2021

Mathematics

[HONOURS]

(CBCS)

(B.Sc. Fifth End Semester Examinations-2021)

MTMH-C501

Full Marks: 60 ***Time: 03 Hrs***

The figures in the right hand margin indicate marks
Candidates are required to give their answers in their own words as far as practicable
Illustrate the answers wherever necessary

Real Analysis - III

1. **Answer any TEN questions:** **10x2=20**

   a) If f is Riemann integrable on $[a,b]$ and $c \in \Re$. We define g on $[a+c, b+c]$ by $g(y) = f(y-c)$. Prove that g is Riemann integrable on $[a+c, b+c]$ and that $\int_{a+c}^{b+c} g = \int_a^b f$

   b) If $f : \Re \to \Re$ is continuous and c>0, define $g : \Re \to \Re$ by $g(x) = \int_{x-c}^{x+c} f(t)\,dt$. Show that g is differentiable on $\Re$ and find $g'(x)$.

c) If $f:[0,1]\to\Re$ is continuous and $\int_0^x f=\int_x^1 f$ for all $x\in[0,1]$. Show that $f(x)=0$ for all $x\in[0,1]$

d) If f is continuous on I=[*a, b*] and assume $f(x)\geq 0\forall x\in I$. Prove that if *L*(*P, f*)= 0 then $f(x)\geq 0\forall x\in I$. where *P* is any particular of *I*

e) A function *f* is integrable on [*a, b*] and $\int_a^b f^2(x)dx=0$. Prove that *f*(*x*)=0 at every point of continuity in [*a, b*]

f) Let $f(x)=x-[x],x\in[0,3]$. Show that f is integrable on [0, 3], Evaluate $\int_0^3 f$.

g) A function [0, 3] by $f(x)=x, 0\leq x<1$

$$=1, 1<x\leq 2$$
$$=x-1\quad 2<x\leq 3$$

Let $F(x)=\int_0^x f(t)dt, x\in[0,3]$. Find F

h) Examine the Convergence of $\int_0^1 \frac{x^p}{1+x}dx$

i) Prove that the improper integral $\int_1^\alpha \frac{1}{x^{1/3}(1+x^{1/2})}dx$ is divergent.

j) Prove that $\int_0^1 \frac{\log_e(1+x)}{x}Sin\frac{1}{x}dx$ is convergent

k) Define Beta function and Gamma function

l) Let $f_n(x)=x^2e^{-nx}, x\geq 0$ is pointwise convergent in $\Re$

m) The sequence of functions $\{f_n\}_n$ defined by $f_n(x)=x^n(1-x^2)\forall x\in[-1,1]$. Find the limit function.

n) Show the series $\sum_{n=1}^{\alpha}\frac{Sin\left(x^2+n^2x\right)}{n(n+1)}$ is uniformly convergent for all real *x*

o) Define uniform Convergent and pointwise convergent of sequence of function $\{f_n(x)\}$. And each n, $f_n:D\to\Re\ D\subseteq\Re$.

2. Answer any FOURquestions **5x4=20**

a) Prove that Weierstrass form of second Mean value theorem is applicable to $\int_a^b \frac{\cos mx}{x}dx$ where $o<a<b<\alpha$ and m is a non-zero real number. Further prove that $\int_a^b \frac{\cos mx}{x}dx$ is bounded. 3+2

b) If $f:[a,b]\to[c,d]$ and $c>0$ is continuous on [*a, b*] and $\int_a^b \log Sin(x)dx=0$. Prove that $f(x)=1\ \forall x\in[a,b]$

c) $\int_{\partial}^{\alpha} \frac{Sin(1-\cos x)}{x^n} dx$ is convergent if $o<n<4$ and absolutely convergent if $1<n<4$

d) Discuss the convergence of $\int_0^{\frac{\pi}{2}} (Cosx)^{\ell} (Sinx)^m dx$.

e) If f is Continuous on $\left[o,\infty^0\right)$ show that $\int_0^{\infty} \frac{f(x)dx}{\sqrt{x\left(1+x^2\right)}}$ is convergent

f) Let $fx(x)=\frac{nx}{1+nx} \forall x\in[0,1]$ show that

i) the sequence $\{f_n\}$ converges to f on [0,1]

ii) f isintegrable on [0,1] and $\lim_{n\to\infty} \int_0^1 f_n = \int_0^1 f$ but still the convergence of the sequence is not uniform on [0,1]

3. Answer any TWO questions **10x2=20**

a) i) If $f:[a,b]\to$□ and $g:[a,b]\to$□ are two Riemann integral functions on $[a,b]$ then prove that for any two positive real numbers λ and μ, $\lambda f+\mu g$ is Riemann integrable on $[a,b]$ and

$$\int_a^b (\lambda f+\mu g)(x)dx = \lambda\int_a^b f(x)dx + \lambda\int_b^b g(x)dx$$

ii) Let $f:[a,b]\to$□ and $g:[a,b]\to$□ be both continuous on $[a,b]$ and $\int_a^b |f-g|=0$. Prove that $f=g$. But f, g are only integrable on $[a,b]$ then $\int_a^b |f-g|=0$ does not imply $f=g$. (4+2)+(2+2)

b) a) i) Let $D\subset$□ and for each $n\in$□, $f_n : D\to$□ is continuous on *D*. If the sequence $\{f_n\}$ be uniformly Convergent on *D* to a function *f*, then *f* is continuous on *D*. Is the converse true? If true, under what conditions it is true.

ii) Let $\{f_n\}$ converges to f on $[a,b]$ and for each $n\in$□, f_n have continuous derivative on $[a,b]$. If the sequence $\{f_n'\}$ .converges uniformly to G on $[a,b]$ then prove that $\{f_n\}$ converges uniformly to on $[a,b]$ and $f'(x)=G(x)\forall x\in[a,b]$

c) a) i) Let a be the only point of singularity of f in $[a,b]$, f, g are integrable on $[a+\in,b]$ for every $\in(o\in<b-a)$ and $f(x)>0,\ g(x)>0$ in $(a,b]$. Prove that

(I) If $\lim_{x \to a^+} \frac{f(x)}{g(x)} = 0$ and $\int_a^b g(x)dx$ converges then $\int_a^b f$ converges.

(II) If $\lim_{x \to a^+} \frac{f(x)}{g(x)} = +\infty$ and $\int_a^b f(x)dx$ diverges then $\int_a^b g$ diverges.

ii) Let $f_n(x) = \begin{cases} n^2 x, & 0 \le x \le 1/n \\ 1/x, & 1/n < x \le 1 \end{cases}$

If $\lim_{n \to \infty} f_n(x) = f(x)$ on [0, 1]. Show that f is point wise convergent but not uniformly convergent. (3+3)+4

[The End]