structure. Schiemer's solution to this is to give up on the idea of defining a structural property on its own, instead defining them relative to some invariant. This relates to Carter's first use of structure in mathematical practice, in which we might work with invariants to determine a property of an object or a set. But of course, different invariants can determine different property structures, where a pair $<S, P>$ (or simply set P) is a property structure of S iff

(i) there exists an invariant $f: S \to N$ and an equivalence relation $R \subseteq S \times S$ such that f determines R; and
(ii) P is the partition of S induced by R, i.e., $P = S/_R$. (Schiemer 2014, 84)

So this is what it means for a set P to be a property structure of another set S relative to an invariant f on S. This is, for instance, the idea behind the Galois group of a polynomial.

For Schiemer, this ultimately has the effect of defining structure in a higher-order set-theoretic fashion, in which structures are identified as classes of equivalence classes determined by some invariant on the objects. Now, whether or not this provides a solution to the problems with structuralism that Carter raises remains open. It does, however, provide a characterization of mathematical structure alternative to category-theoretic structuralism, but one that is nevertheless based in abstract mathematics and a formal definition of structure. One of the differences, however, is that this characterization of structure based in set theory relies on some background interpreted theory such as Zermelo-Fraenkel set theory, as opposed to a category-theoretic approach that, to borrow a turn of phrase from Landry, is structuralist "all the way down."

Schiemer's philosophical structuralism is not as obviously connected to Noether's methodological structuralism as its category-theoretic counterpart, as the latter has direct connections to her ongoing mathematical legacy, while her work on invariants was relatively early in her career. But it does mesh nicely with several aspects of Noether's view. For example, Noether's tendency toward taking a more general perspective on mathematical structures can be accommodated nicely, since these definitions lend themselves to comparisons between property structures, and one structure being more fine-grained than another. This is certainly one way to think of the relationship between ideals in the algebraic integers as Dedekind defined them and ideals on general rings as Noether defined them. If we want to talk about the sense in which the results that Noether proved are the same as Dedekind's, despite being in a different setting, we could consider them as being analogous results in a coarser-grained structure. Rings of algebraic integers are instances of rings that Noether considered, but in the latter setting, they are seen as a more general kind of mathematical object.

5. Conclusion

Ultimately, what version of philosophical structuralism, if any, to which Noether would have subscribed is speculative. While her methodological views are consistently described in structural and conceptual terms by her students, she did not articulate a considered philosophical position in her published work. However, of the various structural views in the literature, two good candidates that we can connect to Noether's work are characterizations of structure based on category theory and invariants. Both articulate the concept of structure in terms of areas of mathematics that Noether either contributed to (in the case of invariant theory) or directly influenced (in the case of category theory). So regardless of the exact field of mathematics that she might have considered to best articulate the concept of structure that she wanted to work with, a formal characterization of structure would likely have been appealing. Given her tendency to articulate concepts as precisely as possible, a metatheoretical articulation of the structure concept in terms of formal mathematics would be natural for her, philosophically.

Noether's methodological inclinations, which we can see borne out in her choices of areas to research, were to generalize given results to more abstract settings. This certainly influenced her students, many of whom went on to develop branches of abstract algebra such as category theory. So when we situate Noether within the history of structuralism as a view, not only can we see her as an excellent example of someone who used structural and axiomatic methods very successfully, we can also see her contributions to some of the mathematical theories underlying contemporary structuralist views, namely to methodological structuralism. In that case, even if it is somewhat open just what kind of structuralist Noether herself would have been, we at least know that she helped made it possible for others even to hold certain kinds of structuralist positions.

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