2021

Applied Mathematics with Oceanology and Computer Programming

[P.G.]

(CBCS)

(M.Sc. First Semester EndExaminations-2021)

MTM – 103

(*ORDINARY DIFFERENTIAL EQUATIONS AND SPECIAL FUNCTIONS*)

Full Marks: 50 ***Time: 02 Hrs***

The figures in the right hand margin indicate marks
Candidates are required to give their answers in their own words as far as practicable
Illustrate the answers wherever necessary

1. **Answer any FOUR questions** **4x2=8**

   a) Consider the second order homogeneous linear differential equation

   $$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x).y=0$$

   Where $a_0(x),\ a_1(x),\ a_2(x)$ are continuous on a real interval $a\le x\le b$ and $a_0(x)\ne 0$ for all $x\in[a,b]$. Let f_1 and $f_2$are solutions of the differential equation. Show that if f_1 and

f_2 have relative maximum at a common point x_0 of the interval $\left[a,b\right]$. Then f_1 and f_2 are linearly dependent on $a \le x \le b$.

b) Discuss the nature of the differential equation

$$z^2 \frac{d^2\omega}{dz^2} + z\,Sinz \frac{d\omega}{dz} + (1-\cos z)\omega = 0 \text{ at } z = 0$$

c) Let $P_0(z)$ be the Legendre's polynomial of degree n. If

$$1 + z^5 = \sum_{n=0}^{5} C_n P_n(z)$$

Then find the value of C_5

d) Prove that

$$F(\alpha,\gamma,v+1;z) = \frac{\gamma}{(\beta-\alpha)z}\left[F(\alpha,\beta-1,\gamma;z) - F(\alpha-1,\beta,\gamma;z)\right]$$

e) What are Bessel's functions of order n ? State for what values of n the solutions are independent of Bessel's equation of order n.

f) Consider the linear system of differential equation

$$\frac{dx}{dt} = a_1 x + b_1 y$$
$$\frac{dy}{dt} = a_2 x + b_2 y$$

Where a_1, b_1, a_2 and a_2 are real constants. Show that the system has two real linearly independent solution of the form $x = Ae^{\lambda t}$ and $y = Be^{\lambda t}$ if $a_2 b_1 > 0$

2. Answer any FOUR questions **8x4=32**

a) i) Let $P_0(x), P_1(x)....P_n(x)$ are continuous on $\left[a,b\right]$ and $y_1(x), y_2(x)....y_n(x)$ are n solutions of the equation $P_0(x)\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1} y}{dx^{n-1}} + ... + P_n(x)y = 0$ then prove that $W(y_1, y_2...y_n)$ is identically zero or nowhere zero in $a \le x \le b$. If $P_0(x), P_1(x)....P_n(x)$ are all polynomial functions of degree n and have one zero at common point $x_0 \in \left[a,b\right]$ show that all solutions are linearly dependent. Where $W\left(y_1, y_2.....y_n\right)$ is the Wronskian of $y_1, y_2.....y_n$

ii) Show that

$$1 + 3P_1(z) + 5P_2(z) + 7P_3(z) + ... + (2n+1)P_n(z) = \frac{d}{dz}\left[P_{n+1}(z) + P_n(z)\right]$$

Where $P_n(z)$ denotes the Legendre's Polynomial of degree n. 3+2+3

b) i) Find the characteristics values and characteristic functions of the Sturm-Liouville problem

$\left(x^3 y'\right) + \lambda xy = 0;\ y(1) = 0,\ y(e) = 0$

ii) Prove that $Sin\left(z\, Sin\theta\right) = 2\sum_{n=1}^{\alpha} J_{2n-1}(z) Sin(2n-1)\theta$ Hence find the series expansion of $Sinz$ in terms of Bessel functions. 4+4

c) i) Solve by using Green function the differential equation $\frac{d^2u}{dx^2} + k^2u = x \quad (k \neq \pi) \rightarrow Pi$ subject to the boundary conditions $u(0) = \alpha \quad u'(1) = \beta$

ii) Show that ordinary Green function does not exists for any arbitrary function for $u'(-1) = u'(1) = 0$ and $\frac{d^2y}{dx^2} = f(x)$

6+2

d) i) If the function $\phi_1, \phi_2 .... \phi_n$ defined as

$$\phi_1 = \begin{pmatrix} \phi_{11} \\ \phi_{21} \\ , \\ \phi_{n1} \end{pmatrix}, \phi_2 = \begin{pmatrix} \phi_{12} \\ \phi_{22} \\ , \\ \phi_{n2} \end{pmatrix} ..... \phi_n = \begin{pmatrix} \phi_{1n} \\ \phi_{2n} \\ , \\ \phi_{nn} \end{pmatrix}$$

be n solution of a homogeneous differential equation $\frac{dx}{dt} = A(t)x$ then prove that if $W(\phi_1, \phi_2 ..... \phi_n) \neq 0$ then solutions are linearly

independent. Where $W(\phi_1, \phi_2 ..... \phi_n)$ is the wronskian of $\phi_1, \phi_2 ..... \phi_n$

ii) Find the solution of non-homogeneous linear equation $\frac{dx}{dt} = Ax + F(t)$ where $A = \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix} F(t) = \begin{pmatrix} e^{5t} \\ 4 \end{pmatrix}$

3+5

e) i) Find the general solution of the equation $2z(1-z)w''(z) + w''(z) + 4w(z) = 0$ by the method of solution in series about $z = 0$, and show that the equation has a solution which is polynomial in z.

ii) Show that $nP_n(z) = zP_n'(z) - P_{n-1}'(z)$ where $P_n(z)$ denotes the Legendre Polynomial of degree n 5+3

f) i) Prove the series solution of the differential equation $(1-z^2)\frac{d^2\omega}{dz^2} - 2z\frac{d\omega}{dz} + n(n-1)\omega = 0$ about $z = 0$ admits polynomial function of n is positive or negative integer.

ii) Prove that $\int_{-1}^{1} P_m(z)P_n(z)dz = \frac{2}{2n+1}\delta_{mn}$ where δ_{mn} and $P_n(z)$ are the Kroneker delta and Legendre's polynomial respectively. 5+3

[Internal Marks – 10]