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Interval-valued fuzzy threshold graph

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ABSTRACT

Interval-valued fuzzy threshold graph is an extension of fuzzy threshold graph. In this paper, interval-valued fuzzy threshold graph, interval-valued fuzzy alternating 4-cycle, and interval-valued fuzzy threshold dimension are defined, and certain properties are studied. It is demonstrated that every interval-valued fuzzy threshold graph can be treated as an interval-valued fuzzy split graph.

1. Introduction

Threshold graph was first introduced by Chvatal and Hammer [4]. These graphs are used in several applied areas, such as psychology, neuro-science, computer science, artificial intelligence, and scheduling theory. The graphs can also be used to control the flow of information between processors, similar to how traffic lights are used in controlling the flow of traffic. Chvatal and Hammer introduced this graph to use in set-packing problems. In 1985, Ordman [13] used this graph in resource allocation problems. A graph $G=(V,E)$ is a threshold graph if there exists non-negative reals w_v $(v \in V)$ and t such that $W(U) \le t$ if and only if $U \subseteq V$ is an independent set, where $W(U)=\sum_{v \in U} w_v$, t is called the threshold. So $G=(V,E)$ is a threshold graph whenever one can assign vertex weights such that a set of vertices is stable if and only if the total of the weights does not exceed a certain threshold. The threshold dimension $t(G)$ of a graph G is the minimum number k of threshold subgraphs $T_1, T_2, \ldots, T_k$ of G that cover the edge set of G. Threshold partition number, denoted by $tp(G)$, is the minimum number of edge disjoint threshold subgraphs needed to cover $E(G)$. An edge cover of a graph G is a set of edges $C \subset E$ such that each vertex is incident with at least one edge in C. The set C is said to cover the vertices of G.

Suppose that there are ATMs that are linked with bank branches. Each ATM has a threshold limit amount. This limit sets the maximum withdrawals per day. The goal is to determine a replenishment schedule for allocating cash inventory at bank branches to service a preassigned subset of ATMs. The problem can be modelled as a threshold graph since each ATM has a threshold of transactions. In reality, each ATM can have a different withdrawal limit. These withdrawal limits can be represented by an interval-valued fuzzy set. The threshold limit can be set such that the branches can replenish the ATMs without ever hampering the flow of transactions in each ATM. Motivated by this example, we investigate use of the interval-valued fuzzy threshold graph to model and solve this type of real problem.

Another graph related to the threshold graph is Ferrers diagraph. Ferrers diagraph was introduced by Peled and Mahadev [14]. A diagraph $\vec{G}=(V,\vec{E})$ is said to be a Ferrers diagraph if it does not contain vertices x,y,z,w, not necessarily distinct, satisfying $\overrightarrow{(x,y)}, \overrightarrow{(z,w)} \in \vec{E}$ and $\overrightarrow{(x,w)}, \overrightarrow{(z,y)} \notin \vec{E}$. For a diagraph $\vec{G}=(V,\vec{E})$, the underlying loop less graph $U(\vec{G})=(V,E)$, where $E=\{(u,v): u,v \in V,\ u \ne v,\ \overrightarrow{(u,v)} \in \vec{E}\}$. Graph theory has had great impact in the real world. After introduction of Euler's graph theory, Rosenfeld [26] generalizes the graph (crisp) as a fuzzy graph. Rosenfeld first considered the fuzzy relation between fuzzy sets and developed various theoretical graph concepts. The field of fuzzy graph theory is growing rapidly because of demands in nature. It has been used to solve such problems as human cardiac functions, fuzzy neural networks, routing problems, traffic light

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problems, and time table scheduling. In fuzzy mathematics, there are different types of fuzzy graphs. There can be graphs with crisp vertex set and fuzzy edge set or fuzzy vertex set and crisp edge set or fuzzy vertex set and fuzzy edge set or crisp vertices and edges with fuzzy connectivity, etc. Problems involving approximate reasoning use fuzzy systems. A *fuzzy set* α on a set X is denoted by $\alpha = (X, \mu)$ and is defined by a function $\mu : X \to [0, 1]$, called the membership function. There are many extensions of fuzzy sets, such as bipolar fuzzy set, interval-valued fuzzy set, and intuitionistic fuzzy set.

In 1971, Zadeh [40] introduced the notion of interval-valued fuzzy sets as a generalization of fuzzy sets in which the membership values are intervals of numbers instead of numbers. As the interval-valued fuzzy set is an interval number, it is more effective in solving uncertainty cases than traditional fuzzy sets. Therefore, it has broader applications, such as fuzzy control, approximate reasoning, medical diagnosis, intelligent control, and multivariate logic.

2. Review of previous research

In 1973, Chvatal and Hammer introduced the threshold graph. Bipartite threshold graphs were studied in [5] under the name difference graphs because they can equivalently be characterized as the graphs (V, E) for which there exist weights $w_v, v \in V$ and a real number t such that $|w_v| < t$ for every v and $(u, v) \in E \Leftrightarrow |w_u - w_v| > t$.

Andelic and Simic [2] studied properties of threshold graphs. Bhutani et al. [3] studied degrees of end nodes and cut nodes in fuzzy graphs. Makwana et al. [6] discussed the extraction of illumination invariant features using a fuzzy threshold based approach. Samanta and Pal [27] have first introduced the fuzzy threshold graph. Mathew and Sunitha [7] defined different types of arcs in a fuzzy graph. Mordeson and Nair [8] gave details of fuzzy graphs and hypergraphs. Pramanik [19] et al. found techniques to find the shortest path in an interval-valued fuzzy hypergraph. Recently, Pramanik et al. [16] defined and studied fuzzy ϕ-tolerance competition graphs. The fuzzy ϕ-tolerance competition graph is an extension of the fuzzy tolerance graph [28]. Samanta and Pal [31,32] worked on fuzzy planar graphs. The interval-valued fuzzy planar graph is a generalization of the fuzzy planar graph introduced in [15] by assigning each vertex and edge to interval-valued fuzzy numbers instead of traditional fuzzy numbers. The bipolar fuzzy hypergraph is a hypergraph in which each vertex and edge is assigned bipolar fuzzy sets. Samanta and Pal [36] have introduced bipolar fuzzy hypergraphs that are important in complex networking systems. Colouring is also a challenging problem. Samanta et al. [38] have found a colouring technique in the approximate sense of fuzzy graphs. The reader may find work on various extensions of fuzzy graphs in [1,21–25]. For further study of fuzzy graphs and variations the literature [17,18,20,29,30,33–35,37] may be very helpful. Nagoorgani and Radha [9] proved some results of regular fuzzy graphs. Nair and Cheng [10] represent cliques and fuzzy cliques in fuzzy graphs. Nair [11] defined perfect and precisely perfect fuzzy graphs. Natarajan and Ayyasawamy [12] described strong and weak domination in fuzzy graphs. Tao [39] et al. described image thresholding using graph cuts.

Multi-processor scheduling, bin packing, and the knapsack problem are variations of set-packing problems; a well-studied problem in combinatorial optimization. These problems have had a large impact on the design and analysis of interval-valued fuzzy threshold graphs. All of these problems involve packing items of different sizes into bins with finite capacities. Consider a parallel system consisting of a set of independent processing units each of which has a set of time-sharable resources, such as a CPU, one or more disks, and network controllers. Here all units have variable capacities as well as resources. Motivated by the problem of packing independent units by optimizing capacities, interval-valued fuzzy threshold graph is introduced. This parallel system can be described by interval-valued fuzzy threshold graph where each of the units and resources represents the vertices of the graph and a task executing on one of the units places requirements on each of the resources that can be best described by edges.

In this paper, we study several properties of an interval-valued fuzzy threshold graph as an extension of a fuzzy threshold graph. We also define, for instance, interval-valued fuzzy split graphs and interval-valued fuzzy threshold dimensions.

3. Preliminaries

A *graph* $G = (V, E)$ consists of a set denoted by V, or by $V(G)$ and a collection E, or $E(G)$, of unordered pairs (u, v) of elements from V. Each element of V is called a *vertex* or a point or a node, and each element of E is called an *edge* or an arc or a line or a link.

An *independent set (stable set)* in a graph $G = (V, E)$ is the set of vertices of V, no two of which are adjacent. That is, $S(\subset V)$ is said to be an independent set if for all $u, v \in S$ $(u, v) \notin V$. The *maximal independent set* is an independent set in which adding any other vertex, causes the new set to fail to be independent. The *maximum independent set* is the maximal independent set with the largest number of vertices.

A *fuzzy set* α on a set X is a mapping $\alpha : X \to [0, 1]$, called the *membership function*. The *support* of α is $\text{supp}(\alpha) = \{x \in X | \alpha(x) \neq 0\}$ and the *core* of α is $\text{core}(\alpha) = \{x \in X | \alpha(x) = 1\}$. The *support length* is $s(\alpha) = |\text{supp}(\alpha)|$ and the *core length* is $c(\alpha) = |\text{core}(\alpha)|$. The height of α is $h(\alpha) = \max\{\alpha(x) | x \in X\}$. The fuzzy set α is said to be *normal* if $h(\alpha) = 1$.

A *fuzzy graph* with a non-empty finite set V as the underlying set is a pair $G = (V, \sigma, \mu)$, where $\sigma : V \to [0, 1]$ is a fuzzy subset of V and $\sigma : V \times V \to [0, 1]$ is a symmetric fuzzy relation on the fuzzy subset σ such that $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$, where $\wedge$ stands for minimum. A fuzzy edge (x, y), $x, y \in V$ is said to be strong [35] if $\mu(x, y) \geq \frac{1}{2} \min\{\sigma(x), \sigma(y)\}$ and is called weak, otherwise.

A *fuzzy path* ρ in a fuzzy graph is a sequence of distinct nodes $x_0, x_1, x_2, \cdots, x_n$ such that $\mu(x_{i-1}, x_i) > 0$, $1 \leq i \leq n$. Here $n \geq 0$ is called the *length of the path*. The consecutive pairs (x_{i-1}, x_i) are called the *fuzzy arcs of the path*. The path ρ is a *fuzzy cycle* if $x_0 = x_n$ and $n \geq 3$. The fuzzy graph without a cycle is called *acyclic fuzzy graph* or *fuzzy forest*.

The *strength of connectedness* between two vertices u and v is $\mu^\infty(u, v) = \sup\{\mu^k(u, v) \big| k = 1, 2, \cdots\}$ where $\mu^k(u, v) = \sup\{\mu(u, u_1) \wedge \mu(u_1, u_2) \wedge \ldots \wedge \mu(u_{k-1}, v) | u_1, u_2, \ldots, u_{k-1} \in V\}$. In a fuzzy graph an arc (u, v) is said to be a *strong arc* [7] or *strong edge*, if $\mu(u, v) \geq \mu^\infty(u, v)$ otherwise it is *weak*.

Two vertices (nodes) in a fuzzy graph are said to be *fuzzy independent* if there is no strong arc between them. The set of all vertices that are mutually (fuzzy) independent is called the *fuzzy independent set*.

An *interval number* D is an interval $[a^-, a^+]$ with $0 \leq a^- \leq a^+ \leq 1$. For two interval numbers $D_1 = [a_1^-, a_1^+]$ and $D_2 = [a_2^-, a_2^+]$ the following are defined:

$$\text{i) } D_1 + D_2 = \left[a_1^-, a_1^+\right] + \left[a_2^-, a_2^+\right] = \left[a_1^- + a_2^- - a_1^- \cdot a_2^-, a_1^+ + a_2^+ - a_1^+ \cdot a_2^+\right],$$

$$\text{ii) } \min\{D_1, D_2\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right],$$

iii) $\max\{D_1, D_2\} = \left[\max\left\{a_1^-, a_2^-\right\}, \max\left\{a_1^+, a_2^+\right\}\right]$,

iv) $D_1 \le D_2 \Leftrightarrow a_1^- \le a_2^-$ and $a_1^+ \le a_2^+$,

v) $D_1 = D_2 \Leftrightarrow a_1^- = a_2^-$ and $a_1^+ = a_2^+$,

vi) $D_1 = D_2 \Leftrightarrow D_1 \le D_2$ and $D_1 \ne D_2$,

vii) $kD_1 = \left[ka_1^-, ka_2^+\right]$, where $0 \le k \le 1$.

An interval-valued fuzzy set A on a set X is a mapping $\mu_A : X \to [0,1] \times [0,1]$, called the *membership function*, i.e., $\mu_A(x) = [\mu_A^-(x), \mu_A^+(x)]$. The *support* of A is $\text{supp}(A) = \{x \in X | \mu_A^-(x) \ne 0\}$ and the *core* of A is $\text{core}(A) = \{x \in X | \mu_A^-(x) = 1\}$. The *support length* is $s(A) = |\text{supp}(A)|$ and the *core length* is $c(A) = |\text{core}(A)|$. The *height* of A is $h(A) = \max\{\mu_A(x) | x \in X\} = [h^-(A), h^+(A)] = [\max\{\mu_A^-(x)\}, \max\{\mu_A^+(x)\}]$, $\forall x \in X$.

Let $F = \{A_1, A_2, \cdots, A_n\}$ be a finite family of interval-valued fuzzy subsets on a set X. The fuzzy intersection of two interval-valued fuzzy sets (IVFSs) A_1 and A_2 is an interval-valued fuzzy set defined by

$$A_1 \cap A_2 = \left\{\left(x, \left[\min\left\{\mu_{A_1}^-(x), \mu_{A_2}^-(x)\right\}, \min\left\{\mu_{A_1}^+(x), \mu_{A_2}^+(x)\right\}\right]\right) : x \in X\right\}.$$

The fuzzy union of two IVFSs A_1 and A_2 is an IVFS defined by

$$A_1 \cup A_2 = \left\{\left(x, \left[\max\left\{\mu_{A_1}^-(x), \mu_{A_2}^-(x)\right\}, \max\left\{\mu_{A_1}^+(x), \mu_{A_2}^+(x)\right\}\right]\right) : x \in X\right\}$$

An *interval-valued fuzzy graph* of a crisp graph $G^* = (V, E)$ is a graph $G = (V, A, B)$, where $A = [\mu_A^-, \mu_A^+]$ is an interval-valued fuzzy set on V and $B = [\mu_B^-, \mu_B^+]$ is an interval-valued fuzzy relation on E. An edge $(x, y), x, y \in V$ in an interval-valued fuzzy graph is said to be strong if $\mu_B^-(x, y) \ge \frac{1}{2}\min\{\mu_A^-(x), \mu_A^-(y)\}$. An interval-valued fuzzy digraph $\overrightarrow{G} = (V, A, \overrightarrow{B})$ is an interval-valued fuzzy graph where the fuzzy relation $\overrightarrow{B}$ is asymmetric.

An interval-valued fuzzy graph $G = (V, A, B)$ is said to be *complete interval-valued fuzzy graph* if $\mu^-(x, y) = \min\{\sigma^-(x), \sigma^-(y)\}$ and $\mu^+(x, y) = \min\{\sigma^+(x), \sigma^+(y)\}$, $\forall x, y \in V$. An interval-valued fuzzy graph is said to be *bipartite* if the vertex set V can be partitioned into two sets V_1 and V_2 such that $\mu^+(u, v) = 0$ if $u, v \in V_1$ or $u, v \in V_2$ and $\mu^+(v_1, v_2) > 0$ if $v_1 \in V_1$ (or V_2) and $v_2 \in V_2$ (or V_1).

A fuzzy graph $G = (V, \sigma, \mu)$ is called a *fuzzy threshold graph* if there exists a non-negative real number t such that $\sum_{u \in U} \sigma(u) \le t$ if and only if $U \subseteq V$ is an independent set in G.

4. Interval-valued fuzzy threshold graph

In this section, we define the interval-valued fuzzy threshold graph and investigate certain properties.

Definition 1 (Interval-valued fuzzy threshold graph (IVFTG)).

An interval-valued fuzzy graph $G = (V, A, B)$ is called an interval-valued fuzzy threshold graph with threshold $t = [t^-, t^+]$ such that $\sum_{u \in U} \mu_A^-(u) \le t^-$ and $\sum_{u \in U} \mu_A^+(u) \le t^+$ if and only if $U (\subseteq V)$ is an independent set in G.

Example 1. Let us consider an example shown in Fig. 1 where the two reals t^- and t^+ are taken as $t^- = 0.2$ and $t^+ = 0.3$.

In this example, one of the independent sets is $U = \{a, b, c, d\}$. For this independent set U

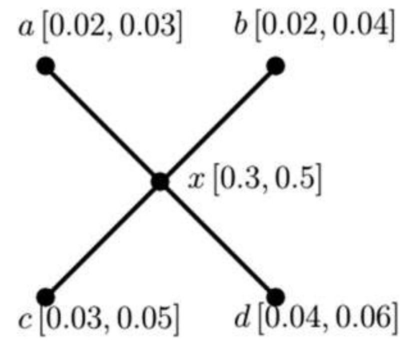

Fig. 1. Example of an IVFTG.

$$\sum_{u \in U} \mu_A^-(u) = 0.02 + 0.02 + 0.03 + 0.04 = 0.11 < 0.2 = t^-$$

$$\sum_{u \in U} \mu_A^+(u) = 0.03 + 0.04 + 0.05 + 0.06 = 0.18 < 0.3 = t^+.$$

Consequently, for non-independent sets $S_1 = \{a, x\}$, $S_2 = \{b, x\}$, $S_3 = \{c, x\}$ and $S_4 = \{d, x\}$,

$$\mu_A^-(a) + \mu_A^-(x) = 0.02 + 0.3 = 0.32 > t^-, \mu_A^+(a) + \mu_A^+(x) = 0.03 + 0.5 = 0.53 > t^+$$

$$\mu_A^-(b) + \mu_A^-(x) = 0.02 + 0.3 = 0.32 > t^-, \mu_A^+(b) + \mu_A^+(x) = 0.04 + 0.5 = 0.54 > t^+$$

$$\mu_A^-(c) + \mu_A^-(x) = 0.03 + 0.3 = 0.33 > t^-, \mu_A^+(c) + \mu_A^+(x) = 0.05 + 0.5 = 0.55 > t^+$$

$$\mu_A^-(d) + \mu_A^-(x) = 0.04 + 0.3 = 0.34 > t^-, \mu_A^+(d) + \mu_A^+(x) = 0.06 + 0.5 = 0.56 > t^+$$

So, the graph shown in Fig. 1 is an IVFTG.

Definition 2 (Interval-valued fuzzy alternating 4-cycle).

Let $G = (V, A, B)$ be an interval-valued fuzzy graph (IVFG). The four vertices say, a, b, c, d of V constitute an interval-valued fuzzy alternating 4-cycle if $\mu_B(a, b) > [0, 0]$ and $\mu_B(c, d) > [0, 0]$, consequently $\mu_B(a, c) = \mu_B(b, d) = [0, 0]$.

Depending on the membership values of the edges (a, d) and (b, c) the four vertices induce $\mathcal{P}_4(\mu_B(a, d) = [0, 0], \mu_B(b, c) > [0, 0]$ or, $\mu_B(a, d) > [0, 0], \mu_B(b, c) = [0, 0])$, $\mathcal{C}_4(\mu_B(a, d) > [0, 0], \mu_B(b, c) > [0, 0])$ or $2\mathcal{K}_2(\mu_B(a, d) = [0, 0], \mu_B(b, c) = [0, 0])$ graphs.

Theorem 1. *An IVFTG does not contain a fuzzy alternating 4-cycle.*

Proof. Let $G = (V, A, B)$ be an IVFTG with threshold $t = [t^-, t^+]$. If possible let, G contain a fuzzy alternating 4-cycle. Thus, $\exists$ $a, b, c, d \in V$ such that $\mu_B(a, b) > [0, 0]$ and $\mu_B(c, d) > [0, 0]$ and $\mu_B(a, c) = \mu_B(b, d) = [0, 0]$. As G is an IVFTG with threshold t, then

$$\mu_A^-(a) + \mu_A^-(b) > t^-, \mu_A^+(a) + \mu_A^+(b) > t^+ \quad (1)$$

$$\mu_A^-(c) + \mu_A^-(d) > t^-, \mu_A^+(c) + \mu_A^+(d) > t^+ \quad (2)$$

$$\mu_A^-(a) + \mu_A^-(d) \le t^-, \mu_A^+(a) + \mu_A^+(d) \le t^+ \quad (3)$$

$$\mu_A^-(b) + \mu_A^-(c) \leq t^-, \mu_A^+(b) + \mu_A^+(c) \leq t^+. \quad (4)$$

From (1) and (3), (2) and (4), we have

$$\mu_A^-(b) - \mu_A^-(d) > t^-, \mu_A^+(b) - \mu_A^+(d) > t^+ \quad (5)$$

$$\mu_A^-(d) - \mu_A^-(b) > t^-, \mu_A^+(d) - \mu_A^+(b) > t^+ \quad (6)$$

But, the in Eq. (5) and (6) are inconsistent. Hence there cannot exist any fuzzy alternating 4-cycle in an IVFTG. ■

Definition 3 (Split graph).

Split graph is a graph in which the vertices can be partitioned into a clique and an independent set.

Definition 4 (Interval-valued fuzzy split graph (IVFSG)).

An interval-valued fuzzy split graph is an IVFG in which the vertices can be partitioned into a fuzzy cycle and a fuzzy independent set.

Example 2. Let us consider an IVFSG $G = (V, A, B)$ which is shown in Fig. 2 and pictorial representation of that IVFSG is shown in Fig. 3. Here the fuzzy clique is $\{a[0.03, 0.05], b[0.04, 0.05], c[0.02, 0.05], d[0.03, 0.06]\}$ and the fuzzy independent set is $\{x[0.4, 0.6], y[0.5, 0.7]\}$.

Theorem 2. *An IVFTG is an IVFSG.*

Proof. Let $G = (V, A, B)$ be an IVFTG. Let $\mathcal{K}$ be the largest fuzzy cycle in G. We claim that G is an IVFSG with fuzzy cycle $\mathcal{K}$ and the fuzzy independent set $V - \mathcal{K}$. Next, we prove that $V - \mathcal{K}$ is a fuzzy independent set.

Suppose, if possible, $V - \mathcal{K}$ is not a fuzzy independent set. Thus, there exists an arc (a, b) in $V - \mathcal{K}$ such that $\mu_B^-(a, b) > 0$ and $\mu_B^+(a, b) > 0$. Since $\mathcal{K}$ is the largest fuzzy cycle in G then for an arc $(c, d) \in \mathcal{K}$, $\mu_B^-(a, c) = \mu_B^-(b, d) = 0$ and $\mu_B^+(a, c) = \mu_B^+(b, d) = 0$. This implies that a, b, c, d form a fuzzy alternating 4-cycle which contradicts that G is *IVFTG*. Therefore, $V - \mathcal{K}$ is a fuzzy independent set. Hence G is an IVFSG. ■

Theorem 3. *Every IVFSG is either an IVFTG or it can be converted to an IVFTG after some modification to the membership values of the vertices.*

Proof. It is shown in Theorem 2 that every IVFTG is an IVFSG. Now, if an IVFSG with fuzzy clique $\mathcal{K}$ and fuzzy independent set $\mathcal{S}$ is not an IVFTG, changes can be made to the membership values of the vertices such that the following inequations hold good for some t^-, t^+:

$$\sum_{u \in \mathcal{S}} \mu_A^-(u) \leq t^-, \sum_{u \in \mathcal{S}} \mu_A^+(u) \leq t^+,$$

$$\mu_A^-(x) + \mu_A^-(y) > t^-, \mu_A^+(x) + \mu_A^+(y) > t^+, \ \forall \ x, y \in \mathcal{K}.$$

The inequations are always consistent because, no $u \in \mathcal{S}$ is adjacent to $x \in \mathcal{K}$, the inequations have no interference. Thus, an IVFSG becomes an IVFTG. ■

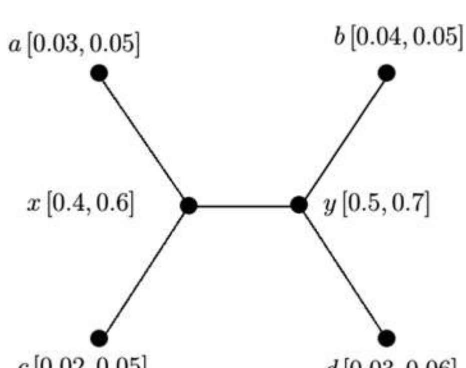

Fig. 2. Example of an IVFSG.

Theorem 4. *An IVFTG can be constructed from a single vertex graph by repeatedly adding a fuzzy isolated vertex or a fuzzy dominating vertex.*

Proof. Consider a single vertex graph with only one vertex x_0. Theorem 3 says that every IVFSG can be made into an IVFTG, the statement can be established if it is shown that after adding a fuzzy isolated vertex or a fuzzy dominating vertex the resultant graph is also an IVFSG.

The single vertex graph can be assumed as an IVFSG with fuzzy clique $\mathcal{K} = \phi$ and fuzzy independent set $\mathcal{S} = \{x_0\}$. Next, a vertex x_1 can be added in two ways, either as an isolated vertex or a dominating vertex. If the vertex x_1 is isolated then $x_1 \in \mathcal{S}$ or if it is dominating then $x_1 \in \mathcal{K}$. The resultant graph remains an interval-valued fuzzy split graph. Hence, the theorem. ■

Definition 5 (Interval-valued fuzzy threshold dimension).

Let $G = (V, A, B)$ be an IVFG. If the minimum k number of interval-valued fuzzy threshold subgraphs of G, say, $G_1, G_2, \ldots, G_k$ cover the edge set of G then the integer k is called the interval-valued fuzzy threshold dimension of G and is denoted by $t(G)$.

Obviously, from the definition, the interval-valued fuzzy threshold dimension of an IVFTG is at least 1.

Theorem 5. *Interval-valued fuzzy threshold dimension of an IVFG is at least 1.*

Proof. To prove, the interval-valued fuzzy threshold dimension of an IVFG is non-zero, show that every IVFG has at least one interval-valued fuzzy threshold subgraph so that it can cover the edge set of the IVFG. If the IVFG is itself an IVFTG then there is nothing to prove. Now, consider IVFG is not an IVFTG.

Theorem 4 gives a clue that every single vertex can form an IVFTG. Therefore, every IVFG has at least one subgraph which is an IVFTG and can cover edge set of the IVFG. Hence, there always exists an interval-valued fuzzy threshold dimension of any IVFG. ■

Theorem 6. *For every IVFG $G = (V, A, B)$ the threshold dimension$(G) \leq n - \alpha(G)$, where $\alpha(G)$ is the cardinality of the maximum independent set of G. In addition, if G is triangle-free then$t(G) = n - \alpha(G)$.*

Proof. Let S be a maximum independent set. Now, we construct the interval-valued fuzzy threshold subgraphs as follows: For each vertex $u \in V - S$ consider a star centred at u. These single vertex star graphs are interval-valued fuzzy threshold graphs. Now we can add one or more fuzzy weak edges to the vertex u and each such interval-valued fuzzy threshold subgraphs along with weak edges covers the graph G. Obviously, $t(G) \leq |V - S| = (n - \alpha(G))$.

If G is triangle-free then for each vertex $u \in V - S$ there exists either a star graph or a star graph together with weak edges which are interval-valued fuzzy threshold graphs and cover the graph G. Then, $t(G) = |V - S| = n - \alpha(G)$. ■

5. Application of interval-valued fuzzy threshold graphs

Consider there are five ATMs and two bank branches. The five ATMs are A, B, C, D and E and the two bank branches are X and Y. The amount of withdrawals for each ATM varies day by day. Suppose, withdrawal limits for each ATM A, B, C, D and E are, respectively, 3 crores to 5 crores, 6 crores to 7 crores, 1 crore to 4 crores, 0.5 crores to 0.8 crores and 0.2 crores to 0.3 crores. The transaction limits for two bank branches X and Y are, respectively, 70 crores to 90 crores and 80 crores to 90 crores.

Next, the problem is find the minimum amount of replenishment cash that should be retained for ATMs from the two branches so that at any time if an ATM faces low cash then either of the branches can transact the required amount to maintain the normal functioning of the ATMs. In reality, depending on many parameters

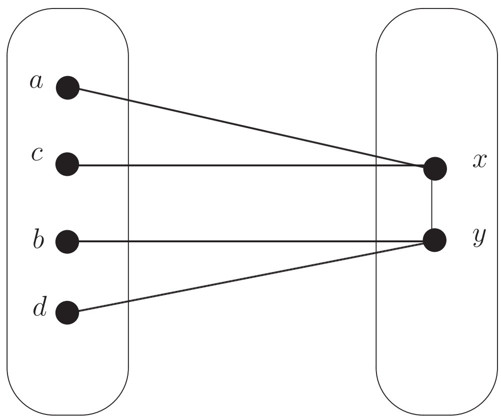

Fig. 3. Pictorial representation of an interval-valued fuzzy split graph shown in Fig. 2.

such as carrying cost, setup cost, bank's daily pay-out etc., this amount may also vary. The problem described here can be modelled as an interval-valued fuzzy threshold graph where ATMs and bank branches are taken as vertices. The membership values of each of these vertices are obtained from the withdrawal limits of ATMs and transaction limits of branches after converting these into equivalent numbers between 0 and 1. The membership value for the threshold limit is the replenishment amount.

First, we find the maximum independent set which is $\{A, B, C, D, E\}$. Next, the graph that is considered is an interval-valued fuzzy split graph. Theorem 2 implies that this will be an interval-valued fuzzy threshold graph with some threshold limit $t = [t^-, t^+]$. Theorem 6 shows that the graph of Fig. 4 has threshold dimension $7 - 5 = 2$ i.e., a minimum of 2 threshold subgraphs covers the edge set of the graph. For a complex system, one can inspect these threshold subgraphs and assign each subgraph a threshold limit so that they form an interval-valued fuzzy threshold graph. Maximum of these threshold limits will be the threshold limit for the graph itself. For this example, the minimum threshold that can be assigned is $[0.107, 0.171]$. Therefore, the replenishment amount varies between 10.7 crores and 17.1 crores.

6. Conclusions

In this paper, we defined the interval-valued fuzzy threshold graph as a generalization of a fuzzy threshold graph. Additionally, we discussed some properties of fuzzy Ferrers digraphs. These graphs help to solve resource allocation problems in a fuzzy system, and these graphs are helpful to control the flow of information using fuzzy set-up. We hope our study will enable us to extend fuzzy graph classes, such as interval-valued fuzzy difference graphs, interval-valued fuzzy matroidal graphs, and interval-valued fuzzy matrogenic graphs. An algorithm can be designed to find the interval-valued fuzzy threshold subgraphs of an interval-valued fuzzy split graph and assign an approximate minimum threshold such that similar types of problems, as stated in Section 5, can be easily solved.

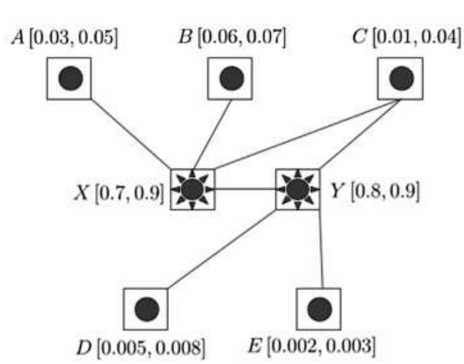

Fig. 4. Cash replenishment problem in ATMs.

References

[1] M. Akram, W.A. Dudek, Interval-valued fuzzy graphs, Comput. Math. Appl. 61 (2) (2011) 289–299.
[2] M. Andelic, S.K. Simic, Some notes on the threshold graphs, Discrete Math. 310 (2010) 2241–2248.
[3] K.R. Bhutani, J. Moderson, A. Rosenfeld, On degrees of end nodes and cut nodes in fuzzy graphs, Iran. J. Fuzzy Syst. 1 (1) (2004) 53–60.
[4] V. Chvatal, P.L. Hammer, Set-packing Problems And Threshold Graphs, CORR 73-21, University of Waterloo, Canada, 1973.
[5] P.L. Hammer, U.N. Peled, X. Sun, Difference graphs, Discrete Appl. Math. 28 (1) (1990) 35–44.
[6] R.M. Makwana, V.K. Thakar, N.C. Chauhan, Extraction of illumination invariant features using fuzzy threshold based approach, IJCA (2011) 25–31.
[7] S. Mathew, M.S. Sunitha, Types of arcs in a fuzzy graph, Inf. Sci. 179 (2009) 1760–1768.
[8] J.N. Mordeson, P.S. Nair, Fuzzy graphs and hypergraphs, Phys. Verl. (2000).
[9] A. Nagoorgani, K. Radha, On regular fuzzy graphs, J. Phys. Sci. 12 (2008) 33–40.
[10] P. S. Nair and S. C. Cheng, Cliques and fuzzy cliques in fuzzy graphs, IFSA World Congress and 20th NAFIPS International Conference 4 (2001) 2277–2280.
[11] P.S. Nair, Perfect and precisely perfect fuzzy graphs, Fuzzy Inf. Process. Soc. (2008) 19–22.
[12] C. Natarajan, S.K. Ayyasawamy, On strong (weak) domination in fuzzy graphs, World Acad. Sci. Eng. Technol. 67 (2010) 526–528.
[13] E. T. Ordman, Threshold coverings and resource allocation, In 16th South-eastern Conference on Combinatorics, Graph Theory and Computing (1985) 99–113.
[14] U.N. Peled, N.V. Mahadev, Threshold Graphs and Related Topics, North Holland, 1995.
[15] T. Pramanik, S. Samanta, M. Pal, Interval-valued fuzzy planar graphs, Int. J. Mach. Learn. Cybern. 7 (4) (2016) 653–664.
[16] T. Pramanik, S. Samanta, B. Sarkar, M. Pal, Fuzzy ϕ-tolerance competition graphs, Soft Comput. (2016), http://dx.doi.org/10.1007/s00500-015-2026-5.
[17] T. Pramanik, M. Pal and S. Mondal, Interval-valued fuzzy competition graph and its application in market competition. Communicated.
[18] T. Pramanik, S. Samanta, M. Pal and S. Mondal, A study on bipolar fuzzy planar graph its application in image shrinking. Communicated.
[19] T. Pramanik, S. Mondal and M. Pal, Shortest path problem on interval-valued fuzzy hypergraphs. Communicated.
[20] T. Pramanik, S. Samanta, M. Pal and M. S. Sunitha, Sum distance in interval-valued fuzzy graphs. Communicated.
[21] H. Rashmanlou, M. Pal, Some properties of highly irregular interval-valued fuzzy graphs, World Appl. Sci. J. 27 (12) (2013) 1756–1773.
[22] H. Rashmanlou, S. Samanta, M. Pal, R.A. Borzooei, A study on bipolar fuzzy graphs, J. Intelligent Fuzzy Syst. 28 (2015) 571–580.
[23] H. Rashmanlou, M. Pal, Isometry on interval-valued fuzzy graphs, Intern. J. Fuzzy Math. Arch. 3 (2014) 28–35.
[24] H. Rashmanlou, M. Pal, Balanced interval-valued fuzzy graphs, J. Phys. Sci. 17 (2013) 43–57.
[25] H. Rashmanlou, M. Pal, Antipodal interval-valued fuzzy graphs, Int. J. Appl. Fuzzy Sets Artif. Intell. 3 (2013) 107–130.
[26] A. Rosenfeld, Fuzzy graphs, in: L.A. Zadeh, K.S. Fu, M. Shimura (Eds.), Fuzzy Sets and Their Applications, Academic Press, New York (, 1975, pp. 77–95.
[27] S. Samanta, M. Pal, Fuzzy threshold graphs, CiiT Int. J. Fuzzy Syst. 3 (2011) 1–9.
[28] S. Samanta, M. Pal, Fuzzy tolerance graphs, Int. J. Latest Trends Math. 1 (2011) 57–67.
[29] S. Samanta, M. Pal, Irregular bipolar fuzzy graphs, International J. Appl. Fuzzy Sets 2 (2012) 91–102.
[30] S. Samanta, M. Pal, Fuzzy k-competition graphs and p-competition fuzzy graphs, Fuzzy Inf. Eng. 5 (2013) 191–204.
[31] S. Samanta, M. Pal, New concepts of fuzzy planar graphs, Int. J. Adv. Res. Artif. Intell. 3 (1) (2014) 52–59.
[32] S. Samanta, M. Pal, Fuzzy planar graph, IEEE Transaction Fuzzy Syst. 23 (6) (2015) 1936–1942.
[33] S. Samanta, M. Pal, Some more results on bipolar fuzzy sets and bipolar fuzzy intersection graphs, J. Fuzzy Math. 22 (2) (2014) 253–262.
[34] S. Samanta, M. Pal, Telecommunication system based on fuzzy graphs, J. Telecommun. Syst. Manage 3 (2013) 1–6.
[35] S. Samanta, M. Akram, M. Pal, m-step fuzzy competition graphs, J. Appl. Math. Comput. 47 (2015) 461–472.
[36] S. Samanta, M. Pal, Bipolar fuzzy hypergraphs, Int. J. Fuzzy Log. Syst. 2 (1) (2012) 17–28.

[37] S. Samanta, M. Pal, A. Pal, Some more results on fuzzy *k*-competition graphs, Int. J. Adv. Res. Artif. Intell. 3 (1) (2014) 60–67.
[38] S. Samanta, T. Pramanik and M. Pal, Fuzzy colouring of fuzzy graphs, Afrika Math..
[39] W. Tao, H. Jin, Y. Zhang, L. Liu, D. Wang, Image thresholding using graph cuts, Syst. Humans 38 (5) (2008) 1181–1195.
[40] L.A. Zadeh, Similarity relations and fuzzy orderings, Inf. Sci. 3 (1971) 177–200.