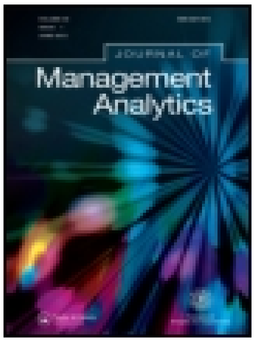

Journal of Management Analytics

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/tjma20

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Journal of Management Analytics, 2021
https://doi.org/10.1080/23270012.2020.1866696

Cloudy fuzzy inventory model under imperfect production process with demand dependent production rate

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(*Received 29 September 2018; revised 21 December 2019; accepted 15 December 2020*)

The aim of this article is an effort to initiate the cloudy fuzzy number in developing classical economic production lot-size model of an item produced in scrappy production process with fixed ordering cost and without shortages. Here, the market value of an item is cloudy fuzzy number and the production rate is demand dependent. In general, fuzziness of any parameter remains fixed over time, but in practice, fuzziness of parameter begins to reduce as time progresses because of collected experience and knowledge that motivates to take cloudy fuzzy number. The model is solved in a crisp, general fuzzy and cloudy fuzzy environment using Yager's index method and De and Beg's ranking index method and comparisons are made for all cases and better results obtained in the cloudy fuzzy model. The model is solved by dominance based Particle Swarm Optimization algorithm to obtain optimal decision and numerical examples and sensitivity analyses are presented to justify the notion.

Keywords: EPL; reliability; De and Beg's ranking index method; cloudy fuzzy number; Dominance Based Particle Swarm Optimization (DBPSO)

1. Introduction

Apparently, a general scenario presents that in the development of economic production lot-size model, usually the demand rate of inventory goods is considered to be constant in nature. But the real scenario shows that quantities involved in inventory will have slight changes from the accurate values. Thus in pragmatic situations, demand variable should be treated as fuzzy in nature. Recently fuzzy concept has been introduced in the production/ inventory problems. Zadeh (1965) first introduced the fuzzy set theory. Later, Bellman and Zadeh (1970) had applied the fuzzy set theoretic approach in making problems. Numerous researches have been done in this area. Researchers like Kauffman and Gupta (1992), Mandal and Maiti (2002), Maiti, Maiti, and Maiti (2014), Maiti and Maiti (2006, 2007), Bera and Maiti (2012), Mahata and Goswami (2007, 2013), De and Sana (2015), Pakhira, Maiti, and Maiti (2018), Garai and Chakraborty (2017), Garai, Chakraborty, and Roy (2018a, 2018b, 2019), Mondal (2018a, 2018b), Mondal, Khan, Vishwakarma, and Saha (2018) Mahata et al. (2018) etc. have investigated extensively over this subject. Kau and Hsu (2002) formulated a lot-size reorder point inventory model with fuzzy demands.

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Initial stage of inventory system reveals ambiguity of demand rate which is high because the decision maker (DM) lacks in any definite information of how many people are accepting the product and how much will be demand rate. At the advancement of time, DM will begin to get major information about the anticipated demand over the process of stock of goods and study whether it is less or more than prior expectation. It is generally observed that when a new product comes into the market, common people will take much more time (no matter what offers/discounts have been declared or what's the quality of product) to understand and to adopt/accept the item. After some days, it is clear to all about the inventory. Gradually, the uncertain domain (cloud) starts getting thinner to DM's mind. In this study, a cloudy fuzzy inventory model is developed depending upon learning from the past experience where fuzziness depends upon time. With advancement of time, fuzziness becomes optimum at the optimum time. This idea is incorporated in the cloudy fuzzy environment. In defuzzification methods, especially on ranking fuzzy numbers, after Yager (1981), several researchers like Ezzati, Allahviranloo, Khezerloo, and Khezerloo (2012), Deng (2014), Zhang, Ignatius, Lim, and Zhao (2014) and others supported the method for ranking of vague numbers using the centre of gravity. Moreover, De and Beg (2016) and De and Mahata (2017, 2019) thought out a new defuzzification method for triangular dense fuzzy set and triangular cloudy fuzzy set, respectively. *Till now*, n*one has addressed this type of realistic production inventory model with cloudy fuzzy demand rate.*

In the classical economic production lot-size (EPL) model, the production rate of single item or multiple items is assumed to be inflexible and predetermined. However, in reality, it is seen that the demand of any goods affected the production process. When the demand tends to gradually high, consumption by the customer is obviously more and to meet the additional requirement of the customer, the manufactures bound to produce more items. Converse is true for reverse situation. In this connection, several researchers' formulated EPL models for single/multiple items considering either uniform or variable production rate (depends on time, demand and/or on hand inventory level). Bhunia and Maiti (1997), Balkhi and Benkherouf (1998), Abad (2000), Mandal and Maiti (2000), Roy, Kar, and Maiti (2010), Das, Roy, and Kar (2010, 2011a, 2011b, 2012) etc. developed their inventory models considering either uniform or variable production rate. However, manufacturing flexibility has become a more important factor in inventory management. In the manufacturing system, different types of flexibility have been judged among which volume flexibility is the most important one. In the manufacturing system, volume flexibility is the ability to change production volume. Cheng (1989) first formulated the demand-dependent production unit cost in the EPQ model; Khouja (1995) introduced volume flexibility and reliability consideration in the EPQ model. Shah and Shah (2014) developed the EPQ model for time declining demand with imperfect production process under inflationary conditions and reliability.

Items are produced using a conventional production process with a certain level of reliability. Higher reliability increases the efficiency of the production process with high expectations. Any production organization targets the goal of achieving production efficiency and ability to operate at an optimum level by reducing the cost of scraps, rework of substandard products, wasted materials, labor hours etc. Many researchers have published a huge number of research papers on imperfect production like Rosenblatt and Lee (1986), Ben-Daya and Hariga (2000), Goyal, Hung, and Chen (2003), Maiti, Bhunia, and Maiti (2006), Sana, Goyal, and Chaudhuri (2007), Das et al. (2011a, 2011b), Manna, Dey, and Mondal (2014), Pal, Sana, and Chaudhuri (2014), etc. Recently, Manna, Das, Dey, and Mondal (2016) considered multi-item EPQ model with learning effect on imperfect production over fuzzy random planning horizon. Khara, Dey, and Mondal (2017) developed an inventory model under development-dependent imperfect production and reliability-dependent demand.

A good number of research papers in inventory control problems were published using soft computing techniques. Several authors use Genetic Algorithm (GA) in different forms to find marketing decisions for their problems. Pal, Maiti, and Maiti (2009) uses GA to solve an EPQ model with price discounted promotional demand in an imprecise planning horizon. Bera and Maiti (2012) used GA to solve a multi-item inventory model incorporating discount. Maiti, Maiti, and Maiti (2009) used GA to solve an inventory model with stochastic lead time and price-dependent demand incorporating advance payment. Mondal and Maiti (2002), Maiti and Maiti (2006, 2007), Jiang, Xu, Wang, and Wang (2009), Maiti et al. (2014) many other researchers use GA in inventory control problems. Also, Bhunia and Shaikh (2015) used PSO to solve a two-warehouse inventory model for deteriorating item under permissible delay in payment. *Here, dominance based particle swarm optimization has been developed to solve this fuzzy inventory model.*

Here, the fuzzy inventory model under imperfect production process with cloudy fuzzy demand rate is developed where the production rate is demand dependent. The model is solved in a crisp, general fuzzy and cloudy fuzzy environment using Yager's index method and De and Beg's ranking index method for defuzzification and the results obtained in the crisp, fuzzy and cloudy fuzzy environment are compared. In this study, the aim is to minimize average total cost to obtain the optimum order quantity and the cycle time using dominance based Particle Swarm Optimization (PSO) algorithm to find optimum decision for the decision maker (DM). The model is justified with some numerical examples and some sensitivity analyses have been presented.

2. Definitions and preliminaries

2.1. *Normalized General Triangular Fuzzy Number (NGTFN):*

A NGTFN $\tilde{A} = (a_1, a_2, a_3)$ (cf. Figure 1) has three parameters a_1, a_2, a_3 where $a_1 < a_2 < a_3$ and is characterized by its continuous membership function $\mu_{\tilde{A}}(x): X \to [0, 1]$, where X is the set and $x \in X$, is defined by

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 \le x \le a_3 \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

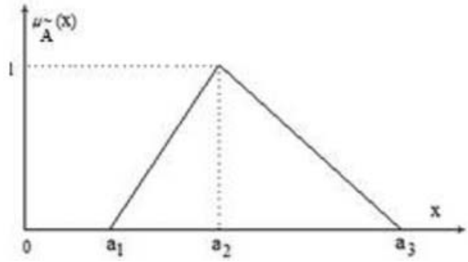

Figure 1. Membership function of a triangular fuzzy number.

2.2. *α-Cut of a fuzzy number*

A α-cut of a fuzzy number $\tilde{A}$ in X is denoted by A_α and is defined as crisp set $A_\alpha=\{x:\mu_{\tilde{A}}(x) \geq \alpha,\ x \in X\}$ where $\alpha \in [0, 1]$. Here, A_α is a non-empty bounded closed interval contained in X and it can be denoted by $A_\alpha=[A_L(\alpha), A_R(\alpha)]$ where $A_L(\alpha) = a_1 + (a_2 - a_1)\alpha$ is called left α-cut and

$$A_R(\alpha) = a_3 - (a_3 - a_2)\alpha \tag{2}$$

is called the right α-cut of $\mu_{\tilde{A}}(x)$, respectively.

2.3. *Yager's ranking index*

If $A_L(\alpha)$ and $A_R(\alpha)$ are the left and right αcuts of a fuzzy number $\tilde{A}$, then the Yager's Ranking index is computed for defuzzification as

$$I(\tilde{A}) = \frac{1}{2}\int_0^1 [A_L(\alpha) + A_R(\alpha)]\,d\alpha = \frac{1}{4}(a_1 + 2a_2 + a_3) \tag{3}$$

Also, the degree of fuzziness (d_f) is defined by the formula $d_f = \dfrac{U_b - L_b}{m}$ where U_b and L_b are the upper and lower bounds of the fuzzy numbers, respectively, and m is their respective mode.

2.4. *Cloudy Normalized Triangular Fuzzy Number (CNTFN) (De and Beg (2016)):*

After infinite time, the normalized triangular fuzzy number $\tilde{A} = (a_1, a_2, a_3)$ becomes a crisp singleton, then fuzzy number $\tilde{A} = (a_1, a_2, a_3)$ is called the cloudy fuzzy number. This means that both $a_1, a_3 \to a_2$ *as* $t \to \infty$.

So, the cloudy fuzzy number takes the following form

$$\tilde{A} = (a_2(1 - \frac{\rho}{1+t}),\ a_2,\ a_2(1 + \frac{\sigma}{1+t}))\text{ for } 0 < \rho, \sigma < 1 \tag{4}$$

It is to be noted that $\lim_{t\to\infty} (1 - \frac{\rho}{1+t})a_2 = a_2$ and $\lim_{t\to\infty} (1 + \frac{\sigma}{1+t})a_2 = a_2$. So, $\tilde{A} \to \{a_2\}$.

Its membership function becomes a continuous function of x and t, defined by

$$\mu(x,t)=\begin{cases}\dfrac{x-a_2\left(1-\dfrac{\rho}{1+t}\right)}{\dfrac{a_2\rho}{1+t}}, & \text{if } a_2\left(1-\dfrac{\rho}{1+t}\right)\le x\le a_2\\[2ex] \dfrac{a_2\left(1+\dfrac{\sigma}{1+t}\right)-x}{\dfrac{a_2\sigma}{1+t}}, & \text{if } a_2\le x\le a_2\left(1+\dfrac{\sigma}{1+t}\right)\\[2ex] 0, & \text{otherwise}\end{cases} \tag{5}$$

The graphical representation of CNTFN is shown in Figure 2. Let left and right α-cut of $\mu(x, t)$ from (5) denoted as $L(\alpha, t)$ and $R(\alpha, t)$, respectively. According to the definition of α-cut given in subsection 2.2,

$$L(\alpha,t)=a_2\left(1-\frac{\rho}{1+t}+\frac{\rho\alpha}{1+t}\right) \text{ and } R(\alpha,t)=a_2\left(1+\frac{\sigma}{1+t}-\frac{\sigma\alpha}{1+t}\right) \tag{6}$$

2.5. *De and Beg's ranking index on CNTFN*

Let left and right α-cut off $\mu(x, t)$ from (5) be denoted as $L(\alpha, t)$ and $R(\alpha, t)$, respectively. Then the defuzzification formula under time extension of Yager's ranking index is given by

$$J(\tilde{A})=\frac{1}{2T}\int_{\alpha=0}^{1}\int_{t=0}^{T}\{L(\alpha,t)+R(\alpha,t)\}\,d\alpha\,dt \tag{7}$$

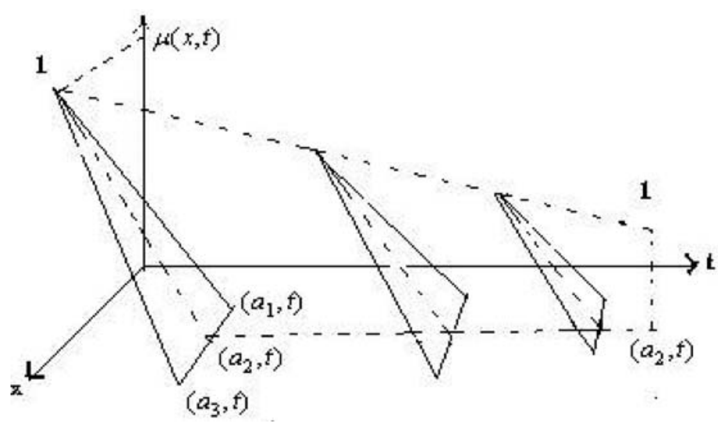

Figure 2. Membership function of CNTFN.

Note that α and t are independent variables. Thus using (5), (6) becomes

$$J(\tilde{A}) = \frac{a_2}{2T}\left[2T + \frac{\sigma - \rho}{2}\log(1+T)\right] \tag{8}$$

Obviously, $\lim_{T\to\infty} \frac{\log(1+T)}{T} = 0$ (Using L'Hopital's rule) and, therefore, $J(\tilde{A}) \to a_2$ *as* $T \to \infty$. Note that $\frac{\log(1+T)}{T}$ is taken as cloud index (CI) (9).

In practice, T is measured in days/months.

2.6. *Arithmetic operations on Normalized General Triangular Fuzzy Number (NGTFN)*

$\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ are two triangular fuzzy numbers, then for usual arithmetic operations $+$, $-$, $\times$, $\div$, respectively, namely addition, subtraction, multiplication and division between $\tilde{A}$ and $\tilde{B}$ are defined as follows:

(i) $\tilde{A} + \tilde{B} = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3)$
(ii) $\tilde{A} - \tilde{B} = (a_1 - b_3,\ a_2 - b_2,\ a_3 - b_1)$

(iii) $\tilde{A} \times \tilde{B} = (a_1 b_1,\ a_2 b_2,\ a_3 b_3)$
(iv) $\frac{\tilde{A}}{\tilde{B}} = (\frac{a_1}{b_3}, \frac{a_2}{b_2}, \frac{a_3}{b_1}),\ b_1, b_2, b_3 \neq 0$
(v) $k\tilde{A} = (ka_1, ka_2, ka_3)$ *if* $k \geq 0$

and $k\tilde{A} = (ka_3, ka_2, ka_1)$ *if* $k < 0$

3. Dominance based particle swarm optimization technique (DBPSO)

During the last decade, nature-inspired intelligence became increasingly popular through the development and utilization of intelligent paradigms in advance information systems design. Among the most popular nature inspired approaches, when task is to optimize with in complex decisions of data or information, PSO draws significant attention. Since its introduction, a very large number of applications and new ideas have been realized in the context of PSO (Marinakis & Marinaki, 2010; Najafi, Niakib, & Shahsavara, 2009). A PSO normally starts with a set of solutions (called swarm) of the decision making problem under consideration. Individual solutions are called particles and food is analogous to optimal solution. In simple terms, the particles are flown through a multi-dimensional search space, where the position of each particle is adjusted according to its own experience and that of its neighbors. The particle i has a position vector $(X_i(t))$, velocity vector $(V_i(t))$, the position at which the best fitness $X_{pbesti}(t)$ encountered by the particle so far and the best position of all particles $X_{gbest}(t)$ in current generation t. In generation $(t+1)$, the position and velocity of the particle are changed to $X_i(t+1)$ and $V_i(t+1)$ using the following rules:

$$V_i(t+1) = w\, V_i(t) + \mu_1 r_1 (X_{pbest\, i}(t) - X_i(t)) + \mu_2 r_2 (X_{gbest}(t) - X_i(t)) \tag{10}$$

$$X_i(t+1) = X_i(t) + V_i(t+1) \tag{11}$$

The parameters μ_1 and μ_2 are set to constant values, which are normally taken as 2, r_1 and r_2 are two random values uniformly distributed in [0,1], w $(0 < w < 1)$ is inertia weight which controls the influence of previous velocity on the new velocity. Here $(X_{pbesti}(t))$ and $(X_{gbest}(t))$ are normally determined by the comparison of objectives due to different solutions. So for optimization problem involving crisp objective, the algorithm works well. If the objective value for the solution X_i dominates the objective value for the solution X_j, we say that X_i dominates X_j. Using this dominance property, PSO can be used to optimize crisp optimization problem. This form of the algorithm is named dominance based PSO (DBPSO) and the algorithm takes the following form. In the algorithm V_{max} represent maximum velocity of a particle, $B_{il}(t)$ and $B_{iu}(t)$ represent lower and upper boundary of the i-th variable, respectively. check_constraint $(X_i(t))$ function check whether solution $X_i(t)$ satisfies the constraints of the problem or not. It returns 1, if the solution $X_i(t)$ satisfies the constraints of the problem, otherwise it returns 0.

3.1. *Proposed DBPSO algorithm*

1. Initialize μ_1, μ_2, w, N and Maxgen.
2. Set iteration counter $t = 0$ and randomly generate initial swarm $P(t)$ of N particles (solutions).
3. Determine objective value of each solution $X_i(t)$ and find $X_{gbest}(t)$ using dominance property.
4. Set initial velocity $V_i(t), \forall X_i(t) \in P(t)$ and set $X_{pbesti}(t) = X_i(t), \forall X_i(t) \in P(t)$.
5. While $(t <$ Maxgen) do
6. For $i = 1{:}N$ do
7. $V_i(t+1) = w\, V_i(t) + \mu_1 r_1 (X_{pbest\, i}(t) - X_i(t)) + \mu_2 r_2 (X_{gbest}(t) - X_i(t))$
8. If $(V_i(t+1) > V_{\max})$ then set $V_i(t+1) = V_{\max}$.
9. If $(V_i(t+1) < -V_{\max})$ then set $V_i\,(t+1) = -V_{\max}$
10. $X_i(t+1) = X_i(t) + V_i(t+1)$
11. If $(X_i(t+1) > B_{iu}(t))$ then set $X_i(t+1) = B_{iu}(t)$.
12. If $(X_i(t+1) < B_{il}(t))$ then set $X_i(t+1) = B_{il}(t)$.
13. If check_constraint $(X_i(t+1)) = 0$
14. Set $X_i(t+1) = X_i(t),\ V_i(t+1) = V_i(t)$
15. Else
16. If $X_i(t+1)$ dominates $X_{pbesti}\,(t)$ then set $X_{pbesti}\,(t+1) = X_i(t+1)$.
17. If $X_i(t+1)$ dominates $X_{gbest}\,(t)$ then set $X_{gbest}\,(t+1) = X_i(t+1)$.
18. End If.
19. End For.
20. Set $t = t+1$.
21. End While.
22. Output: $X_{gbest}(t)$.
23. End Algorithm

3.2. *Implementation of DBPSO*

(a) **Representation of solutions:** A n-dimensional real vector $X_i = (x_{i1}, x_{i2}, \ldots \ldots, x_{in})$ is used to represent i-th solution, where $x_{i1}, x_{i2}, \ldots \ldots, x_{in}$ represent n decision variables of the decision making problem under consideration.

(b) **Initialization:** N such solutions $X_i=(x_{i1}, x_{i2}, \ldots \ldots, x_{in})$, $i=1,2, \ldots, N$, are randomly generated by random number generator within the boundaries for each variable $[B_{jl}, B_{ju}]$, $j=1,2, \ldots \ldots, n$. Initialize ($P(0)$) sub function is used for this purpose.

(c) **Dominance property:** For crisp maximization problem, a solution X_i dominates a solution X_j if the objective value of X_i is greater than that of X_j.

(d) **Implementation:** With the above function and values the algorithm is implemented using C-programming language. Different parametric values of the algorithm used to solve the model are as follows (Engelbreach, 2005),$\mu_1 = 1.49618, \ \mu_2 = 1.49618, \ w = 0.7298$.

4. Notations and assumptions

The following notations and assumptions are adopted to develop the proposed inventory model.

4.1. *Notations*

k Production rate per cycle.
d Demand rate per cycle ($d < k$).
r Production process reliability.
$q(t)$ Instantaneous inventory level
Q Maximum inventory level (decision variable)
T Cycle length (decision variable).
t_1 Production period (decision variable)
c Production cost per unit.
c_3 Setup cost per cycle.
h Inventory carrying cost per unit quantity per unit time.
Z Average total inventory cost.
Q^* Optimum value of Q.
T^* Optimum value of T.
Z^* Optimum value of Z.
t_1^* Optimum value of t_1.

4.2. *Assumptions*

(i) Replenishment occurs instantaneously on placing of order quantity, so lead time is zero.
(ii) The inventory is developed for a single item in an imperfect production process.
(iii) Shortages are not allowed.
(iv) The time horizon of the inventory system is infinite.

(v) The production rate k is demand dependent and is of the form

$$k = a + b/d \tag{12}$$

where a and b are positive constants.

(vi) At the beginning of the inventory system, ambiguity of demand rate is high because the decision maker (DM) has no any definite information how many people are accepting the product and how much will be demand rate. As the time progresses, DM will begin to get more information about the expected demand over the process of inventory and learn whether it is below- or over-expected. It is generally observed that when a new product comes into the market, people will take much more time (no matter what offers/discounts have been declared or what's the quality of product) to adopt/accept the item. Gradually, the uncertain region (cloud) gets thinner in DM's mind. In this respect, demand rate is assumed to be cloudy fuzzy (§ 2.4).

5. Model development and analysis

The process reliability r means that among the items produced in a production run, only r percent is acceptable that can be used to meet the customer's demand. Initially, the production process starts with zero inventories with production rate k and demand rate d. During the interval $[0, t_1]$, inventory level gradually built up at a rate $rk - d$ and reached at its maximum level Q at the end of production process. The inventory level gradually depleted during the period $[t_1, T]$ due to customer's demand and ultimately became zero at $t = T$. The graphical representation of this model is shown in Figure 2. The instantaneous state of $q(t)$ describing the differential equations in the interval $[0, T]$ of that item is given by

$$\begin{aligned} \frac{dq(t)}{dt} &= rk - d, \quad 0 \le t \le t_1 \\ &= -d, \qquad t_1 \le t \le T \qquad \text{where } rk - d > 0 \end{aligned} \tag{13}$$

with boundary condition

$$q(0) = 0, \;\; q(t_1) = Q, \;\; q(T) = 0 \tag{14}$$

The solution of the differential equation (13) using the boundary condition (14) is given by

$$q(t) = \begin{cases} (rk - d)t, & 0 \le t \le t_1 \\ d(T - t), & t_1 \le t \le T \end{cases} \tag{15}$$

The length of each cycle is

$$T = \frac{Q}{rk - d} + \frac{Q}{d} = \frac{Qrk}{d(rk - d)} \tag{16}$$

Total holding cost for each cycle is given by

$$h\,H_1(Q,\,r,\,k) \tag{17}$$

where $H_1(Q,\,r,\,k) = \int_0^T q(t)\,dt = \int_0^{t_1} (rk-d)t\,dt + \int_{t_1}^T d(T-t)dt = \dfrac{Q^2 r\,k}{2d(rk-d)}.$

Total production cost per cycle is

$$c\,P_c(Q,\,r,\,k) \tag{18}$$

where $P_c(Q,\,r,\,k) = \int_0^{t_1} kdt = k\,t_1 = k\dfrac{Q}{rk-d}$ where $Q = (rk-d)t_1$.

$$\begin{aligned}\text{Total cost } &= \text{ Production cost } + \text{ Set up cost } + \text{ Holding cost}\\ &= c\,P_c(Q,\,r,\,k) + c_3 + h\,H_1(Q,\,r,\,k)\\ &= \frac{c\,k\,Q}{rk-d} + c_3 + \frac{hQ^2 r\,k}{2d(rk-d)}\end{aligned}$$

Therefore, the total average cost is

$$\begin{aligned}Z &= \left[\frac{ckQ}{rk-d} + c_3 + \frac{hQ^2rk}{2d(rk-d)}\right]/T\\ &= \frac{cd}{r} + \frac{c_3}{T} + \frac{hT(rk-d)d}{2rk}\\ &= \frac{cd}{r} + \frac{c_3}{T} + \frac{hdT(a\,r + (b\,r-1)\,d)}{2(a+b\,d)r}\end{aligned}$$

Hence, our problem is given by Minimizing Z= $\dfrac{cd}{r} + \dfrac{c_3}{T} + \dfrac{hdT(a\,r + (b\,r-1)\,d)}{2(a+b\,d)r}$

$$\text{subject to } \; d(T-t_1) = (rk-d)t_1 \;\; i.e.\; r\,k\,t_1 = d\,T,\;\; Q = d(T-t_1) \tag{19}$$

Now, the problem is reduced to minimize the average cost Z and to find the optimum value of Q and T for which $Z\ (Q,\ T)$ is minimum and the corresponding value of t_1 . The average cost is minimized by DBPSO.

5.1. *Fuzzy mathematical model*

Initially, when production process starts, the demand rate of an item is ambiguous. Naturally, demand rate is assumed to be general fuzzy over the cycle length. Then fuzzy demand rate $\tilde{d}$ is as follows $\tilde{d} = (d_1,\ d_2,\ d_3)$ for NGTFN.

Therefore, the problem (19) becomes a fuzzy problem, as given by

$$\begin{aligned}&\text{Minimize } \tilde{Z} = \frac{c\tilde{d}}{r} + \frac{c_3}{T} + \frac{h\tilde{d}T(a\,r + (b\,r-1)\,\tilde{d})}{2(a+b\,\tilde{d})r}\\ &\text{subject to } r\,\tilde{k}\,t_1 = \tilde{d}\,T,\;\; \tilde{Q} = \tilde{d}\,(T-t_1)\end{aligned} \tag{20}$$

Now, using (1), the membership function of the fuzzy objective, fuzzy order quantity and fuzzy production rate under NGTFN are given by

$$\mu_1(Z)=\begin{cases}\dfrac{Z-Z_1}{Z_2-Z_1}, & Z_1\le Z\le Z_2\\ \dfrac{Z_3-Z}{Z_3-Z_2}, & Z_2\le Z\le Z_3\\ 0, & \text{otherwise}\end{cases}\quad \text{where}\begin{cases}Z_1=\dfrac{c\,d_1}{r}+\dfrac{c_3}{T}+\dfrac{h\,d_1T\{a\,r+(b\,r-1)d_1\}}{2r(a+b\,d_3)}\\ Z_2=\dfrac{c\,d_2}{r}+\dfrac{c_3}{T}+\dfrac{h\,d_2T\{a\,r+(b\,r-1)d_2\}}{2r(a+b\,d_2)}\\ Z_3=\dfrac{c\,d_3}{r}+\dfrac{c_3}{T}+\dfrac{h\,d_3T\{a\,r+(b\,r-1)d_3\}}{2r(a+b\,d_1)}\end{cases}\tag{21}$$

$$\mu_2(Q)=\begin{cases}\dfrac{Q-Q_1}{Q_2-Q_1}, & Q_1\le Q\le Q_2\\ \dfrac{Q_3-Q}{Q_3-Q_2}, & Q_2\le Q\le Q_3\\ 0, & \text{otherwise}\end{cases}\quad \text{where}\begin{cases}Q_1=d_1\,(T-t_1)\\ Q_2=d_2\,(T-t_1)\\ Q_3=d_3\,(T-t_1)\end{cases}\tag{22}$$

$$\mu_3(k)=\begin{cases}\dfrac{k-k_1}{k_2-k_1}, & k_1\le k\le k_2\\ \dfrac{k_3-k}{k_3-k_2}, & k_2\le k\le k_3\\ 0, & \text{otherwise}\end{cases}\quad \text{where}\begin{cases}r\,k_1\,t_1=d_1\,T\\ r\,k_2\,t_1=d_2\,T\\ r\,k_3\,t_1=d_3\,T\end{cases}\tag{23}$$

The index value of the fuzzy objective, fuzzy order quantity and fuzzy production rate are respectively, obtained using (2) and (3) as

$$\begin{cases}I(\tilde{Z})=\dfrac{1}{4}(Z_1+2Z_2+Z_3)\\ \qquad=\dfrac{c(d_1+2d_2+d_3)}{4r}+\dfrac{c_3}{T}+\dfrac{hT}{8r}\\ \left[\dfrac{d_1\{a\,r+(b\,r-1)d_1\}}{a+b\,d_3}+\dfrac{2d_2\{a\,r+(b\,r-1)d_2\}}{a+b\,d_2}+\dfrac{d_3\{a\,r+(b\,r-1)d_3\}}{a+b\,d_1}\right]\\ I(\tilde{Q})=\dfrac{1}{4}(Q_1+2Q_2+Q_3)=\dfrac{(T-t_1)}{4}(d_1+2d_2+d_3)\\ I(\tilde{k})=\dfrac{1}{4}(k_1+2k_2+k_3)=\dfrac{T}{4\,r\,t_1}(d_1+2d_2+d_3)\ \text{[using (21), (22) and (23)]}\end{cases}\tag{24}$$

5.1.1. *Particular cases*

Subcase-4.1.1.1: If $d_1,\ d_2,\ d_3\to d$ then $I(\tilde{Z})\to\dfrac{cd}{r}+\dfrac{c_3}{T}+\dfrac{hdT(a\,r+(b\,r-1)\,d)}{2(a+b\,d)r}$

$$I(\tilde{Q})\to d(T-t_1)$$

$$\text{and } I(\tilde{k})\to\frac{d\,T}{r\,t_1}$$

This is a classical EPQ model with process reliability ***r*.**

Subcase-4.1.1.2: If $r \to 1,\ b \to 0$ then $I(\tilde{Z}) \to cd + \frac{c_3}{T} + \frac{hdT}{2a}(a-d)$

$$I(\tilde{Q}) \to d(T - t_1)$$

$$I(\tilde{k}) \to \frac{dT}{t_1}$$

Also, this is a classical EPQ model with production rate ***a*.**

5.2. *Cloudy fuzzy mathematical model*

Initially, when production process starts, the demand rate of an item is ambiguous. As the time progresses, hesitancy of demand rate tends to a certain demand rate over the cycle length. Then fuzzy demand rate $\tilde{d}$ becomes cloudy fuzzy following the equation (4)

Now, using (5), the membership function of the fuzzy objective, fuzzy order quantity and fuzzy production rate under CNTFN are given by

$$\chi_1(Z,T) = \begin{cases} \dfrac{Z - Z_{11}}{Z_{12} - Z_{11}}, & Z_{11} \le Z \le Z_{12} \\ \dfrac{Z_{13} - Z}{Z_{13} - Z_{12}}, & Z_{12} \le Z \le Z_{13} \\ 0, & \text{otherwise} \end{cases} \quad \text{where} \begin{cases} Z_{11} = \dfrac{c(1 - \frac{\rho}{1+T})d}{r} + \dfrac{c_3}{T} \\ \quad + \dfrac{hTd(1 - \frac{\rho}{1+T})}{2r} \left[\dfrac{ar + (br-1)d(1 - \frac{\rho}{1+T})}{a + bd(1 + \frac{\sigma}{1+T})} \right] \\ Z_{12} = \dfrac{cd}{r} + \dfrac{c_3}{T} + \dfrac{hdT\{ar + (br-1)d\}}{2r(a+bd)} \\ Z_{13} = \dfrac{c(1 + \frac{\sigma}{1+T})d}{r} + \dfrac{c_3}{T} \\ \quad + \dfrac{hTd(1 - \frac{\rho}{1+T})}{2r} \left[\dfrac{ar + (br-1)d(1 + \frac{\sigma}{1+T})}{a + bd(1 - \frac{\rho}{1+T})} \right] \end{cases} \tag{25}$$

$$\chi_2(Q,T) = \begin{cases} \dfrac{Q - Q_{11}}{Q_{12} - Q_{11}}, & Q_{11} \le Q \le Q_{12} \\ \dfrac{Q_{13} - Q}{Q_{13} - Q_{12}}, & Q_{12} \le Q \le Q_{13} \\ 0, & \text{otherwise} \end{cases} \quad \text{where} \begin{cases} Q_{11} = d(1 - \frac{\rho}{1+T})(T - t_1) \\ Q_{12} = d(T - t_1) \\ Q_{13} = d(1 + \frac{\sigma}{1+T})(T - t_1) \end{cases} \tag{26}$$

$$\chi_3(k,T)=\begin{cases}\dfrac{k-k_{11}}{k_{12}-k_{11}}, & k_{11}\le k\le k_{12}\\ \dfrac{k_{13}-k}{k_{13}-k_{12}}, & k_{12}\le k\le k_{13}\\ 0, & \text{otherwise}\end{cases} \quad \text{where} \begin{cases}k_{11}=d(1-\dfrac{\rho}{1+T})\dfrac{T}{r\,t_1}\\ k_{12}=\dfrac{d\,T}{r\,t_1}\\ k_{13}=d(1+\dfrac{\sigma}{1+T})\dfrac{T}{r\,t_1}\end{cases} \tag{27}$$

Using (7) the index value of the fuzzy objective, fuzzy order quantity and fuzzy production rate are respectively are given by

$$\begin{aligned} J(\tilde{Z}) &= \frac{1}{4\tau}\int_{T=0}^{\tau}(Z_{11}+2Z_{12}+Z_{13})\,dT = \frac{1}{4\tau}\int_0^{\tau}\left[\frac{c\,d}{r}(4+\frac{\sigma-\rho}{1+T})+\frac{4\,c_3}{T}\right]dT\\ &+\frac{1}{4\tau}\int_0^{\tau}\frac{h\,d\,T}{2\,r}\left[(1-\frac{\rho}{1+T})\frac{a\,r+(b\,r-1)\,d(1-\frac{\rho}{1+T})}{a+\,b\,d(1+\frac{\sigma}{1+T})}+2\frac{a\,r+(b\,r-1)\,d}{a+b\,d}\right.\\ &\left.+(1+\frac{\sigma}{1+T})\frac{a\,r+(b\,r-1)\,d(1+\frac{\sigma}{1+T})}{a+\,b\,d(1-\frac{\rho}{1+T})}\right]dT \quad \text{[Using (25)]}\\ &= I_1+\frac{h\,d}{8\,\tau r}(I_2+I_3+I_4)\end{aligned} \tag{28}$$

The expression of I_1 , I_2, I_3 *and* I_4 are given in Appendix

$$\begin{aligned} J(\tilde{Q}) &= \frac{1}{\tau}\int_0^{\tau}\frac{1}{4}(Q_{11}+2\,Q_{12}+Q_{13})\,dT = \frac{d}{4\tau}\int_0^{\tau}\left(4+\frac{\sigma-\rho}{1+T}\right)(T-t_1)\,dT \quad \text{[Using (26)]}\\ &= \frac{d}{4\tau}[2\tau^2-4\,\tau\,t_1+(\sigma-\rho)\,(\tau-(1+t_1)\ln|1+\tau|)]\end{aligned} \tag{29}$$

$$\begin{aligned} J(\tilde{k}) &= \frac{1}{\tau}\int_0^{\tau}\frac{1}{4}(k_{11}+2k_{12}+k_{13})\,dT = \frac{1}{4\tau}\int_0^{\tau}\frac{d\,T}{r\,t_1}\left\{4+\frac{(\sigma-\rho)}{1+T}\right\}dT \quad \text{[Using (27)]}\\ &= \frac{d}{4\tau r\,t_1}[2\,\tau^2+(\sigma-\rho)(\tau-\ln|1+\tau|)]\end{aligned} \tag{30}$$

Stability analysis and particular cases

(i) If $\rho,\sigma\to 0$ *then* $p\to q$ *and* $u\to v$ Also, $I_2\to\dfrac{p}{2u}\tau^2,\ \ I_4\to\dfrac{p}{2u}\tau^2,\ I_3=\dfrac{p}{u}\tau^2$

So, $J(\tilde{Z})\to\dfrac{c\,d}{r}+\dfrac{c_3}{\tau}\ln\left|\dfrac{\tau}{\varepsilon}\right|+\dfrac{h\,d}{4r\tau}\dfrac{p\tau^2}{u},\ J(\tilde{Q})\to d(\dfrac{\tau}{2}-t_1)\ ,\ \ J(\tilde{k})\to\dfrac{d}{2\,rt_1}\tau$

(ii) If $\rho,\ \sigma \to 0$then the model reduces to (i). The above expressions deduced in (i) are in the form of classical EPQ model. Thus we choose εin such a way that above expressions reduced to the classical EPQ model.

Hence, $\frac{c\,d}{r} + \frac{c_3}{\tau}\ln\left|\frac{\tau}{\varepsilon}\right| + \frac{h\,d}{4r\tau}\frac{p\tau^2}{u} \cong \frac{c\,d}{r} + \frac{c_3}{T} + \frac{hdT}{2}\frac{p}{ur}$. Comparing we have $\frac{1}{T} = \frac{1}{\tau}\ln\left|\frac{\tau}{\varepsilon}\right|,\ \ T = \frac{\tau}{2}$

From these, we get $\varepsilon = \frac{2T}{e^2}$. Also, if $\tau = 2$ *then* $T = 1$. *Hence*, $\varepsilon \to 2e^{-2} << 1$

Since $2 < e \Rightarrow \frac{2T}{e^2} < \frac{T}{2} \Rightarrow \varepsilon < \frac{T}{2}$

6. Numerical illustration

The following values of inventory parameters are used to calculate the minimum values of average cost function (Z*) along with the optimum inventory level (Q*), optimum production period (t_1*) and optimum cycle length (T*) $a = 100$, $b = 1.22$, $c_3 = \$300$, h = \$ 1.5 per unit, $c = \$$ 3 per unit, $r = .8$, $d = 500$ units for the crisp model; for fuzzy model demand rate < d_1, d_2, d_3> = <460, 500, 600> units keeping other inventory parameters are the same as taken in the crisp model and that for the cloudy fuzzy model, $\sigma = 0.16$, $\rho = 0.13$, $\varepsilon = 0.6$. Optimum results are obtained via dominance based particle swarm optimization and presented in Table 1.

It is noted that for computation of degree of fuzziness, apply formula $d_f = \frac{U_b - L_b}{m}$ where U_b, L_b, respectively are the upper and lower bounds of fuzzy components and m is the Mode which is obtained using the formula Mode(m) = 3-×Median-2×Mean. For fuzzy demand rate <460, 500, 600>, Median=500, Mean = 520, $U_b = 600$, $L_b = 460$, $m = 460$.

From the above results, it has been observed that minimum cost is obtained in the cloudy fuzzy model and the value of optimum cost Rs. 2115.33 after the completion of 2.22 months. In the cloudy fuzzy environment, degree of fuzziness is less than the general triangular number as the hesitancy of fuzzy gradually decreases due to the taking experience over time.

6.1. *Sensitivity analysis of cloudy fuzzy model*

Using the above numerical illustration, the effect of under- or over-estimation of various parameters on average cost is studied. Here using $\Delta z = \frac{(z' - z)}{z} \times 100\%$ as a measure of sensitivity where z is the true value and z′ is the estimated value. The sensitivity analysis is shown by increasing or decreasing the parameters by 5%, 10% and 15%, taking one at a time and keeping the others as true values. The results are presented in Table 2.

It is observed form Table 2 that the parameters d and c are highly sensitive. For the changes of demand at −15%, average inventory cost reduces to −13.32% and for 15%, the average inventory cost increases at +13.29%, respectively. This result shows that production rate increases with the increase of demand which, in turn, increases of

Table 1. Optimum values of the EPL model by DBPSO.

Model	t_1* (months)	T* (months)	Q* units	Z* ($)	$d_f = \frac{U_b - L_b}{m}$	$CI = \frac{\log(1+T)}{T}$
Crisp	1.5	1.704	102.00	2127.56		
Fuzzy	1.9	2.58	346.30	2164.49	0.304	
Cloudy Fuzzy	1.85	2.22	183.03	2115.33		0.227

average cost. Also, for the changes of a and b from −15% to +15%, production rate also increases. Thus average cost moderately increases due to the increase of demand. Again, the same results were observed for the changes of unit production cost. These phenomena agree with reality. But for the changes of c_3, h, σ and $\rho$from −15% to +15%, there are moderately variations on the average cost. This sensitivity table reveals that the observations done on inventory model are more realistic and more practicable.

6.2. *Effect of changing cycle time*

Comparing the results obtained in the crisp, general fuzzy and cloudy fuzzy environment, it has been observed from the graphical illustration (Figure 3) that the cloudy fuzzy model predicts the minimum cost 2068.57 ($) and the minimum cost is obtained at cycle time 4 months, which is shown in Figure 4. In Figure 4, the curve shows U-shaped pattern under the cloudy fuzzy model. So the curve is convex. So, it is interesting to note that the cloudy fuzzy model is more reliable.

6.3. *Effect of changing reliability*

Relibility is the most important factor in the manufacturing system as reliability is defined to be the capability of manufacturing units without breakdown of the system. It has been observed from the graphical illustration (Figure 5) that as the reliability increases, average cost gradually decreases as the increase of reliabilty resulted in the increase of production rate. So, the cost of finished good consistently decreases.

Also, the performance level, as measured by reliability, can significantly improve the manufacturing system. Since the present model is minimization problem, averege cost decreases with the increase of reliability.

6.4. *Comparison of average cost under different cycle time*

Difference in average inventory cost of the crisp model and the general fuzzy model with respect to the cloudy fuzzy model for different value of cycle time is shown in Table 3. From this Table 3, it is seen that the cloudy fuzzy model that gives the minimum average inventory cost at time 4 months which is the better choice of inventory practinioner and decision maker.

Table 2. Sensitivity analysis for the cloudy fuzzy model.

Parameters	% Change	Average cost (z^*)	$\frac{(z' - z)}{z} \times 100\%$
d	−15%	1833.44	−13.32
	−10%	1927.49	−8.88
	−5%	2021.45	−4.44
	5%	2209.13	+4.43
	10%	2302.87	+8.86
	15%	2396.55	+13.29
a	−15%	2099.51	−0.75
	−10%	2104.86	−0.49
	−5%	2110.13	−0.25
	5%	2120.45	+0.24
	10%	2125.51	+0.48
	15%	2130.48	+0.69
b	−15%	2006.40	−5.15
	−10%	2046.12	−3.27
	−5%	2082.28	−1.56
	5%	2145.66	+1.43
	10%	2173.58	+2.75
	15%	2199.39	+3.97
c_3	−15%	2108.56	−0.32
	−10%	2110.82	−0.21
	−5%	2113.07	−0.11
	5%	2122.09	+0.32
	10%	2128.87	+0.64
	15%	2135.63	+0.96
c	−15%	1833.27	−13.37
	−10%	1927.29	−8.90
	−5%	2021.31	−4.44
	5%	2209.35	+4.44
	10%	2303.37	+8.89
	15%	2397.38	+13.33
h	−15%	2100.38	−0.71
	−10%	2105.36	−0.47
	−5%	2110.35	−0.23
	5%	2120.31	+0.23
	10%	2125.28	+0.47
	15%	2130.28	+0.71
σ	−15%	2111.02	−0.20
	−10%	2112.46	−0.14
	−5%	2113.90	−0.07
	5%	2116.77	+0.07
	10%	2118.20	+0.14
	15%	2119.64	+0.20
ρ	−15%	2118.80	+0.16
	−10%	2117.64	+0.11
	−5%	2116.49	+0.05
	5%	2114.18	−0.05
	10%	2113.02	−0.11
	15%	2111.86	−0.16

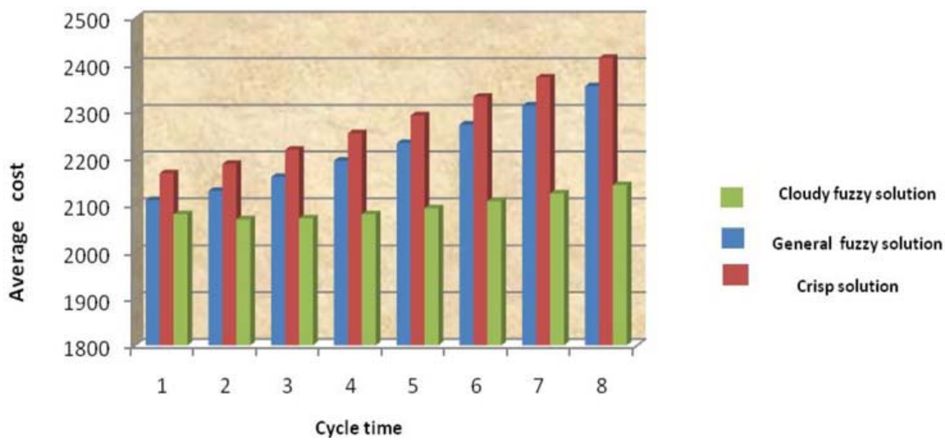

Figure 3. Average cost vs. cycle time.

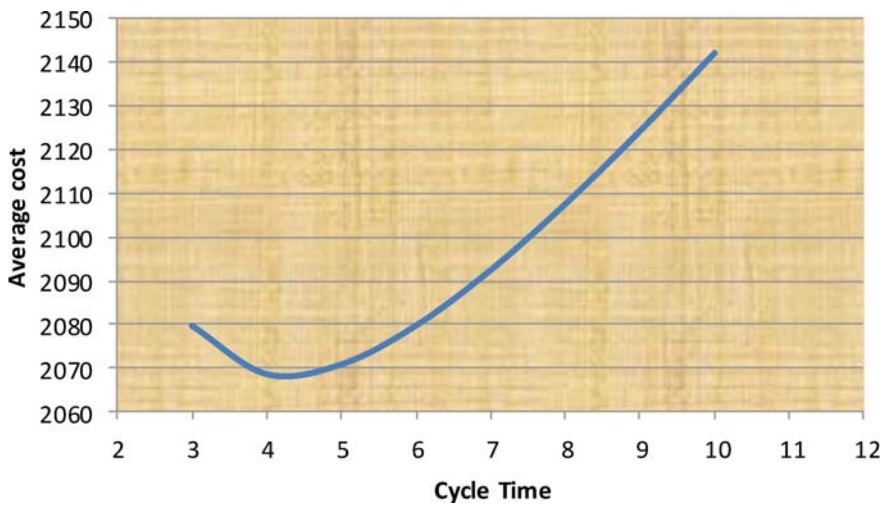

Figure 4. Average cost vs. cycle time for the cloudy fuzzy model.

7. Conclusion and future research

The defuzzification of cloudy fuzzy is the well-established (De & Mahata, 2017, 2019) and latest methodology in fuzzy environment, but a few research works have been done in this area. For the first time, the fuzzy inventory model under imperfect production process with cloudy fuzzy demand rate is developed where the production rate is demand dependent. The model has been discussed over a crisp, fuzzy and cloudy fuzzy environment exclusively and cloudy fuzzy defuzzification using Yager's index method and De and Beg's ranking index method for comparing the results obtained in crisp, fuzzy and cloudy fuzzy environment. It has been observed that average cost is 2068.57 (Rupees/Dollar) which is minimum at cycle length 4 (months/years) for the cloudy fuzzy environment where as average cost obtained in

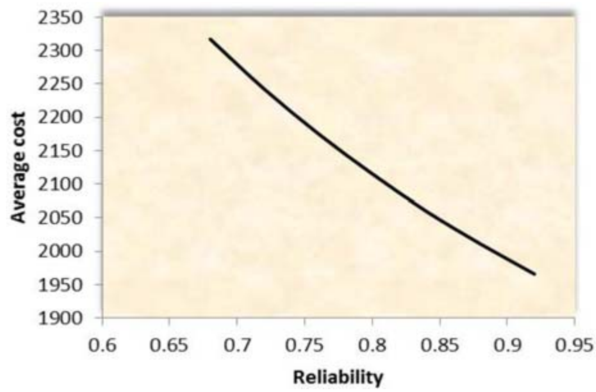

Figure 5. Average cost vs. reliability for the cloudy fuzzy model.

Table 3. Average cost under different model.

Crisp model				Genral fuzzy model			Cloudy fuzzy model		
Cycle time T	t_1*	Q*	Z*	t_1*	Q*	Z*	t_1*	Q*	Z*
3	2.64	179.58	2109.86	2.68	164.80	2167.25	1.35	74.68	2079.64
4	3.52	239.46	**2129.58**	3.62	195.70	**2187.59**	1.80	99.52	**2068.57**
5	4.41	299.29	2159.47	4.55	231.75	2217.92	2.21	149.04	2070.79
6	5.20	359.15	2194.36	5.51	252.35	2253.35	2.69	154.26	2079.68
7	6.11	419.01	2232.11	6.45	283.25	2291.45	3.13	189.16	2092.40
8	7.04	478.87	2271.65	7.37	323.42	2331.43	3.59	203.02	2107.53
9	7.92	538.73	2312.33	8.34	339.90	2372.59	4.04	228.91	2124.26
10	8.81	598.59	2353.95	9.20	394.49	2414.60	4.53	238.13	2142.13

Bold face in Table-3 indicates that objective value is minimum at cycle time 4 in crisp model, general fuzzy model and cloudy fuzzy model respectively. Here comparison is made for three models and cloudy fuzzy model provide minimum cost.

general fuzzy environment is 2187.59 (Rupees/Dollar) at the same cycle length which is more than that of the cloudy fuzzy environment. Changes in different inventory parameters and fuzzy variables predict that cloudy fuzzy number is more reliable than general fuzzy number and hesitancy of fuzzy gradually decreases in the cloudy fuzzy environment due to gathering experience over time. Further extension of this model can be done considering some realistic situations such as multi-item, quantity discount, price and reliability-dependent demand, learning effect etc. Moreover, in future, this model can be formulated with random planning horizon, fuzzy planning horizon in stochastic, fuzzy stochastic environments. Sensitivity analyses reveal the superiority of the cloudy fuzzy environment with respect to that of the general fuzzy model. The managerial insights are observed as follows:

- The cloudy fuzzy model gives average minimum cost
- Parameters involved in this model are not equally responsible for minimization of cost function.
- It is not really true that less fuzziness guarantees minimum cost.

Acknowledgements

The author expresses heartfelt thanks and gratitude to the Editor and anonymous reviewers for their valuable comments and suggestions which helped immensely to improving the research paper. Author would like to thank Mr. Uday Kumar Maiti for improving the English language.

Disclosure statement

No potential conflict of interest was reported by the author.

Funding

This work was supported by the University Grant Commission (UGC), New Delhi, India under the research grant [grant number PSW-132/14-15(ERO)].

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Appendix

The expressions of I_1, I_2, I_3 and I_4 are given below.

$$I_1 = \frac{1}{4\tau}\int_0^\tau \left\{ \frac{c\,d}{r}\left(4 + \frac{\sigma - \rho}{1+T}\right) + \frac{4\,c_3}{T} \right\} dT \quad = \frac{c\,d}{r}\left(1 + \frac{\sigma - \rho}{4\tau}\ln|1+\tau|\right) + \frac{c_3}{\tau}\ln\left|\frac{\tau}{\varepsilon}\right|$$

$$I_2 = \int_0^\tau T\left(1 - \frac{\rho}{1+T}\right) \frac{\left\{ a\,r + (b\,r - 1)\,d\left(1 - \frac{\rho}{1+T}\right)\right\}}{a + b\,d\left(1 + \frac{\sigma}{1+T}\right)}\, dT$$

$$= \int_0^\tau T\left(1 - \frac{\rho}{1+T}\right) \frac{T(a\,r + (b\,r-1)d) + a\,r + (b\,r - 1)\,d(1-\rho)}{T(a + b\,d) + a + b\,d(1+\sigma)}\, dT$$

$$= \int_0^\tau T\left(1 - \frac{\rho}{1+T}\right)\frac{pT + q}{T\,u + v}\, dT$$

$$[p = a\,r + (b\,r - 1)\,d,\ \ q = a\,r + (b\,r-1)\,d\,(1-\rho),\ \ u = a + b\,d,\ \ v = a + b\,d(1+\sigma)]$$

$$= p\int_0^\tau \frac{T^2}{T\,u + v}\, dT + q\int_0^\tau \frac{T}{T\,u+v}\, dT - \rho p\int_0^\tau \frac{T^2}{(T\,u+v)(1+T)}\, dT - \rho q\int_0^\tau \frac{T}{(T\,u+v)(1+T)}\, dT$$

$$= I_{21} + I_{22} - I_{23} - I_{24}$$

$$\text{where } I_{21} = p\int_0^\tau \frac{T^2}{T\,u+v}\, dT = \frac{p}{u^3}\left[\frac{u^2\,\tau^2}{2} - u\,v\,\tau + v^2\,\ln\left|\frac{v + u\,\tau}{v}\right|\right]$$

$$I_{22} = q\int_0^\tau \frac{T}{T\,u+v}\, dT = \frac{q}{u}\left[\tau - \frac{v}{u}\ln\left|\frac{v+\tau u}{v}\right|\right]$$

$$I_{23} = \rho p\int_0^\tau \frac{T^2}{(T\,u+v)(1+T)}\, dT = \frac{\rho p}{u - v}\left[\tau - \ln|1+\tau| - \frac{v\,\tau}{u} + \frac{v^2}{u^2}\ln\left|\frac{v + \tau u}{v}\right|\right]$$

$$I_{24} = \rho q\int_0^\tau \frac{T}{(T\,u+v)(1+T)}\, dT = \frac{\rho\,q}{v-u}\left[\frac{v}{u}\ln\left|\frac{v+\tau u}{v}\right| - \ln|1+\tau|\right]$$

$$I_3 = 2\int_0^\tau \frac{a\,r + (b\,r-1)\,d}{a + b\,d}\, dT = \frac{p}{u}\,\tau^2$$

$$I_4 = \int_0^\tau T\left(1 + \frac{\sigma}{1+T}\right) \frac{\left\{a\,r + (b\,r-1)\,d\left(1 + \frac{\sigma}{1+T}\right)\right\}}{a + b\,d\left(1 - \frac{\rho}{1+T}\right)}\, dT$$

$$= \int_0^\tau T\left(1 + \frac{\sigma}{1+T}\right) \frac{T(ar + (br-1)d) + ar + (br-1)\,d(1+\sigma)}{T(a+bd) + a + bd(1-\rho)}\,dT$$

$$= \int_0^\tau T\left(1 + \frac{\sigma}{1+T}\right) \frac{pT + y}{Tu + s}\,dT$$

$$[p = ar + (br-1)\,d,\;\; y = ar + (br-1)\,d\,(1+\sigma),\;\; u = a + bd,\;\; s = a + bd(1-\rho)]$$

$$= p\int_0^\tau \frac{T^2}{Tu+s}\,dT + y\int_0^\tau \frac{T}{Tu+s}\,dT + \sigma p\int_0^\tau \frac{T^2}{(Tu+s)(1+T)}\,dT + \sigma y\int_0^\tau \frac{T}{(Tu+s)(1+T)}\,dT$$

$$= I_{41} + I_{42} + I_{43} + I_{44}$$

$$\text{where } I_{41} = p\int_0^\tau \frac{T^2}{Tu+s}\,dT = \frac{p}{u^3}\left[\frac{u^2\tau^2}{2} - us\tau + s^2\ln\left|\frac{s+u\tau}{s}\right|\right]$$

$$I_{42} = y\int_0^\tau \frac{T}{Tu+s}\,dT = \frac{y}{u}\left[\tau - \frac{s}{u}\ln\left|\frac{s+\tau u}{s}\right|\right]$$

$$I_{43} = \sigma p\int_0^\tau \frac{T^2}{(Tu+s)(1+T)}\,dT = \frac{\sigma p}{u-s}\left[\tau - \ln|1+\tau| - \frac{s\tau}{u} + \frac{s^2}{u^2}\ln\left|\frac{s+\tau u}{s}\right|\right]$$

$$I_{44} = \sigma y\int_0^\tau \frac{T}{(Tu+s)(1+T)}\,dT = \frac{\sigma y}{s-u}\left[\frac{s}{u}\ln\left|\frac{s+\tau u}{s}\right| - \ln|1+\tau|\right]$$