APPENDIX A

Selected Proofs and Additional Material

A.1 Dedekind continuity

Postulate 2.4 of the existence of infima for lower-bounded chains (p. 36) guarantees that a maximal chain in $\langle W, < \rangle$ is continuous in the sense of Dedekind: a Dedekind cut of a maximal chain has neither gaps nor jumps. A Dedekind cut completely partitions a maximal chain C into two non-empty sub-chains A and B such that every element of A is strictly below every element of B. There is a gap if A has no maximal element and B has no minimal element, whereas there is a jump if A has a maximal element and B has a minimal element. Clearly, a jump is excluded by density and the chain's maximality. Further, a gap is excluded as well (by Postulate 2.4), which is the topic of the following Fact: given Postulate 2.4, maximal chains are continuous.

Fact A.1. *Postulate 2.4 implies that a Dedekind cut $\{A,B\}$ of a maximal chain C has no gaps.*

Proof. Let $\{A,B\}, A < B$ be a Dedekind cut of the maximal chain $C = A \cup B$. Since $A \neq \emptyset$, B is lower bounded, so B has the infimum $\inf B$ by Postulate 2.4. Clearly, $\inf B \leqslant B$ and $A \leqslant \inf B$ since every element of A is a lower bound for B (by the definition of infima). Hence $\inf B \in C$, as otherwise C would not be a *maximal* chain. Since $\{A,B\}$ completely partitions C, either $\inf B \in A$ or $\inf B \in B$. If the former, by the definition of infima $\inf B$ is a maximal element of A; if the latter, $\inf B$ is a minimal element of B. This proves that there is no gap. □

Since a maximal chain is a subset of some history (see Fact 2.1(3)), Postulate 2.5 of the existence of history-relative suprema provides another guarantee that a maximal chain is Dedekind continuous; the argument is analogous to the proof of Fact A.1.

A.2 Formal details of the interrelation of BST_{92} and $\mathrm{BST}_{\mathrm{NF}}$

This section complements Chapter 3 by offering details of the interrelation between the two alternative BST frameworks, BST_{92} and $\mathrm{BST}_{\mathrm{NF}}$, that were described there. Among other things, we provide proofs to the translatability results for BST_{92} and $\mathrm{BST}_{\mathrm{NF}}$ that were formulated in Chapter 3.6.2. We establish two translation mappings, the Λ- and the Υ-transform, which will be shown to preserve the basic indeterministic structure: given a BST_{92} structure, its Υ-transform replaces each choice point by the set of transitions from that choice point, leading to a choice set and, accordingly, to a $\mathrm{BST}_{\mathrm{NF}}$ structure. In the other direction, the Λ transform operates on a $\mathrm{BST}_{\mathrm{NF}}$ structure, replacing each choice set by a single point, which will be a choice point in the resulting BST_{92} structure. In this way, as announced, BST_{92} and $\mathrm{BST}_{\mathrm{NF}}$ can be seen as two alternative representations of an underlying indeterministic structure. This means that we can represent indeterministic scenarios without really having to decide between the different prior choice principles of BST_{92} and BST_{NF}, and that we can take a pragmatic attitude toward the topological consequences of BST_{92} vs. BST_{NF} as well.

A.2.1 Characterizing the transition structure of a BST_{92} structure

We are often only interested in indeterministic transitions, as deterministic transitions make no difference to the branching of histories[1]. In the present context, however, it is important to consider *all* transitions, including those that are trivial from the point of view of indeterminism. We repeat the definition of the Υ *transform* of a BST_{92} structure:

Definition 3.17 (The Υ transform as the full transition structure of a BST_{92} structure.)**.** Let $\langle W, < \rangle$ be a BST_{92} structure. Then we define the transformed structure, $\Upsilon(\langle W, < \rangle)$, to be the full transition structure (including trivial transitions) together with the transition ordering $\prec$ from Def. 3.10, as follows:

$$\Upsilon(\langle W, < \rangle) =_{\text{df}} \langle W', \prec \rangle, \text{ where } \quad W' =_{\text{df}} \text{TR}_{\text{full}}(W) = \{e \rightarrowtail H \mid e \in W, H \in \Pi_e\}.$$

From here on we will mark transformed elements with primes.

Having defined the transition structure, we now characterize its properties. It turns out that the full transition structure $\Upsilon(\langle W, < \rangle)$ looks very much like the original BST_{92} structure $\langle W, < \rangle$, except for what happens at the choice points. In fact, we will be able to show that, apart from the prior choice postulate, all defining properties of BST_{92}; that is, the whole list of properties of a common BST structure from Def. 2.10, continue to hold; see Lemma 3.3, which was stated in Chapter 3 and which we will repeat and prove below. With respect to the choice points, the difference is the following. In BST_{92}, the branching of histories is from a choice point, shared among the histories that branch, to the immediate possibilities for the future at that choice point. There are no first points in these different possible futures, and this fact leads to the failure of local Euclidicity in the global BST_{92} topology (see Section 4.4). In $\Upsilon(\langle W, < \rangle)$, on the other hand, each choice point is replaced by all the transitions that have that choice point as an initial. Therefore, where in BST_{92} there was a last point that was shared between the different possibilities, in the transition structure there are now multiple first points characterizing these different possibilities, and there is no longer any last shared point.[2] In the structures of Figure 3.1 (p. 44), the move from (a) to (b) exactly corresponds to the move from the BST_{92} structure M_a to its transition structure M_b.[3] For the topological consequences, see Section 4.4.

In order to prove that $\Upsilon(\langle W, < \rangle)$ is a common BST structure, we first need to establish the form that histories (i.e., maximal directed sets), have in that ordering. Their form is quite intuitive, even though it turns out that the proof of that fact is somewhat lengthy. We first establish a useful general fact about directed sets of transitions:

[1] For a study along those lines, see Müller (2010).

[2] This image of fanning out the transitions from a choice point motivates our notation, Υ.

[3] To be precise, the transition structure of M_a is order-isomorphic to M_b. See Chapter A.2.3 for a formal discussion.

Fact A.2. *Let $T \subseteq \Upsilon(\langle W,<\rangle)$ be a set of transitions, and let there be $e \in W$ and $H_1, H_2 \in \Pi_e$, $H_1 \neq H_2$, such that both $\tau_1 = e \rightarrowtail H_1$ and $\tau_2 = e \rightarrowtail H_2$ are members of T. Then T is not directed.*

Proof. Assume otherwise; that is, assume that there is some $\tau^* = e^* \rightarrowtail H^* \in T$ for which $\tau_1 \preccurlyeq \tau^*$ and $\tau_2 \preccurlyeq \tau^*$. By Fact 3.11(1), this implies $H_{e^*} \subseteq H_1$ and $H_{e^*} \subseteq H_2$. But as H_1 and H_2 are different elements of the partition Π_e, we have $H_1 \cap H_2 = \emptyset$, contradicting $H_{e^*} \subseteq H_1 \cap H_2$. (Note that $H_{e^*} \neq \emptyset$ by Fact 2.1(3).) □

Now we can tackle the form of histories in $\Upsilon(\langle W,<\rangle)$.

Lemma A.1. *Let $\langle W,<\rangle$ be a BST$_{92}$ structure, and let $\langle W',\prec\rangle =_{\text{df}} \Upsilon(\langle W,<\rangle)$. The histories (maximal directed sets) in $\langle W',\prec\rangle$ are exactly the sets*

$$T_h =_{\text{df}} \{e \rightarrowtail \Pi_e\langle h\rangle \mid e \in h\}$$

for h in Hist(W).

Proof. First we establish that such sets are indeed histories in $\langle W',\prec\rangle$. Thus, take some $h \in \text{Hist}(W)$, and let $T_h =_{\text{df}} \{e \rightarrowtail \Pi_e\langle h\rangle \mid e \in h\} \subseteq W'$. The set T_h is directed: take $e_1 \rightarrowtail \Pi_{e_1}\langle h\rangle$ and $e_2 \rightarrowtail \Pi_{e_2}\langle h\rangle$ from T_h, whence $e_1, e_2 \in h$. As h is directed, there is $e_3 \in h$ such that $e_1, e_2 \leqslant e_3$. By construction, $e_3 \rightarrowtail \Pi_{e_3}\langle h\rangle \in T_h$. And as to the ordering, $H_{e_3} \subseteq \Pi_{e_1}\langle h\rangle$ because $e_3 \in h$ and $e_1 \leqslant e_3$. Analogously, $H_{e_3} \subseteq \Pi_{e_2}\langle h\rangle$. So indeed (noting Fact 3.11(1)), $e_i \rightarrowtail \Pi_{e_i}\langle h\rangle \preccurlyeq e_3 \rightarrowtail \Pi_{e_3}\langle h\rangle$ $(i = 1,2)$, establishing the common upper bound. Moreover, T_h is maximal directed. To prove this, take some $\tau^* \in (W' \setminus T_h)$; this transition has the form $\tau^* = e^* \rightarrowtail H^*$ for some $e^* \in W$, $H^* \in \Pi_{e^*}$. There are two cases.

Case 1: There is some $\tau = e \rightarrowtail \Pi_e\langle h\rangle \in T_h$ for which $e = e^*$, i.e., $e^* \in h$. Then, as $\tau \neq \tau^*$, by Fact A.2, $T_h \cup \{\tau^*\}$ cannot be directed.

Case 2: There is no $\tau = e \rightarrowtail \Pi_e\langle h\rangle \in T_h$ for which $e = e^*$, i.e., $e^* \notin h$. Then we have $e^* \in h'$ for a different $h' \in \text{Hist}(W)$, and by the BST$_{92}$ prior choice principle, there is some $c \in h \cap h'$ such that $c < e^*$ and $h \perp_c h'$. As $c \in h$, we have $\tau_c =_{\text{df}} c \rightarrowtail \Pi_c\langle h\rangle \in T_h$. We can now show that $T_h \cup \{\tau^*\}$ is not directed: there can be no common upper bound for τ_c and τ^* in W'. Assume for reductio that there is some $\tau' = e' \rightarrowtail H' \in T_h \cup \{\tau^*\}$ for which $\tau_c \preccurlyeq \tau'$ and $\tau^* \preccurlyeq \tau'$. We can rule out $\tau' = \tau^*$: we have $\tau_c \neq \tau^*$ by $c < e^*$, and $\tau_c \not\prec \tau^*$ as $H_{e^*} \not\subseteq \Pi_c\langle h\rangle$ (as $H_{e^*} \subseteq \Pi_c\langle h'\rangle$). So we must have $\tau' \in T_h$. By the definition of $\prec$, the assumed ordering relations imply $H_{e'} \subseteq \Pi_c\langle h\rangle$ and $H_{e'} \subseteq H^* \subseteq \Pi_c\langle h'\rangle$. But we have $\Pi_c\langle h\rangle \cap \Pi_c\langle h'\rangle = \emptyset$.

So, having shown that the sets T_h are indeed histories in $\langle W',\prec\rangle$, we need to show that *all* histories in $\langle W',\prec\rangle$ are of that form. Thus, let $g \subseteq W'$ be a history in $\langle W',\prec\rangle$, maximal directed with respect to $\prec$. By Fact A.2, there is no $e \in W$ for which g contains two transitions $e \rightarrowtail H_1$ and $e \rightarrowtail H_2$, $H_1 \neq H_2$, so that we can write

$$g = \{e \rightarrowtail H(e) \mid e \in E\}$$

for some set $E \subseteq W$, where $H(e) \in \Pi_e$. We first show that E is directed: Take $e_1, e_2 \in E$, so that $\tau_i =_{\text{df}} e_i \rightarrowtail H(e_i) \in g$. By directedness of g, there is some $\tau_3 = e_3 \rightarrowtail H(e_3) \in g$ for which $\tau_i \preccurlyeq \tau_3$ $(i = 1,2)$, which implies $e_3 \in E$, $e_1 \leqslant e_3$, and $e_2 \leqslant e_3$. This proves that E is directed, and therefore there is some history $h \in \text{Hist}(W)$ for which $E \subseteq h$. We now show that for all $e \in E$, we have $h \in H(e)$. Thus, take some $e \in E$, which is the initial of some $\tau = e \rightarrowtail H(e) \in g$. There are two cases.

Case 1: τ is maximal in g. By Fact 2.1(9), τ therefore is also maximal in W', which implies that e is maximal in W. By Fact 2.1(10), there is a unique history containing e, and as $e \in h$, that unique history must be our h. We therefore have $\Pi_e = \{\{h\}\}$. As $H(e) \in \Pi_e$, this implies $H(e) = \{h\}$, whence $h \in H(e)$.

Case 2: τ is not maximal in g, i.e., there is some $\tau' = e' \rightarrowtail H(e') \in g$ for which $\tau \prec \tau'$. By Fact 3.11(1), this implies that $H_{e'} \subseteq H(e)$, and as $e' \in E \subseteq h$, we have $h \in H_{e'}$ and therefore also $h \in H(e)$.

As $h \in H(e)$ for all $e \in E$, we have $H(e) = \Pi_e\langle h\rangle$ for all $e \in E$. This implies $g \subseteq T_h$, and by Fact 2.1(7), we have established $g = T_h$. □

Given these facts, we can now prove that switching from a BST_{92} structure to its full transition structure preserves the common BST structure axioms. We will need to assume, however, that the BST_{92} structure contains no minima otherwise the historical connection might fail: If a minimal element $e \in W$ is a choice point, its image under the Υ transformation consists of two or more minimal elements in the resulting order, which accordingly have no common lower bound.

Lemma 3.3. *Let $\langle W, <\rangle$ be a BST_{92} structure without minima. Then its full transition structure $\Upsilon(\langle W, <\rangle)$ is still a common BST structure according to Definition 2.10.*

Proof. We need to check that $\langle W', \prec\rangle =_{\text{df}} \Upsilon(\langle W, <\rangle)$ satisfies all the properties (1)–(7) listed in Definition 2.10.

1. W' is non-empty. See Fact 3.10(1).
2. $\langle W' \prec\rangle$ is a strict partial ordering. See Fact 3.10(2).
3. $\prec$ is dense.
   Let $(e_1 \rightarrowtail H_1) \prec (e_3 \rightarrowtail H_3)$, which means that $e_1 < e_3$ and $H_3 \subseteq H_1$. By density of $<$, there is $e_2 \in W$ for which $e_1 < e_2 < e_3$. Take some $h \in H_3$, so that $\{e_1, e_2, e_3\} \subseteq h$. Let $H_2 =_{\text{df}} \Pi_{e_2}\langle h\rangle$. We claim that the transition $e_2 \rightarrowtail H_2$ is $\prec$-sliced between the two transitions above. We have to show that $H_2 \subseteq H_1$ and $H_3 \subseteq H_2$. For the former, take some $h_2 \in H_2$; we have $h_2 \equiv_{e_1} h$ as witnessed by e_2. As $H_1 \in \Pi_{e_1}$, therefore, $h_2 \in H_1$ iff $h \in H_1$. Now as $h \in H_3$ and $H_3 \subseteq H_1$, we have $h \in H_1$, so that indeed, $H_2 \subseteq H_1$. The latter claim is established analogously.
4. Any lower bounded chain in $\langle W', \prec\rangle$ has an infimum in $\prec$.
   Let $l' = \{e_i \rightarrowtail H_i \mid i \in \Gamma\}$ (Γ some index set) be a chain that is lower bounded by $e^* \rightarrowtail H^*$. Then the set $l =_{\text{df}} \{e_i \mid i \in \Gamma\}$ of initials of l' is a chain lower bounded by e^*, and there is a history $h \subseteq W$ for which $l \subseteq h$. By the BST_{92} postulate of infima, l has an infimum v in $<$. The infimum v gives rise to the transition $v' =_{\text{df}} v \rightarrowtail \Pi_v\langle h\rangle \in W'$. Let $e_i \rightarrowtail H_i \in l'$. We have $v \leqslant e_i$ (as $e_i \in l$), and $H_i \subseteq \Pi_v\langle h\rangle$ because $e_i \in h$ and $v \leqslant e_i$. Thus, $v' \preccurlyeq (e_i \rightarrowtail H_i)$, so v' is a lower bound of l'. Let now $e \rightarrowtail H$ be any lower bound of l', whence e is a lower bound of l. As v is the infimum of l, we have $e \leqslant v$, and as $l \subseteq h$, we have $H = \Pi_e\langle h\rangle$, which implies $H_v \subseteq H$. Thus $e \rightarrowtail H \preccurlyeq v'$, i.e., v' is indeed the greatest lower bound of l'.
5. Any upper-bounded chain in $\langle W', \prec\rangle$ has a history-relative supremum in each history to which it belongs.
   Let the chain l' be upper bounded by u' in $\langle W', \prec\rangle$ and $l' \cup \{u'\} \subseteq h'$ for $h' \in \text{Hist}(W', \prec)$. Given the form of histories in $\text{Hist}(W', \prec)$ (see Lemma A.1), $h' = \{e \rightarrowtail \Pi_e\langle h\rangle \mid e \in h\}$ for some $h \in \text{Hist}(W)$. It follows that for the set l of initials of l' and for u the initial of u', $l \cup \{u\} \subseteq h$; additionally, $l \leqslant u$. By the BST_{92}

axiom of history-relative suprema, there is a history-relative supremum $s = \sup_h l$ of l in h. Consider now the transition $s' = s \rightarrowtail \Pi_s\langle h\rangle \in h'$. That transition is an upper bound of l': for any $e \rightarrowtail H \in l'$, we have $e \leqslant s$ and $H_s \subseteq H = \Pi_e\langle h\rangle$. Furthermore, s' is the least upper bound (i.e., the supremum) of l' in h': let $s^{*\prime} \in h'$ be an upper bound of l' in h'; by the form of histories, $s^{*\prime} = s^* \rightarrowtail \Pi_{s^*}\langle h\rangle$with $s^* \in h$. Thus, $s \leqslant s^*$ (as s is the h-relative supremum of l), and as $s', s^{*\prime} \in h'$, by Fact 3.11(1), we have $s' \preccurlyeq s^{*\prime}$.

Summing up, we have established that for $l' \subseteq h'$ an upper-bounded chain in a W'-history $h' = \{e \rightarrowtail \Pi_e\langle h\rangle \mid e \in h\}$ (where h is the corresponding W-history), the initials satisfy $l \subseteq h$, and there exists the h'-relative supremum

$$\sup_{h'} l' = s \rightarrowtail \Pi_s\langle h\rangle,$$

where $s = \sup_h l$.

6. $\langle W', \prec\rangle$ satisfies Weiner's postulate.
We will employ the claim established at the end of the previous item (5), and the fact that $\langle W, <\rangle$ satisfies Weiner's postulate.
Let $h'_1, h'_2 \in \text{Hist}(W')$, and let $h'_i = \{e \rightarrowtail \Pi_e\langle h_i\rangle \mid e \in h_i\}$ for $h_i \in \text{Hist}(W)$ $(i = 1,2)$. We consider two chains $l', k' \subseteq h'_1 \cap h'_2$, their respective chains of initials $l, k \subseteq h_1 \cap h_2$, and their history-relative suprema $s'_i = s_i \rightarrowtail \Pi_{s_i}\langle h_i\rangle = \sup_{h'_i}(l')$ and $c'_i = c_i \rightarrowtail \Pi_{c_i}\langle h_i\rangle = \sup_{h'_i}(k')$, where $s_i = \sup_{h_i} l$ and $c_i = \sup_{h_i} k$, for $i = 1,2$. Suppose that $s'_1 \preccurlyeq c'_1$, i.e., $s_1 \leqslant c_1$ and $\Pi_{c_1}\langle h_1\rangle \subseteq \Pi_{s_1}\langle h_1\rangle$ (see Fact 3.11(3)). By Weiner's postulate of BST_{92} applied to the chains l and k, from $s_1 \leqslant c_1$ we may infer $s_2 \leqslant c_2$. Note that $s_2 \in h_2$. Hence $\Pi_{c_2}\langle h_2\rangle \subseteq \Pi_{s_2}\langle h_2\rangle$. In terms of the transition ordering, this means that $s'_2 \preccurlyeq c'_2$.
7. Historical connection. Note that by assumption, W has no minima. Let $h'_1, h'_2 \in \text{Hist}(W')$ be histories, which by Lemma A.1 correspond to $h_1, h_2 \in \text{Hist}(W)$. By historical connection for W, there is some $e \in h_1 \cap h_2$, and by no minima, there is some $e^* \in W$ for which $e^* < e$. It follows that $e^* \in h_1 \cap h_2$. Let $\tau =_{\text{df}} e^* \rightarrowtail \Pi_{e^*}\langle h_1\rangle$. By Lemma A.1, we have $\tau \in h'_1$. Now $e > e^*$ is a witness for $h_1 \equiv_{e^*} h_2$, so that $\Pi_{e^*}\langle h_1\rangle = \Pi_{e^*}\langle h_2\rangle$, i.e., $\tau \in h'_2$ as well. □

A.2.2 BST_{92} transition structures are BST_{NF} structures

We can now show that the full transition structure of a BST_{92} structure without minima is indeed a BST_{NF} structure. The only thing that is still missing is to show that the new prior choice principle PCP_{NF} is satisfied. To this end we need an auxiliary fact that shows how BST_{92} choice points give rise to BST_{NF} choice sets.

Fact A.3. *Let $\langle W, <\rangle$ be a BST_{92} structure without minima, and let $h_1 \perp_c h_2$ for $h_1, h_2 \in$* Hist(W). *Then $c'_1 =_{\text{df}} c \rightarrowtail \Pi_c\langle h_1\rangle$ and $c'_2 =_{\text{df}} c \rightarrowtail \Pi_c\langle h_2\rangle$ belong to $\Upsilon(\langle W, <\rangle)$ and are elements of a choice set $\ddot{c}$ for which $h'_1 \perp_{\ddot{c}} h'_2$, where $h'_i = \{e \rightarrowtail \Pi_e\langle h_i\rangle \mid e \in h_i\}$ $(i = 1,2)$ are the histories in $\Upsilon(\langle W, <\rangle)$ corresponding to h_1 and h_2.*

Proof. Let $\langle W', <'\rangle =_{\text{df}} \Upsilon(\langle W, <\rangle)$, let $h_1, h_2 \in \text{Hist}(W)$, and let $c \in h_1 \cap h_2$ be such that $h_1 \perp_c h_2$. By Lemma A.1, $h'_i = \{e \rightarrowtail \Pi_e\langle h_i\rangle \mid e \in h_i\} \in \text{Hist}(W')$ $(i = 1,2)$. Let $c'_i =_{\text{df}} c \rightarrowtail \Pi_c\langle h_i\rangle$, so that $c'_i \in h'_i$ $(i = 1,2)$. In order to show that c'_1, c'_2 are elements of a choice set $\ddot{c}$

that fulfills $h'_1 \perp_{\ddot{c}} h'_2$, we need to show that every chain $l' \in \mathscr{C}_{c'_1}$, for which $\sup_{h'_1} l' = c'_1$, has c'_2 as another history-relative supremum, and vice versa. Since c is not a minimal element of W, $\mathscr{C}_{c'_1} \neq \emptyset$. Pick an arbitrary chain $l' \in \mathscr{C}_{c'_1}$, and note that it has the form $l' = \{e \rightarrowtail \Pi_e\langle h_1\rangle \mid e \in l\}$ for some chain $l \subseteq h_1$, with $c = \sup_{h_1} l$. Since $h_1 \perp_c h_2$, $l \subseteq h_2$ as well, and as $l < c$, we have that for every $e \in l$, $\Pi_e\langle h_1\rangle = \Pi_e\langle h_2\rangle$. Hence $l' \subseteq h'_1 \cap h'_2$. It follows that $\sup_{h'_1} l' = c'_1$ and $\sup_{h'_2} l' = c'_2$ (note that the $c'_i \in h'_i$ are upper bounds of l' and that their initial, c, is the h_1- as well as the h_2-relative supremum of l). Since l' is an arbitrary chain in $\mathscr{C}_{c'_1}$, we showed that every chain in $\mathscr{C}_{c'_1}$ has at least two history-relative suprema, c'_1 and c'_2, i.e., there is a choice set $\ddot{c}'$ such that $\{c'_1, c'_2\} \subseteq \ddot{c}'$. Since $h'_1 \cap \ddot{c}' = c'_1 \neq c'_2 = h'_2 \cap \ddot{c}'$, we have $h'_1 \perp_{\ddot{c}'} h'_2$. □

Given this auxiliary fact, we can establish our lemma:

Lemma 3.4. *Let $\langle W,<\rangle$ be a BST$_{92}$ structure without minima. Then that structure's full transition structure $\langle W',<'\rangle =_{\text{df}} \Upsilon(\langle W,<\rangle)$ satisfies the PCP$_{NF}$ as in Definition 3.14.*

Proof. Let l' be a chain in $\langle W',<'\rangle$ that is lower bounded by u', and let $h'_1, h'_2 \in \text{Hist}(W')$ be such that (†) $l' \subseteq h'_1$ but (‡) $l' \cap h'_2 = \emptyset$. By Lemma A.1, $h'_i = \{e \rightarrowtail \Pi_e\langle h_i\rangle \mid e \in h_i\}$ for some $h_i \in \text{Hist}(W)$, $i = 1,2$. By (†) we have that $l' = \{e \rightarrowtail \Pi_e\langle h_1\rangle \mid e \in l\}$ for a chain $l \subseteq h_1$ that is lower bounded by u, where u is the initial of u'. By (‡), for every $e \rightarrowtail \Pi_e\langle h_1\rangle \in l'$, either $e \notin h_2$, or ($e \in h_2$ but $\Pi_e\langle h_1\rangle \neq \Pi_e\langle h_2\rangle$). There are now four cases, depending on the form of l': (i) l' has a minimal element $v' = v \rightarrowtail \Pi_v\langle h_1\rangle$, $v \in h_1 \cap h_2$, and $\Pi_v\langle h_1\rangle = \Pi_v\langle h_2\rangle$, or (ii) l' has a minimal element $v' = v \rightarrowtail \Pi_v\langle h_1\rangle$, $v \in h_1 \cap h_2$, but $\Pi_v\langle h_1\rangle \neq \Pi_v\langle h_2\rangle$, or (iii) l' has a minimal element $v' = v \rightarrowtail \Pi_v\langle h_1\rangle$, and $v \in h_1 \setminus h_2$, or (iv) l' has no minimal element at all. Case (i) is impossible as it contradicts (‡). Consider then case (ii): since $h_1 \perp_v h_2$, by Fact A.3 and no minimal elements in $\langle W,<\rangle$, the two transitions, $v'_1 = v \rightarrowtail \Pi_v\langle h_1\rangle$ and $v'_2 = v \rightarrowtail \Pi_v\langle h_2\rangle$ are distinct elements of a choice set $\ddot{v}'$ at which histories h'_1 and h'_2 split, $h'_1 \perp_{\ddot{v}'} h'_2$. Furthermore, as v'_1 is the minimal element of l', $v'_1 \preccurlyeq l'$, as required by PCP$_{\text{NF}}$. In cases (iii) and (iv), no element $e \in l$ can belong to h_2: (iii) if l' has a minimum $v' = v \rightarrowtail \Pi_v\langle h_1\rangle$ with $v \in h_1 \setminus h_2$, no point above v can belong to h_2 (Fact 2.1(6)), and (iv) in case l' has no minimum at all, the assumption that $e \in l \cap h_2$ implies that there is also some other $e_1 \in l \cap h$ with $e_1 < e$, and hence $e_1 \rightarrowtail \Pi_{e_1}\langle h_2\rangle \in l' \cap h'_2$, contradicting (‡). Thus, $l \cap h_2 = \emptyset$ in cases (iii) and (iv). Applying the PCP of BST$_{92}$ to the chain $l \subseteq h_1 \setminus h_2$ that is lower bounded by u, we get $v \in W$ such that $v < l$ and $h_1 \perp_v h_2$. Exactly as in case (ii) we thus invoke the assumption of no minimal elements in $\langle W,<\rangle$ and Fact A.3 to produce the sought-after choice set $\ddot{v}'$ containing $v \rightarrowtail \Pi_v\langle h_1\rangle$ and $v \rightarrowtail \Pi_v\langle h_2\rangle$, for which $h'_1 \perp_{\ddot{v}'} h'_2$. Since $v < l$ and given the form of l', we have $v \rightarrowtail \Pi_v\langle h_1\rangle \preccurlyeq l'$ as well. □

Given this result, we have shown that the full transition structure of a BST$_{92}$ structure is a BST$_{\text{NF}}$ structure:

Theorem 3.2. *Let $\langle W,<\rangle$ be a BST$_{92}$ structure without minima. Then that structure's full transition structure $\Upsilon(\langle W,<\rangle)$ is a BST$_{NF}$ structure.*

Proof. From Lemma 3.3 and Lemma 3.4. □

For ease of reference, we repeat as a separate numbered Fact the result of Exercise 3.2, the proof of which is given in Appendix B.3:

Fact A.4. *Let $\langle W,<\rangle$ be a BST$_{92}$ structure without maxima or minima. Then its Υ-transform, $\langle W',<'\rangle$, has no maxima or minima either.*

A.2.3 From new foundations BST_{NF} to BST_{92}

We have seen how the move from a BST_{92} structure to its full transition structure brings us from BST_{92} to BST_{NF}. In the other direction, there is also a fairly simple translation, viz., combining all the elements of a choice set to form a single point.

The Λ transformation from BST_{NF} to BST_{92} is defined in Chapter 3, Def. 3.18, which we repeat here for convenience's sake before we establish some pertinent facts:

Definition 3.18 (The Λ transformation from BST_{NF} to BST_{92}.)**.** Let $\langle W, < \rangle$ be a BST_{NF} structure. Then we define the companion Λ-transformed ("collapsed") structure as follows:

$$\Lambda(\langle W, < \rangle) =_{\text{df}} \langle W', <' \rangle, \quad \text{where}$$
$$W' =_{\text{df}} \{\ddot{e} \mid e \in W\};$$
$$\ddot{e}_1 <' \ddot{e}_2 \quad \text{iff} \quad e_1' < e_2' \quad \text{for some} \quad e_1' \in \ddot{e}_1, e_2' \in \ddot{e}_2.$$

We extend the Λ notation to points and subsets of W setting $\Lambda(e) =_{\text{df}} \ddot{e}$ for $e \in W$, and $\Lambda(A) =_{\text{df}} \{\ddot{e} \mid e \in A\}$ for $A \subseteq W$.

Fact A.5 (Facts about Definition 3.18)**.** *The following holds:*

1. *Let $e_1, e_2 \in W$ and let $e_1 < e_2$. Then $\Lambda(e_1) <' \Lambda(e_2)$.*
2. *Let $t \subseteq W$ be a chain (with respect to $<$). Then $\Lambda(t)$ is a chain (with respect to $<'$).*
3. *Let $E \subseteq W$ be directed. Then $\Lambda(E)$ is also directed.*

Proof. (1): holds by the definition of $<'$. (2) and (3) follow immediately. □

Fact A.6 (Justification of the notation in Definition 3.18)**.** *In BST_{NF}: (1) If $e_1 < e_2$ and $e_2' \in \ddot{e}_2$, then $e_1 < e_2'$. So we can write $e_1 <' \ddot{e}_2$. (2) If $e_1 <' \ddot{e}_2$ and $e_1^* \in \ddot{e}_1$, $e_1 \neq e_1^*$, then $e_1^* \not<' \ddot{e}_2$. So given $\ddot{e}_1 <' \ddot{e}_2$, there is a unique $e_1 \in \ddot{e}_1$ for which $e_1 <' \ddot{e}_2$. (3) If $\ddot{e}_1 <' \ddot{e}_2$, then there are no $e_i^* \in \ddot{e}_i$ ($i = 1, 2$) for which $e_2^* < e_1^*$.*

Proof. (1): By Fact 3.23.

(2): Let $e_1 <' \ddot{e}_2$ as witnessed by e_2 (i.e., $e_1 < e_2$), and $e_1^* \in \ddot{e}_1$, $e_1 \neq e_1^*$. Assume for reductio $e_1^* <' \ddot{e}_2$, then there has to be some witness $e_2^* \in \ddot{e}_2$ for which $e_1^* < e_2^*$. Then by (1), we also have $e_1^* < e_2$, so that (by downward closure of histories) there is a history containing e_1 and e_1^*, contradicting Fact 3.13(1).

(3): Let $\ddot{e}_1 <' \ddot{e}_2$ as witnessed by e_1 and e_2 (i.e., $e_1 \in \ddot{e}_1$, $e_2 \in \ddot{e}_2$, and $e_1 < e_2$). Assume for reductio that there are $e_1^* \in \ddot{e}_1$ and $e_2^* \in \ddot{e}_2$ for which $e_2^* < e_1^*$. Then by (1), we have $e_1 < e_2^*$, and by transitivity, $e_1 < e_1^*$. Thereby e_1 and e_1^*, being different elements of $\ddot{e}_1$ (by irreflexivity of $<$), would have to belong to one history, contradicting Fact 3.13(1). □

Similarly to what we established about the properties of the transition structure of a BST_{92} structure, we can characterize the Λ-transform of a BST_{NF} structure. It transpires that, as announced, the Λ-transform leads us back to BST_{92}. As above, we split the proof into a number of steps.

Fact A.7. *Let $\langle W, < \rangle$ be a BST_{NF} structure. Then its Λ-transform, $\langle W', <' \rangle =_{\text{df}} \Lambda(\langle W, < \rangle)$, is (1) non-empty and (2) a strict partial ordering.*

Proof. (1) By construction, W' is non-empty (given that W was non-empty).

(2) Asymmetry follows from Fact A.6(3). For transitivity, let $\ddot{e}_1 <' \ddot{e}_2$ and $\ddot{e}_2 <' \ddot{e}_3$. Then by Fact A.6(2), there is a unique $e_2 \in \ddot{e}_2$ for which $e_2 <' \ddot{e}_3$, and a unique $e_1 \in \ddot{e}_1$ for which $e_1 <' \ddot{e}_2$. So by $e_2 \in \ddot{e}_2$ and by transitivity of $<$ we have $e_1 <' \ddot{e}_3$, which proves $\ddot{e}_1 <' \ddot{e}_3$. □

Before we can establish the history-relative suprema of upper bounded chains, we need to prove a lemma about the form of histories in W'. From here on, we need to work under the assumption that the BST_{NF} structures under consideration have no maxima, as otherwise the transformed structure may contain fewer histories.

Lemma A.2. *Let $\langle W, <\rangle$ be a BST_{NF} model without maxima, and let $\langle W', <'\rangle =_{df} \Lambda(\langle W, <\rangle)$. The histories (maximal directed sets) in $\langle W', <'\rangle$ are exactly the sets $\Lambda(h)$, for $h \in \mathrm{Hist}(W)$. That is, (1) for $h \in \mathrm{Hist}(W)$, the set $\Lambda(h)$ is maximal directed and (2) for any maximal directed set $h' \in \mathrm{Hist}(W')$ there is a unique history $h \in \mathrm{Hist}(W)$ such that $h' = \Lambda(h)$.*

Proof. (1) The set $\Lambda(h) \subseteq W'$ is directed by Fact A.5(3), so there is some maximal directed $h' \in \mathrm{Hist}(W')$ for which $\Lambda(h) \subseteq h'$. By Fact A.7, $\langle W', <'\rangle$ is a non-empty strict partial ordering. Thus, by Fact 2.1(9), h' cannot have a maximum. This allows us to define a function $f : h' \mapsto W$ that establishes the converse of Λ on h', in the following sense: (i) for any $\ddot{e} \in h'$, $\Lambda(f(\ddot{e})) = \ddot{e}$, (ii) for any $\ddot{e}_1, \ddot{e}_2 \in h'$, we have $\ddot{e}_1 <' \ddot{e}_2$ iff $f(\ddot{e}_1) < f(\ddot{e}_2)$, and (iii) for any $e \in h$, $f(\Lambda(e)) = e$. (Note that the primed histories and the primed ordering refer to $\Lambda(\langle W, <\rangle)$, not to the BST_{NF} structure.)

To define f, let $\ddot{e}_1 \in h'$. Let $\ddot{e}_2 \in h'$ such that $\ddot{e}_1 <' \ddot{e}_2$; such an element exists as h' has no maxima. By Fact A.6(2), there is a unique $v \in W$ for which $\ddot{e}_1 = \Lambda(v)$ and $v <' \ddot{e}_2$. That v is, moreover, independent of the chosen upper bound $\ddot{e}_2 \in h'$: let $v^* \in \ddot{e}_1$ be such that $v^* <' \ddot{e}_3$ for some $\ddot{e}_3 \in h'$ for which $\ddot{e}_1 <' \ddot{e}_3$. Then by the directedness of h', there is some common upper bound $\ddot{e}_4$ of $\ddot{e}_2$ and $\ddot{e}_3$, and again invoking Fact A.6(2), we have $v^* = v$. So we can set $f(\ddot{e}_1) = v$ as specified. Note that thereby, $f(\ddot{e}_1) \in \ddot{e}_1$. Constraint (i) holds by construction, as $\Lambda(f(\ddot{e})) = \Lambda(v) = \ddot{e}$. For (ii, "⇒"), let $\ddot{e}_1, \ddot{e}_2 \in h'$ satisfy $\ddot{e}_1 <' \ddot{e}_2$. Then $f(\ddot{e}_1) <' \ddot{e}_2$ by construction (as $\ddot{e}_2$ is an upper bound of $\ddot{e}_1$ in h'), and the claim follows by Fact A.6(1), noting that $f(\ddot{e}_2) \in \ddot{e}_2$. For (ii, "⇐"), let $\ddot{e}_1, \ddot{e}_2 \in h'$ be such that $f(\ddot{e}_1) < f(\ddot{e}_2)$. By (i) and by the definition of the ordering $<'$, this implies $\ddot{e}_1 <' \ddot{e}_2$. For (iii), let $e_1 \in h$. As h contains no maxima, there is some $e_2 \in h$ for which $e_1 < e_2$. Let $\ddot{e}_i = \Lambda(e_i)$ $(i = 1, 2)$, so that (by the definition of $<'$) we have $\ddot{e}_1 <' \ddot{e}_2$. By the definition of f, $f(\ddot{e}_1) = e^*$ for the unique member $e^* \in \ddot{e}_1$ for which $e^* <' \ddot{e}_2$. Given $e_1 < e_2$, we have $e^* = e_1$, i.e., $f(\Lambda(e_1)) = f(\ddot{e}_1) = e_1$.

Now for maximality of the directed set $\Lambda(h) \subseteq h'$, assume for reductio that $h' = \Lambda(h) \cup A'$ with $\Lambda(h) \cap A' = \emptyset$ and $A' \neq \emptyset$ (i.e., assume that $\Lambda(h)$ is not maximal directed). Let $A =_{df} \{f(\ddot{e}) \mid \ddot{e} \in A'\}$, so that $A \neq \emptyset$ and $A \cap h = \emptyset$. By property (ii) of f, $h \cup A$ is directed, violating the maximality of h.

(2) Let $h' \in \mathrm{Hist}(W')$, and define f as above. Let $E =_{df} \{f(\ddot{e}) \mid \ddot{e} \in h'\}$, so that by (i), $\Lambda(E) = h'$. By (ii), E is directed, so that there is some $h \in \mathrm{Hist}(W)$ with $E \subseteq h$. It follows that $h' = \Lambda(E) \subseteq \Lambda(h)$. By (1), we have that $\Lambda(h) = h''$ for some $h'' \in \mathrm{Hist}(W')$. So we have two histories $h', h'' \in \mathrm{Hist}(W')$ for which $h' \subseteq h''$, whence, by Fact 2.1(7), $h' = h''$. This means that we have found some $h \in \mathrm{Hist}(W)$ for which $\Lambda(h) = h'$. For uniqueness of h, let $h_1, h_2 \in \mathrm{Hist}(W)$ be such that $\Lambda(h_1) = \Lambda(h_2) = h'$. Then $h_1 = f(h')$ and $h_2 = f(h')$, establishing $h_1 = h_2$. □

We can now prove the two main Lemmas and the resulting Theorem for this translatability direction, as claimed in Chapter 3.

Lemma 3.5. *Let $\langle W,<\rangle$ be a BST_{NF} structure without maxima. Then its Λ-transform, $\langle W',<'\rangle =_{\text{df}} \Lambda(\langle W,<\rangle)$, is a common BST structure.*

Proof. Our task is to show that $\langle W',<'\rangle$ satisfies the postulates of Definition 2.10 of a common BST structure.

1. Non-emptiness: By Fact A.7(1).
2. Partial ordering: By Fact A.7(2)
3. $<'$ is dense.
   Let $\ddot{e}_1 <' \ddot{e}_2$, so by Fact A.6(2), there is a unique $e_1 \in \ddot{e}_1$ for which $e_1 <' \ddot{e}_2$. Let $e_2 \in \ddot{e}_2$; in particular, $e_1 < e_2$. By density of $<$, we have $e^* \in W$ such that $e_1 < e^* < e_2$. By the definition of the $<'$ ordering, this establishes $\ddot{e}_1 = \Lambda(e_1) <' \Lambda(e^*) <' \Lambda(e_2) = \ddot{e}_2$, which proves density of $<'$.
4. Lower bounded chains have infima in $<'$.
   Let $t' \subseteq W'$ be a lower bounded chain, and let $\ddot{b} \in W'$ be a lower bound for t'. The elements of t' are of the form $\ddot{e} = \Lambda(e)$ with $e \in W$. We distinguish two cases. (a) If t' has a least element (which covers the case that t' has only one element), then that least element is the infimum of t' with respect to $<'$, by definition. (b) If t' has no least element, pick some $\ddot{e} \in t'$, and let $t^{*\prime} =_{\text{df}} \{x \in t' \mid x <' \ddot{e}\}$. We have $\inf t^{*\prime} = \inf t'$ by the definition of the infimum. And by Fact A.6(2), for all $\ddot{e}_1 \in t^{*\prime}$ there are unique $e_1 \in W$ for which $e_1 \in \ddot{e}_1$ and $e_1 < \ddot{e}$, and there is a unique $b^* \in \ddot{b}$ for which $b^* < \ddot{e}$. So there is a unique set $t^* \subseteq W$ given by
   $$t^* = \{e_1 \in W \mid e_1 < \ddot{e} \wedge \ddot{e}_1 \in t^{*\prime}\},$$
   which is a chain since $t^{*\prime}$ is a chain; furthermore, t^* is lower bounded by $b^* \in W$. By the properties of BST_{NF}, t^* therefore has an infimum $a =_{\text{df}} \inf t^*$, $a \in W$. We claim that $\ddot{a}$ is the infimum of t' with respect to $<'$. As $a < t^*$, we have $\ddot{a} <' t'$ by the definition of $<'$. Now let $\ddot{c} \leqslant' t'$. Again by Fact A.6(2), there is a unique $c \in \ddot{c}$ for which $c < \ddot{e}$. By the fact that a is the infimum of t^*, we have $c \leqslant a$, which implies $\ddot{c} \leqslant' \ddot{a}$. So $\ddot{a}$ is indeed the greatest lower bound (i.e., the infimum), of t'.
5. Upper bounded chains have history-relative suprema in $<'$.
   Let t' be an upper bounded chain with $\ddot{b}$ an upper bound, and let $\ddot{b} \in h'$ for h' some history in $\langle W',<'\rangle$, so that $t' \subseteq h'$ as well. As h' has a unique pre-image h under Λ (by Lemma A.2), also $\ddot{b}$ and t' have unique pre-images $b \in h$ and $t \subseteq h$. So by the BST_{NF} axioms, t has an h-relative supremum $s \in h$. By Fact A.5(1), we have $t' \leqslant' \ddot{s}$, and for any $\ddot{a} \in h'$ for which $t' \leqslant' \ddot{a}$, we can consider the unique pre-image $a \in h \cap \ddot{a}$, for which $t \leqslant a$. By the fact that s is the h-relative supremum of t, we have $s \leqslant a$, which translates into $\ddot{s} \leqslant' \ddot{a}$; that is, $\ddot{s}$ is the least upper bound in h', and therefore the h'-relative supremum, of t'.
6. Weiner's postulate.
   Consider two histories $h'_1, h'_2 \in \text{Hist}(W')$, two chains $l', k' \subseteq h'_1 \cap h'_2$, and their history-relative suprema $\ddot{s}_i = sup_{h'_i} l'$ and $\ddot{c}_i = sup_{h'_i} k$ $(i = 1,2)$. Assume that $\ddot{s}_1 \leqslant' \ddot{c}_1$. We denote the unique pre-images of $h'_1, h'_2, l', k', \ddot{s}_1, \ddot{s}_2, \ddot{c}_1$, and $\ddot{c}_2$ under Λ by h_1, h_2, l, k, s_1, s_2, c_1, and c_2, respectively. By the uniqueness of pre-images

and properties of $<'$, we have $l,k \subseteq h_1 \cap h_2$, $s_i = sup_{h_i}(l)$, $c_i = sup_{h_i}(k)$ $(i = 1,2)$, and $s_1 \leqslant c_1$. Then by Weiner's postulate of BST_{NF}, $s_2 \leqslant c_2$. Since $\ddot{s}_2 = \Lambda(s_2)$ and $\ddot{c}_2 = \Lambda(c_2)$, we have $\ddot{s}_2 \leqslant' \ddot{c}_2$.

7. Historical connection. Pick h'_1, h'_2 histories in $\langle W', <' \rangle$, and consider their unique pre-images h_1, h_2 by Λ, which by Lemma A.2 are histories in the BST_{NF} structure $\langle W, < \rangle$. By historical connection for BST_{NF} structures, we have $h_1 \cap h_2 \neq \emptyset$. Hence $\Lambda(h_1) \cap \Lambda(h_2) \neq \emptyset$, i.e., $h'_1 \cap h'_2 \neq \emptyset$. □

Lemma 3.6. *The Λ-transform $\Lambda(\langle W, < \rangle)$ of a BST_{NF} structure without maxima $\langle W, < \rangle$ satisfies the BST_{92} prior choice principle.*

Proof. Let h'_1, h'_2 be histories in $\langle W', <' \rangle$, and let $t' \subseteq h'_1 \setminus h'_2$ be a lower bounded chain in h'_1 that contains no element of h'_2. We have to find a maximal element $c \in h'_1 \cap h'_2$ that lies below t', $c <' t'$, and for which $h'_1 \perp_c h'_2$. The histories h'_1, h'_2 have as unique pre-images the $\langle W, < \rangle$-histories h_1, h_2. As $t' \subseteq h'_1$, the unique pre-image $t \subseteq h_1$. Furthermore, $t \cap h_2 = \emptyset$, for an element $e \in t \cap h_2$ would give rise to $\ddot{e} \in t' \cap h'_2$, violating our assumption about t'. So $t \subseteq h_1 \setminus h_2$. From the BST_{NF} prior choice principle, we have a choice set $\ddot{s}$ and $s_1 \in h_1 \cap \ddot{s}$ for which $s_1 \leqslant t$, while there is some $s_2 \in \ddot{s} \cap h_2$. Let $c' =_{\text{df}} \Lambda(s_1) = \ddot{s}$; we claim that c' is the sought-for choice point. (a) By Lemma 3.2 we have $\ddot{s}_1 = \ddot{s}_2$, and as $s_i \in h_i$, we have $\ddot{s}_i \in h'_i$ $(i = 1,2)$, so that $c' = \ddot{s}_1 = \ddot{s}_2 \in h'_1 \cap h'_2$. (b) As c' lies at the intersection of h'_1 and h'_2, it cannot be that $\ddot{s}_1 \in t'$. This excludes $s_1 \in t$, so that in fact $s_1 < t$. This in turn implies $c' = \ddot{s}_1 <' t'$. (c) For the maximality of c' in $h'_1 \cap h'_2$, assume that there is $\ddot{a} \in h'_1 \cap h'_2$ for which $c' < \ddot{a}$. Then we have a unique pre-image $a_1 \in h_1 \cap h_2$ for which both $s_1 < a_1$ and $s_2 < a_1$, so that both s_1 and s_2 belong to history h_1. This contradicts Fact 3.13(1). So $c' = \ddot{s}$ is in fact maximal in $h'_1 \cap h'_2$. (d) By the definition of $\perp_{c'}$, we therefore have $h'_1 \perp_{c'} h'_2$. □

Theorem 3.3. *The Λ-transform $\Lambda(\langle W, < \rangle)$ of a BST_{NF} structure without maxima $\langle W, < \rangle$ is a BST_{92} structure.*

Proof. By Lemma 3.5 and Lemma 3.6. □

For the Λ transform we can prove, just like for the Υ transform, that a structure without maxima or minima is transformed into one that has no maxima or minima either.

Fact A.8. *Let $\langle W, < \rangle$ be a BST_{NF} structure without maxima and minima. Then its Λ-transform, $\langle W', <' \rangle$, has no maxima and minima either.*

Proof. Let $\ddot{e} \in W'$. There is some $e_1 \in W$ for which $\ddot{e} = \Lambda(e_1)$. As W has no maximal nor minimal elements, there are $e_2, e_3 \in W$ for which $e_2 < e_1 < e_3$. Then by the definition of the ordering, $\ddot{e}_2 <' \Lambda(e_1) = \ddot{e}$, establishing that $\ddot{e}$ cannot be a minimum, and $\Lambda(e_1) = \ddot{e} <' \ddot{e}_3$, establishing that $\ddot{e}$ cannot be a maximum. □

A.2.4 Going full circle

We have now established that there is a way to get from BST_{92} structures without minima to BST_{NF} structures and from BST_{NF} structures without maxima to BST_{92} structures. This leads to the question of where we land when we concatenate these transformations (restricted to structures without minima and maxima). We can show that, as one might hope, we end up where we started: the resulting structures are order-isomorphic to the ones we started with.

A.2.4.1 From BST_{92} to BST_{NF} to BST_{92}

We can prove the following Theorem, already presented in Chapter 3:

Theorem A.4. *The function $\Lambda \circ \Upsilon$ is an order isomorphism of BST_{92} structures without maximal or minimal elements: Let $\langle W_1, <_1 \rangle$ be a BST_{92} structure without maximal or minimal elements, let $\langle W_2, <_2 \rangle =_{\text{df}} \Upsilon(\langle W_1, <_1 \rangle)$, and let $\langle W_3, <_3 \rangle =_{\text{df}} \Lambda(\langle W_2, <_2 \rangle)$. Then there is an order isomorphism φ between $\langle W_1, <_1 \rangle$ and $\langle W_3, <_3 \rangle$, i.e., a bijection between W_1 and W_3 that preserves the ordering. Accordingly, $\langle W_3, <_3 \rangle$ has no minima and no maxima.*

Proof. We claim that we can use the mapping φ, defined for $e \in W_1$ to be

$$\varphi(e) =_{\text{df}} \{e \rightarrowtail H \mid H \in \Pi_e\}.$$

We have to show (1) that φ is indeed a mapping from W_1 to W_3, (2) that φ is injective, (3) that φ is surjective, and (4) that φ preserves the ordering.

(1) Mapping: We have to show that for any $e \in W_1$, $\varphi(e) = \{e \rightarrowtail H \mid H \in \Pi_e\} \in W_3$. Since $\langle W_1, <_1 \rangle$ has no minimal elements, the set W_2 is the full transition structure of $\langle W_1, <_1 \rangle$, so that for $e \in W_1$ and for any $H \in \Pi_e$, the transition $e \rightarrowtail H \in W_2$. Thus, for $e \in W_1$, the set $\varphi(e) \subseteq W_2$. The set W_3 contains, for any $\tau \in W_2$, the set $\Lambda(\tau) = \ddot{\tau} \in W_3$, and $\ddot{\tau} \subseteq W_2$ as well. Let now $e \in W_1$, and pick some $H^* \in \Pi_e$,which fixes some $\tau = e \rightarrowtail H^* \in W_2$. We claim that

$$\ddot{\tau} = \{e \rightarrowtail H \mid H \in \Pi_e\},$$

which establishes $\ddot{\tau} = \varphi(e)$, so that indeed, $\varphi(e) \in W_3$. The claim is an equality between subsets of W_2, so that we show inclusion both ways.

"$\subseteq$": Let $\tau' = e' \rightarrowtail H' \in \ddot{\tau}$; we have to show that $\tau' \in \{e \rightarrowtail H \mid H \in \Pi_e\}$. We have $\tau \in h$ and $\tau' \in h'$ for $h, h' \in \text{Hist}(W_2)$. By Lemma A.1 we know that these histories are of the form

$$h = \{e_1 \rightarrowtail \Pi_{e_1}\langle h_1 \rangle \mid e_1 \in h_1\}; \quad h' = \{e'_1 \rightarrowtail \Pi_{e'_1}\langle h'_1 \rangle \mid e'_1 \in h'_1\}$$

for some $h_1, h'_1 \in \text{Hist}(W_1)$. The set $\ddot{\tau}$ is defined as the intersection of all sets of history-relative suprema of any chain $l \subseteq W_2$ ending in, but not containing τ ($l \in \mathscr{C}_\tau$), so that for any $l \in \mathscr{C}_\tau$, we have $\sup_h l = \tau$ and $\sup_{h'} l = \tau'$, since $\tau' \in \ddot{\tau}$. As $\tau = e \rightarrowtail H^* \in h$, we have $e \in h_1$. We now claim that $e \in h'_1$ as well. Assume otherwise, so that $e \in h_1 \setminus h'_1$. By PCP_{92}, there is then some $c <_1 e$ for which $h_1 \perp_c h'_1$. Let $\tau_c =_{\text{df}} c \rightarrowtail \Pi_c\langle h_1 \rangle$, so that $\tau_c \in h$ and $\tau_c <_2 \tau$. There is thus some chain $l \in \mathscr{C}_\tau$ for which $\tau_c \in l$. As $\sup_{h'} l = \tau'$ (by $\tau' \in \ddot{\tau}$, see above), we have $l \subseteq h'$, which implies $\tau_c \in h'$ and $c \in h'_1$. Elements of h' are of the form $e'_1 \rightarrowtail \Pi_{e'_1}\langle h'_1 \rangle$; we therefore must have $\Pi_c\langle h_1 \rangle = \Pi_c\langle h'_1 \rangle$, contradicting $h_1 \perp_c h'_1$. So indeed $e \in h_1 \cap h'_1$.

Now take some $l \in \mathscr{C}_\tau$; we have $l \subseteq W_2$ and indeed $l \subseteq h \cap h'$. Let l_1 be the set of initials of the elements of l, i.e., $l_1 \subseteq W_1$ and $l = \{e_1 \rightarrowtail \Pi_{e_1}\langle h_1 \rangle \mid e_1 \in l_1\}$. Note that $\sup_{h_1} l_1 = e = \sup_{h'_1} l_1$ from $\sup_h l = \tau$ and $e \in h_1 \cap h'_1$, and as $l \subseteq h \cap h'$, we have $\Pi_{e_1}\langle h_1 \rangle = \Pi_{e_1}\langle h'_1 \rangle$ for all $e_1 \in l_1$. We now claim that $\sup_{h'} l = \tau'' =_{\text{df}} e \rightarrowtail \Pi_e\langle h'_1 \rangle$. We have $\tau'' \in h'$ because

$e \in h'_1$, and $l <_2 \tau''$ because $l_1 <_1 e$, so τ'' is an upper bound of l in h'. Let now $\tau^* = e^* \rightarrowtail \Pi_{e^*}\langle h'_1\rangle \in h'$ be some upper bound of l in h'. Then e^* is an upper bound of l_1 in h'_1, and thus $e \leqslant_1 e^*$ as $\sup_{h'_1} l_1 = e$, so that $\tau'' \leqslant_2 \tau^*$, proving that τ'' is the h'-relative supremum of l. So we have shown that $\tau'' = e \rightarrowtail \Pi_e\langle h'_1\rangle = \sup_{h'} l = \tau'$. So indeed, $\tau' \in \{e \rightarrowtail H \mid H \in \Pi_e\}$.

"$\supseteq$": Given $\tau = e \rightarrowtail H^*$, consider an arbitrary $\tau' \in \{e \rightarrowtail H \mid H \in \Pi_e\}$, i.e., $\tau' = e \rightarrowtail H$ for the e in question and for some $H \in \Pi_e$. We have to show that $\tau' \in \ddot{\tau}$. We have $\tau \in h$ and $\tau' \in h'$ for some $h, h' \in \mathrm{Hist}(W_2)$, which are again of the form

$$h = \{e_1 \rightarrowtail \Pi_{e_1}\langle h_1\rangle \mid e_1 \in h_1\}; \quad h' = \{e'_1 \rightarrowtail \Pi_{e'_1}\langle h'_1\rangle \mid e'_1 \in h'_1\}$$

for some $h_1, h'_1 \in \mathrm{Hist}(W_1)$, so that $\tau' = e \rightarrowtail \Pi_e\langle h'_1\rangle$.

Let $l \in \mathscr{C}_\tau$; we have $\sup_h l = \tau = e \rightarrowtail H^*$ by assumption. We now claim that $\sup_{h'} l = \tau'$, which establishes $\tau' \in \ddot{\tau}$. To prove that τ' is the h'-relative supremum of l, as above, let l_1 be the set of initials of the elements l, so that $l_1 \subseteq W_1$ and $l = \{e_1 \rightarrowtail \Pi_{e_1}\langle h_1\rangle \mid e_1 \in l_1\}$. Again as above, $l_1 <_1 e$, and thus τ' is an upper bound of l in h'. Let now $\tau^* = e^* \rightarrowtail \Pi_{e^*}\langle h'_1\rangle \in h'$ be some upper bound of l in h'. Then $e^* \in h'_1$ is an upper bound of l_1 in h'_1, and thus $e \leqslant_1 e^*$ as $\sup_{h'_1} l_1 = e$ (note that $e \in h'_1$ as $\tau' \in h'$). Therefore $\tau' \leqslant_2 \tau^*$, proving that τ' is the h'-relative supremum of l. As l was an arbitrary chain from $\mathscr{C}_\tau$, we have indeed $\tau' \in \ddot{\tau}$.

(2) Injectivity: Let $e, e' \in W_1$ with $e \neq e'$. Then $\varphi(e) \neq \varphi(e')$. This is clear as the sets $\varphi(e)$ and $\varphi(e')$ have different members.

(3) Surjectivity: Let $a \in W_3$. We have to find some $e \in W_1$ for which $\varphi(e) = a$. As $a \in W_3$, we have $a = \ddot{\tau}$ for some $\tau = e \rightarrowtail H \in W_2$, where $e \in W_1$ and $H \in \Pi_e$. Above under (1) we have established that for $\tau = e \rightarrowtail H \in W_2$, we have $\ddot{\tau} = \{e \rightarrowtail H \mid H \in \Pi_e\}$, i.e., $a = \ddot{\tau} = \varphi(e)$.

(4) Order preservation: We have to show that for $e_1, e_2 \in W_1$, $e_1 <_1 e_2$ iff $\varphi(e_1) <_3 \varphi(e_2)$. (The claim about equality follows from the fact that φ is a bijection.) We know from the definition of φ that $\varphi(e_i) = \{e_i \rightarrowtail H \mid H \in \Pi_{e_i}\}$ $(i = 1, 2)$.

"$\Rightarrow$": Let $e_1, e_2 \in W_1$ with $e_1 <_1 e_2$, and let $h_2 \in H_{e_2}$. Let $\tau_1 =_{\mathrm{df}} e_1 \rightarrowtail \Pi_{e_1}\langle h_2\rangle$ and $\tau_2 =_{\mathrm{df}} e_2 \rightarrowtail H$ for some $H \in \Pi_{e_2}$; we have $\tau_1, \tau_2 \in W_2$ and $\varphi(e_i) = \ddot{\tau}_i$ $(i = 1, 2)$. By the definition of the transition ordering $<_2$, we have $\tau_1 <_2 \tau_2$, and by the definition of $<_3$ in terms of instances, we thus have $\ddot{\tau}_1 <_3 \ddot{\tau}_2$, i.e., $\varphi(e_1) <_3 \varphi(e_2)$.

"$\Leftarrow$": Let $\varphi(e_1) <_3 \varphi(e_2)$, i.e., there are some $\tau_1 \in \varphi(e_1)$, $\tau_2 \in \varphi(e_2)$ for which $\tau_1 <_2 \tau_2$. These transitions have the form $\tau_i = e_i \rightarrowtail H_i$ for some $e_i \in W_1$ and $H_i \in \Pi_{e_i}$ $(i = 1, 2)$. Thus, in particular, from $\tau_1 <_2 \tau_2$ we have that $e_1 <_1 e_2$. □

A.2.4.2 From $\mathrm{BST_{NF}}$ to $\mathrm{BST_{92}}$ to $\mathrm{BST_{NF}}$

Before we can tackle the main Theorem below (which has also been presented in Chapter 3), we need to establish an additional fact.

Fact A.9. *Let $\langle W_1, <_1\rangle$ be a BST_{NF} structure without maxima and $\langle W_2, <_2\rangle =_{\mathrm{df}} \Lambda(\langle W_1, <_1\rangle)$ the corresponding BST_{92} structure. Then for any $h_1, h_2 \in \mathrm{Hist}(W_1)$, we have $h_1 \perp^1_{\ddot{e}} h_2$ iff $\Lambda(h_1) \perp^2_{\ddot{e}} \Lambda(h_2)$, where $\perp^i_{\ddot{e}}$ is the relation of splitting for histories in W_i.*

Proof. "$\Rightarrow$" Let $h_1, h_2 \in \mathrm{Hist}(W_1)$, and let $h_1 \perp^1_{\ddot{e}} h_2$. Then there are $e_1, e_2 \in \ddot{e}$ such that $e_1 \neq e_2$ and $h_i \cap \ddot{e} = \{e_i\}$ $(i = 1, 2)$. Then $\Lambda(e_1) = \Lambda(e_2) = \ddot{e}$, so $\ddot{e} \in \Lambda(h_1) \cap \Lambda(h_2)$. Moreover, $\ddot{e}$ is maximal in $\Lambda(h_1) \cap \Lambda(h_2)$, which establishes $\Lambda(h_1) \perp^2_{\ddot{e}} \Lambda(h_2)$. To prove this, assume for reductio that there is some $\ddot{e}' >_2 \ddot{e}$ in the intersection of $\Lambda(h_1)$ and $\Lambda(h_2)$. This means that there are some $e'_1, e'_2 \in \ddot{e}'$ with $e'_1 \in h_1$, $e'_2 \in h_2$, and $\Lambda(e'_1) = \Lambda(e'_2) = \ddot{e}'$. The ordering $\ddot{e} <_2 \ddot{e}'$ implies that for some $e^* \in \ddot{e}$, $e^* <_2 \ddot{e}'$, which further implies (by

Fact A.6(1)) that $e^* <_1 e'_1$ and $e^* <_1 e'_2$. But then $e^* \in h_1 \cap h_2$, and by Fact A.6(2), it must be that $e^* = e_1 = e_2$, which contradicts $h_1 \perp^1_{\ddot{e}} h_2$.

"$\Leftarrow$" Let $\Lambda(h_1) \perp^2_{\ddot{e}} \Lambda(h_2)$, which implies that $\ddot{e} \in \Lambda(h_1) \cap \Lambda(h_2)$. Note that there are e_i such that $e_i \in h_i$ and $\Lambda(e_i) = \ddot{e}$ $(i = 1,2)$. Therefore, $h_i \cap \ddot{e} \neq \emptyset$, so that h_1 and h_2 fulfill the precondition for either $h_1 \equiv^1_{\ddot{e}} h_2$ or $h_1 \perp^1_{\ddot{e}} h_2$ (see Def. 3.13). For reductio, assume the former, which means that $h_1 \cap \ddot{e} = h_2 \cap \ddot{e}$, i.e., $e_1 = e_2$. As there are no maxima in the intersection of histories in BST_{NF} (Fact 3.17), there is some $e^* \in h_1 \cap h_2$ for which $e_1 <_1 e^*$. Now for $\ddot{e}^* =_{\text{df}} \Lambda(e^*)$ we have $\ddot{e}^* \in \Lambda(h_1) \cap \Lambda(h_2)$, and $\ddot{e} <_2 \ddot{e}^*$. This, however, contradicts the maximality of $\ddot{e}$ implied by $\Lambda(h_1) \perp^2_{\ddot{e}} \Lambda(h_2)$. So in fact, we have $h_1 \perp^1_{\ddot{e}} h_2$. Note that by contraposing the above Fact (and making a simple observation) we have that for any $h_1, h_2 \in \text{Hist}(W_1)$, $h_1 \equiv^1_{\ddot{e}} h_2$ iff $\Lambda(h_1) \equiv^2_{\ddot{e}} \Lambda(h_2)$. □

Theorem 3.5. *The function $\Upsilon \circ \Lambda$ is an order isomorphism of BST_{NF} structures without maximal or minimal elements: Let $\langle W_1, <_1 \rangle$ be a BST_{NF} structure without maximal or minimal elements, let $\langle W_2, <_2 \rangle =_{\text{df}} \Lambda(\langle W_1, <_1 \rangle)$, and let $\langle W_3, <_3 \rangle =_{\text{df}} \Upsilon(\langle W_2, <_2 \rangle)$. Then there is an order isomorphism φ between $\langle W_1, <_1 \rangle$ and $\langle W_3, <_3 \rangle$, i.e., a bijection between W_1 and W_3 that preserves the ordering. Accordingly $\langle W_3, <_3 \rangle$ has no minima and no maxima.*

Proof. We claim that we can use the mapping φ, defined for $e \in W_1$ to be

$$\varphi(e) =_{\text{df}} \ddot{e} \rightarrowtail \Pi_{\ddot{e}} \langle \Lambda(h) \rangle \text{ for arbitrary } h \in H_e \subseteq \text{Hist}(W_1).$$

First we show that $\varphi(e)$ is well-defined. Thus, let $h, h' \in H_e$; we need to show that $\Pi_{\ddot{e}} \langle \Lambda(h) \rangle = \Pi_{\ddot{e}} \langle \Lambda(h') \rangle$. By Lemma A.2 (1), $\Lambda(h), \Lambda(h') \in \text{Hist}(W_2)$. Also, by Fact 3.13, $h \equiv_{\ddot{e}} h'$, and so by Fact A.9, $\Pi_{\ddot{e}} \langle \Lambda(h) \rangle = \Pi_{\ddot{e}} \langle \Lambda(h') \rangle$.

We now have to show (1) that φ is indeed a mapping from W_1 to W_3, (2) that φ is injective, (3) that φ is surjective, and (4) that φ preserves the ordering.

(1) Mapping: We have to show that for any $e \in W_1$, $\varphi(e) = \ddot{e} \rightarrowtail \Pi_{\ddot{e}} \langle \Lambda(h) \rangle \in W_3$, where $h \in H_e \subseteq \text{Hist}(W_1)$. The set W_3 is defined via W_2, and the set $W_2 = \Lambda[W_1]$, which means that for every $e \in W_1$, $\ddot{e} \in W_2$. By Lemma A.2, $\Lambda(h)$ is a history in $\langle W_2, <_2 \rangle$ for any $h \in \text{Hist}(W_1)$. Since for any $h \in H_e$, $\ddot{e} = \Lambda(e) \in \Lambda(h)$, we get that $\Pi_{\ddot{e}} \langle \Lambda(h) \rangle$ is an basic outcome of $\ddot{e}$, so indeed $\ddot{e} \rightarrowtail \Pi_{\ddot{e}} \langle \Lambda(h) \rangle \in W_3$.

(2) Injectivity: Let $e, e' \in W_1$ and $e \neq e'$. If $\ddot{e} \neq \ddot{e}'$, then obviously $\varphi(e) \neq \varphi(e')$, as these two transitions then have different initials. If $\ddot{e} = \ddot{e}'$ but $e \neq e'$, then e and e' are incompatible elements of the choice set $\ddot{e}$, and moreover, for any $h, h' \in \text{Hist}(W_1)$, if $e \in h, e' \in h'$, then $h \perp_{\ddot{e}} h'$, and hence by Fact A.9, $\Lambda(h) \perp_{\ddot{e}} \Lambda(h')$. Accordingly, $\Pi_{\ddot{e}} \langle \Lambda(h) \rangle \neq \Pi_{\ddot{e}} \langle \Lambda(h') \rangle$, and hence $\varphi(e) \neq \varphi(e')$.

(3) Surjectivity: Let $a \in W_3$. We have to find some $e \in W_1$ for which $\varphi(e) = a$. As $a \in W_3$, we have $a = \ddot{e}' \rightarrowtail H$, where $\ddot{e}' \in W_2$ and $H \in \Pi_{\ddot{e}'}$. Since $\langle W_2, <_2 \rangle$ is the result of Λ-transform applied to $\langle W_1, <_1 \rangle$, there is (possibly more than one) $e^* \in W_1$ for which $\Lambda(e^*) = \ddot{e}'$. We need to find which of these is the sought-after e. Clearly, there is some $h^* \in \text{Hist}(W_2)$ for which $H = \Pi_{\ddot{e}'} \langle h^* \rangle$. By Lemma A.2(2), there is a unique $h \in \text{Hist}(W_1)$ such that $h^* = \Lambda(h)$, and hence $H = \Pi_{\ddot{e}'} \langle \Lambda(h) \rangle$. For the sought-after e we thus take the unique $e \in \ddot{e}' \cap h$; clearly $\ddot{e}' = \ddot{e}$. It follows that $\varphi(e) = \ddot{e} \rightarrowtail H$, where $H = \Pi_{\ddot{e}} \langle \Lambda(h) \rangle$.

(4) Order preservation: We have to show that for $e_1, e_2 \in W_1$, $e_1 <_1 e_2$ iff $\varphi(e_1) <_3 \varphi(e_2)$. (The claim about equality follows from the fact that φ is a bijection.)

"$\Rightarrow$": Let $e_1, e_2 \in W_1$ with $e_1 <_1 e_2$. We show that $\varphi(e_1) <_3 \varphi(e_2)$. Since for $\varphi(e_1)$ we may pick an arbitrary member of H_{e_1}, we pick $h_2 \in H_{e_2} \subseteq H_{e_1}$, so that $e_1, e_2 \in h_2$. We get, as required, $\ddot{e}_1 <_2 \ddot{e}_2$ and hence, as the basic outcomes $\varphi(e_i)$ are defined by the same history $\Lambda(h_2)$, we get $H_{\ddot{e}_2} \subseteq \Pi_{\ddot{e}_1}\langle\Lambda(h_2)\rangle$. Hence, $\ddot{e}_1 \rightarrowtail \Pi_{\ddot{e}_1}\langle\Lambda(h_2)\rangle <_3 \ddot{e}_2 \rightarrowtail \Pi_{\ddot{e}_2}\langle\Lambda(h_2)\rangle$.

"$\Leftarrow$": Let $\varphi(e_1) <_3 \varphi(e_2)$, i.e., $(\ddot{e}_1 \rightarrowtail \Pi_{\ddot{e}_1}\langle\Lambda(h_1)\rangle <_3 (\ddot{e}_2 \rightarrowtail \Pi_{\ddot{e}_2}\langle\Lambda(h_2)\rangle)$, for $h_i \in H_{e_i}, e_i \in \ddot{e}_i$. Hence for some $e_1' \in \ddot{e}_1$: (i) $e_1' <_2 \ddot{e}_2$, and hence $H_{\ddot{e}_2} \subseteq H_{e_1'}$, so $h_2 \in H_{e_1'}$ (because $H_{e_2} \subseteq H_{\ddot{e}_2}$). Since $h_2 \in H_{e_2} \subseteq H_{e_1}$, and it is impossible that $\{e_1, e_1'\} \subseteq h_2$ (by Fact 3.13(1)), it must be that $e_1 = e_1'$ and hence $e_1 < e_2$ (by (i)). □

A.2.5 The translatability of some notions pertaining to MFB

We assume in this section that there is a BST_{NF} structure $\mathscr{W} = \langle W, < \rangle$ with no maximal elements, and we consider its Λ-transform, $\mathscr{W}' = \langle W', <' \rangle =_{df} \Lambda(\mathscr{W})$, which is a BST_{92} structure. We use relational symbols with primes for relations on $\mathscr{W}'$, and relational symbols without primes for relations on $\mathscr{W}$.

We define what we will claim to be the transform of basic transitions in $\mathscr{W}$ to basic transitions in $\mathscr{W}'$. Note that for set-theoretical reasons, since $\Lambda(\ddot{c}) = \{\ddot{c}\}$ rather than $\ddot{c}$, we cannot identify the transform of basic transitions with our standard transform Λ. For this new transform we use $\tilde{\Lambda}$. We write transitions $X \rightarrowtail Y$ as pairs $\langle X, Y\rangle$ for clarity here.

Definition A.1. Let $\tau = \langle \ddot{c}, H_{c'}\rangle$, with $c' \in \ddot{c}$, be a basic transition in $\mathscr{W}$. Then $\tilde{\Lambda}(\tau) =_{df} \langle \ddot{c}, \Lambda(H_{c'})\rangle = \langle \ddot{c}, \{\Lambda(h) \mid c' \in h\}\rangle$. We extend this notation to sets of basic transitions, so we write $\tilde{\Lambda}(T) = \{\tilde{\Lambda}(\tau) \mid \tau \in T\}$.

Observe that in the deterministic case $\ddot{c}$ is a singleton, $\ddot{c} = \{c\}$, so $\tau = \langle\{c\}, H_c\rangle$, and its transform is $\tilde{\Lambda}(\tau) = \langle\{c\}, \Lambda(H_c)\rangle$. We next prove the claim announced above:

Fact A.10. *Let $\tau = \langle \ddot{c}, H_{c'}\rangle$ with $c' \in \ddot{c}$ be a basic transition in $\mathscr{W}$. Then $\tilde{\Lambda}(\tau)$ is a basic transition in $\mathscr{W}'$.*

Proof. We note that for every $h \in H_{c'}$ we have $\ddot{c} \in \Lambda(h)$. Next, we observe that no histories in $\Lambda(H_{c'})$ split at $\ddot{c}$ in the sense of $\perp'_{\ddot{c}}$. This follows from Fact A.9, since no histories in $H_{c'}$ split at $\ddot{c}$ in the sense of $\perp_{\ddot{c}}$. □

Our next lemma says that MFB-related notions translate between $\mathscr{W}$ and $\Lambda(\mathscr{W})$. To recall, while each: consistency, downward closure of a set of transitions, explanatory funny business, and combinatorial funny business is defined exactly the same in BST_{92} and BST_{NF}, combinatorial consistency is defined somewhat differently, by Def. 5.5 for BST_{92} and Def. 5.11 for BST_{NF}.

Lemma A.3. *Let $\mathscr{W} = \langle W, < \rangle$ be a BST_{NF} structure with no maximal elements and let $\mathscr{W}' = \langle W', <' \rangle =_{df} \Lambda(\mathscr{W})$ be the corresponding BST_{92} structure. Let $T \subseteq \mathrm{TR}(W)$ be a set of basic transitions, and let $T' =_{df} \{\tilde{\Lambda}(\tau) \mid \tau \in T\} \subseteq \mathrm{TR}(W')$. Then:*

1. *T is consistent iff T' is consistent.*
2. *For $\tau_1, \tau_2 \in \mathrm{TR}(W)$, $\tau_1 \prec \tau_2$ iff $\tilde{\Lambda}(\tau_1) \prec' \tilde{\Lambda}(\tau_2)$. Hence τ belongs to the downward extension of T iff $\tilde{\Lambda}(\tau)$ belongs to the downward extension of T'.*
3. *T is combinatorially consistent in the sense of Def. 5.11 iff T' is combinatorially consistent in the sense of Def. 5.5.*

4. *T is a case of combinatorial funny business (CFB) iff T' is a case of CFB (see Def. 5.6).*
5. *T is a case of explanatory funny business (EFB) iff T' is a case of EFB (see Def. 5.8).*

Proof. (1) By the form of histories in $\mathscr{W}'$, h witnesses the consistency of $T \subseteq \text{TR}(W)$ iff $\Lambda(h)$ witnesses the consistency of $T' \subseteq \text{TR}(W')$, from which the claim follows.

(2) Let $\tau_i = \langle \ddot{e}_i, H_{e'_i}\rangle$ for $e'_i \in \ddot{e}_i$ and $i = 1,2$. Since the initial of τ_i is the same as the initial of $\tilde{\Lambda}(\tau_i)$, we need only look at their outcomes. However, $H_{e'_1} \subseteq H_{e'_2}$ is equivalent to $\Lambda(H_{e'_1}) \subseteq \Lambda(H_{e'_2})$, so the claim follows.

(3) We need to check whether the following equivalences hold: a pair $\tau_1, \tau_2 \in \text{TR}(W)$ satisfies a clause of Def. 5.11 iff the pair $\tilde{\Lambda}(\tau_1), \tilde{\Lambda}(\tau_2) \in \text{TR}(W')$ satisfies the corresponding clause of Def. 5.5. To begin with clause (1), since τ_i and $\tilde{\Lambda}(\tau_i)$ share the same initial, the absence of blatant inconsistency in the former pair means the absence of blatant inconsistency in the latter pair. Turning to clause (2), its antecedent in Def. 5.11 is equivalent to its antecedent in Def. 5.5, since the $\tilde{\Lambda}$ transform leaves the initials intact. The consequent of clause (2) in Def. 5.11 is $H_{\ddot{e}_2} \subseteq H_{e'_1}$, which holds iff e'_1 is the unique member of $\ddot{e}_1$ for which all members e'_2 of $\ddot{e}_2$ satisfy $e'_1 < e'_2$ (see Fact 4.9). Furthermore, by Theorem 3.1 we have, for all $e'_2, e''_2 \in \ddot{e}_2$, that $e'_1 < e'_2$ iff $e'_1 < e''_2$. So $H_{\ddot{e}_2} \subseteq H_{e'_1}$ iff $H_{e'_2} \subseteq H_{e'_1}$, where $H_{e'_2}$ is the given outcome of τ_2. Clause (2) of Def. 5.5 follows as $H_{e'_2} \subseteq H_{e'_1}$ iff $\Lambda(H_{e'_2}) \subseteq \Lambda(H_{e'_1})$. Thus, a pair $\tau_1, \tau_2 \in \text{TR}(W)$ satisfies clause (2) (or (3), which is proved in exactly the same way) of Def. 5.11 iff the pair $\tilde{\Lambda}(\tau_1), \tilde{\Lambda}(\tau_2) \in \text{TR}(W')$ satisfies the corresponding clause of Def. 5.5. Turning to condition (4), it is satisfied because of the following equivalence: for any $e'_1 \in \ddot{e}_1$ and $e'_2 \in \ddot{e}_2$, e'_1 and e'_2 are incomparable in the sense of $<$ and $e'_1, e'_2 \in h$ for some history h iff $\ddot{e}_1$ and $\ddot{e}_2$ are incomparable in the sense of $<'$ and $\ddot{e}_1, \ddot{e}_2 \in \Lambda(h)$.

(4) Let T be a case of CFB (i.e., it is combinatorially consistent but inconsistent). Then by (1) and (3) above, T' is combinatorially consistent as well, but inconsistent (i.e., a case of CFB). The opposite direction follows analogously.

(5) To recall, EFB means inconsistency plus no downward extension being blatantly inconsistent. From (2) we have that T^* is a downward extension of T iff $\tilde{\Lambda}(T^*)$ is a downward extension of T'. By item (3) of this Lemma we have that there is a blatantly inconsistent pair in T iff there is a blatantly inconsistent pair in T'. Together with (1) this implies that T is a case of EFB iff T' is a case of EFB. □

A.3 Proof of Theorem 5.1

In Chapter 5, we defined two different notions of modal funny business, namely combinatorial funny business (Def. 5.6) and explanatory funny business (Def. 5.8). We discussed their interrelation in Chapter 5.2.4, announcing the main Theorem 5.1 that states that the two notions are equivalent at the level of BST_{92} structures. Here we present a proof of that theorem. First, we repeat the definitions for convenience's sake.

Definition 5.6 (Combinatorial funny business)**.** A set of basic transitions T constitutes a case of *combinatorial funny business* (CFB) iff T is combinatorially consistent (Def. 5.5), but inconsistent ($H(T) = \emptyset$).

Definition 5.8 (Explanatory funny business)**.** A set T of transitions is a case of *explanatory funny business* (EFB) iff (1) T is inconsistent ($H(T) = \emptyset$) and (2) there is no downward extension T^* of T that is blatantly inconsistent.

Theorem 5.1 (There is combinatorial funny business iff there is explanatory funny business)**.** *Let $\langle W, < \rangle$ be a BST$_{92}$ structure. For its set of basic indeterministic transitions,* $\mathrm{TR}(W)$*, the following holds: There is a subset $T_1 \subseteq \mathrm{TR}(W)$ exhibiting combinatorial funny business iff there is a subset $T_2 \subseteq \mathrm{TR}(W)$ exhibiting explanatory funny business.*

Proof. "$\Rightarrow$": This has been established via Lemma 5.3.

"$\Leftarrow$": Assume that there is no CFB in the given BST$_{92}$ structure. We will show that the assumption that there is EFB in that structure leads to a contradiction. Thus, assume for reductio that a set of transitions T in the given structure witnesses EFB (i.e., it is inconsistent), but no downward extension of T is blatantly inconsistent. Since there is no CFB, T must be combinatorially inconsistent. Let T^* be the maximal downward extension of T, which contains all transitions τ^* for which for some $\tau \in T$, we have $\tau^* \preccurlyeq \tau$. By Lemma 5.2, T^* is also inconsistent and combinatorially inconsistent. Combinatorial inconsistency means that there are transitions $\tau_1 = e_1 \rightarrowtail H_1$ and $\tau_2 = e_2 \rightarrowtail H_2$ in T^* that fail at least one of the four clauses of Definition 5.5. By our assumption that T witnesses EFB, T^* is not blatantly inconsistent. The fact that T^* is not blatantly inconsistent shows that the first three clauses of Definition 5.5 cannot be what accounts for the combinatorial inconsistency: If $e_1 = e_2$, then by no blatant inconsistency, $H_1 = H_2$ (i.e., $\tau_1 = \tau_2$). And if $e_1 < e_2$, then $\tau =_{\mathrm{df}} e_1 \rightarrowtail \Pi_{e_1}(e_2) \in T^*$ by the fact that T^* is maximally downward extended (note that $\tau \prec \tau_2$), and as τ_1 also has the initial e_1 and T^* is not blatantly inconsistent, we have $\tau = \tau_1$, so that $\tau_1 \prec \tau_2$. For $e_2 < e_1$ we argue in the same way. Thus, the combinatorial inconsistency of T^* must be due to the existence of some $\tau_1 = e_1 \rightarrowtail H_1$ and $\tau_2 = e_2 \rightarrowtail H_2$ in T^* for which e_1 and e_2 are not order related and do not share any history (are not SLR)—all other ways for a set of transitions to witness combinatorial inconsistency are excluded.

We let E^* be the set of initials of transitions from T^*. By no blatant inconsistency, for any $e \in E^*$ there is exactly one transition $\tau = e \rightarrowtail H \in T^*$. We will denote that transition by τ_e. Let now $e_1, e_2 \in E$ be two initials of transitions from T^* that witness its combinatorial inconsistency; that is, e_1, e_2 are incomparable and not SLR, so there is no history $h \supseteq \{e_1, e_2\}$. We will now find a set of transitions $T_C \subseteq T^*$ that is combinatorially consistent but inconsistent, violating our initial assumption of no CFB (by Lemma 5.2). By this we will have established the right to left direction of our theorem. Note that, to establish the combinatorial consistency of a set of transitions $T_C \subseteq T^*$, it suffices to show that all initials of transitions from T_C are pairwise consistent (they share a history), as having inconsistent initials of transitions was the only way for T^*, and therefore for any of its subsets to be combinatorially inconsistent.

Let now C_i be the set of choice points in the past of e_i splitting off some e_i-history from some non-e_i-history ($i = 1, 2$).[4] That is, we define

$$C_i = \{c < e_i \mid \exists h_i \in H_{e_i} \exists h \notin H_{e_i} [h_i \perp_c h]\}.$$

We have $C_1 \cup C_2 \subseteq E^*$ as T^* is downward maximal, and we know that C_1 and C_2 are each consistent, and thus in particular, if $e, e' \in C_i$, then there exists h such that $e, e' \in h$: since $C_i < e_i$, for any h_i containing e_i we have $C_i \subseteq h_i$ ($i = 1, 2$). We set (again, for $i = 1, 2$)

[4] In previous papers on MFB, the term "past causal loci" was used for the members of C_i, with the notation $C_i = pcl(e_i)$.

$$T_i =_{\rm df} \{\tau_c \mid c \in C_i\}, \quad \tau_c = c \rightarrowtail \Pi_c(e_i),$$

where the form of τ_c (which, to recall, was unique in T^* for any $e \in E^*$) follows from the fact that $e_i \in E^*$ and T^* is downward maximal.

We will now define a maximal pairwise consistent subset C of $C_1 \cup C_2$ (which might be equal to $C_1 \cup C_2$), as follows. Let

$$A =_{\rm df} \{c \in C_2 \mid \forall c_1 \in C_1 \exists h \in \text{Hist}\,[c \in h \wedge c_1 \in h]\}.$$

We claim that $C =_{\rm df} C_1 \cup A$ is pairwise consistent, and maximally so as a subset of $C_1 \cup C_2$. For pairwise consistency, let $e, e' \in C$. There are three cases: if $e, e' \in C_1$, the claim follows by the consistency of C_1; similarly for $e, e' \in A \subseteq C_2$ by the consistency of C_2, and for $e \in C_1, e' \in A$, the claim follows from the definition of A. For maximality, let $e \in (C_1 \cup C_2) \setminus C$, i.e., $e \in C_2 \setminus A$. By the definition of A, there is then some $c_1 \in C_1$ for which there is no $h \in$ Hist containing both e and c_1, i.e., $C \cup \{e\}$ is not pairwise consistent.

We now let $T_C =_{\rm df} \{\tau_c \mid c \in C\}$, which implies $T_C \subseteq T_1 \cup T_2$. As C is pairwise consistent and $T_C \subseteq T^*$, the set T_C is combinatorially consistent. We claim that T_C is inconsistent. Once we have established this, then our work here is done. So assume for reductio that T_C is consistent, so that there is some $h \in H(T_C)$. We can write $C = C_1' \cup C_2'$ with $C_i' \subseteq C_i$, via $C_i' = C_i \cap C$ $(i = 1, 2)$. The fact that $h \in H(T_C)$ then implies that

$$h \in \bigcap_{c \in C_1'} \Pi_c(e_1) \cap \bigcap_{c \in C_2'} \Pi_c(e_2),$$

by the form of T_1 and T_2 noted above. Note that $C \subseteq h$, as C constitutes the set of initials for the transitions from T_C. We now show that $e_1 \in h$ and $e_2 \in h$, which is the sought-for contradiction, as no history can contain both e_1 and e_2. To establish $e_i \in h$ $(i = 1, 2)$, assume that $e_i \notin h$, so that $e_i \in h_i \setminus h$ for some $h_i \in H_{e_i}$. Now by PCP$_{92}$, there is some $c^* < e_i$ for which $h_i \perp_{c^*} h$. By the definition of C_i, we have $c^* \in C_i$. As $C \subseteq h$ and $c^* \in h$, we have that $C \cup \{c^*\}$ is consistent (as witnessed by h), and thereby also pairwise consistent. As C was maximally pairwise consistent, we thus must have $c^* \in C$, implying $c^* \in C_i'$. The fact that $h_i \perp_{c^*} h$ implies that $h \notin \Pi_{c^*}(h_i) = \Pi_{c^*}(e_i)$. But then $h \notin H(T_C)$ after all, contradicting our assumption.

So, bringing all of these disparate strands together, we have shown that T_C, a set of transitions in our BST$_{92}$ structure, exhibits CFB, contrary to our initial assumption. So the set of transitions T cannot witness EFB after all, and we have shown that if there is no CFB in a BST$_{92}$ structure, there is also no EFB. □

A.4 Additional material for Chapter 8

A.4.1 Extensions by one point or by multiple points?

The Bell-Aspect setup discussed in Chapter 8.4.4 contains four cases of PFB. In this section we justify our decision to analyze this setup using only a single new choice point $\langle e^*, 0\rangle$. The alternative option would be to consider a surface structure with four candidates for new choice points, e^*_{ij}, one for each case of PFB. Each e^*_{ij} would then need to be placed below a_i and below b_j. This choice of "one vs. many" is the BST version of the distinction between multiple separate screener-off systems and a single common screener-off system in the purely probabilistic framework of Hofer-Szabó, Rédei, and Szabó (see Hofer-Szabó, 2008).[5]

Let us investigate the option with many e^*_{ij}'s, that is, let us suppose that in a surface structure for the Bell-Aspect experiment there are four point events e^*_{ij}, each intended to take care of a single case of PFB. Observe that e^*_{ij} cannot be above L or R; otherwise it would prohibit the occurrence of one of the settings a_i or b_j. By the same observation, every point e^*_{ij} must belong to every history to which $a_i \cap b_j$ belongs, for every $i = 1, 2$, $j = 3, 4$. That means that all these points have to be *SLR* to or below both selection events L and R.

In addition, if one of the e^*_{ij} is below some other $e^*_{i'j'}$, this makes the bottom one irrelevant for PFB, so they all need to be pairwise *SLR*. And as they are introduced in order to explain instances of PFB, they had better not lead to additional cases of PFB in the multiplied structure. The latter structure is produced by making four subsequent multiplications, each with respect to a different e^*_{ij}. Given that e^*_{ij} is associated with an N_{ij}-multiplication, we produce the $N_{13} \cdot N_{14} \cdot N_{23} \cdot N_{24}$-multiplication. The *SLR* relation between any two e^*_{ij} and $e^*_{i'j'}$ ensures that a subsequent multiplication leaves $\langle e^*_{ij}, \emptyset\rangle$ intact, so eventually we obtain an extended event $E^*_0 =_{\text{df}} \{\langle e^*_{ij}, \emptyset\rangle \mid i = 1, 2, j = 3, 4\}$ with $N_{13} \cdot N_{14} \cdot N_{23} \cdot N_{24}$ elementary outcomes. Each elementary outcome of E^*_0 is identified with the intersection $H^n_{13} \cap H^{n'}_{23} \cap H^{n''}_{14} \cap H^{n'''}_{24}$ of some four outcomes of events $\langle e^*_{13}, \emptyset\rangle, \langle e^*_{14}, \emptyset\rangle, \langle e^*_{23}, \emptyset\rangle$, and $\langle e^*_{24}, \emptyset\rangle$, respectively. (Here the superscripts n, n', n'', n''' point to elementary outcomes of events $\langle e^*_{ij}, \emptyset\rangle$, i.e., n points to the n-th outcome of $\langle e^*_{13}, \emptyset\rangle$, n' points to the n'-th outcome of $\langle e^*_{14}, \emptyset\rangle$, etc.). These intersections, being elementary outcomes, provide the most fine-grained partition of outcomes involved in the cases of PFB, as offered by a $N_{13}N_{14}N_{23}N_{24}$-multiplied BST_{92} structure.

Now, if Outcome Independence is satisfied with respect to the elementary outcomes of E^*_0, we just have an $N_{13}N_{14}N_{23}N_{24}$-multiplied probabilistic structure with one extended event E^*_0 that serves the role of a single hidden variable. Its values are given by transitions $E^*_0 \rightarrowtail H^n_{13} \cap H^{n'}_{23} \cap H^{n''}_{14} \cap H^{n'''}_{24}$. The difference between this structure and an N-multiplied structure with a single *point* event $\langle e^*, 0\rangle$ is inessential.

On the remaining option, Outcome Independence is not satisfied by the elementary outcomes of E^*_0, but is satisfied by the elementary outcomes of each $\langle e^*_{ij}, 0\rangle$. However, an elementary outcome of $\langle e_{ij}, 0\rangle$ is identifiable with a *non-elementary* outcome of E^*_0. Accordingly, Outcome Independence is not satisfied by elementary outcomes of E^*_0, but is satisfied by some non-elementary outcomes of it. We may re-phrase this fact in terms of partitions: the condition is satisfied on a less than maximally fine-grained level, while failing at the most fine-grained level. This looks like a fluke, and in any case, does not explain the four cases of PFB we started with.

To sum up, in the context of the Bell-Aspect experiment, the construction of a structure with many hidden variables for PFB either reduces to the construction with a single hidden variable, or abandons any explanation of PFB.[6]

[5] The distinction was, however, first introduced and argued for in the BST framework (see Belnap and Szabó, 1996), where it was phrased it in terms of common causes and *common* common causes. The framework of Hofer-Szabó, Rédei, and Szabó was introduced in Hofer-Szabó et al. (1999). For a recent presentation of their results in this framework, see Hofer-Szabó et al. (2013).

[6] For a similar diagnosis in a purely probabilistic framework, see Wroński et al. (2017, p. 95).

A.4.2 Proofs for Chapter 8

Lemma 8.4. *Let $\mathscr{W} = \langle W, <, \mu, e^*, E, C\rangle$ be a probabilistic BST_{92} surface structure and $\mathscr{W}'$—the N-multiplied structure corresponding to $\mathscr{W}$. Then*

(1) For every history $h \in \mathrm{Hist}(W)$ the set $\varphi^n(h)$ is a maximal directed subset of W', i.e., a history in W'.

(2) For every maximal directed subset $A' \subseteq W'$ there is a history $h \in \mathrm{Hist}(W)$ and $n \in \{1,\dots,N\}$ for which $A' = \varphi^n(h)$.

Proof. (1) It is easy to see that φ^n is an order-preserving bijection between h and $\varphi^n(h)$, which implies that $\varphi^n(h)$ is directed. To establish maximal directedness, take a directed set $A' \subseteq W'$ for which $\varphi^n(h) \subseteq A'$. Note that by the definition of the ordering, there are no upper bounds for elements $\langle x_1, n_1\rangle$ and $\langle x_2, n_2\rangle$ if $n_1, n_2 \in \{1,\dots,N\}$ and $n_1 \neq n_2$. It follows that A' can be written as the union of $A' = \{\langle x,0\rangle \mid e^* \not< x \wedge x \in W\} \cup \{\langle x,n\rangle \mid e^* < x \wedge x \in W\}$ for some $n \in \{1,\dots,N\}$. We claim now that the set $A =_{\mathrm{df}} \{x \in W \mid \langle x,0\rangle \in A'\} \cup \{x \in W \mid \langle x,n\rangle \in A'\}$ is directed: Let $e_1, e_2 \in A$, so that there are unique $\langle e_1, n_1'\rangle, \langle e_2, n_2'\rangle \in A'$, with $n_1', n_2' \in \{0,1,\dots,N\}$. By directedness of A', these elements have a common upper bound $\langle e_3, n_3'\rangle \in A'$, so that $e_3 \in A$, and by the definition of the ordering, $e_1 \leqslant e_3$ and $e_2 \leqslant e_3$, so A is directed, indeed. Finally, by $\varphi^n(h) \subseteq A'$ we have $h \subseteq A$. Now as h is maximal directed, it must be that $A = h$, whence $A' = \varphi^n(h)$.

(2) Let A' be a maximal directed subset of W'. In (1) above we established that $A' = \{\langle x,0\rangle \mid e^* \not< x \wedge x \in W\} \cup \{\langle x,n\rangle \mid e^* < x \wedge x \in W\}$ for some $n \in \{1,\dots,N\}$ and that the set $A =_{\mathrm{df}} \{x \in W \mid \langle x,0\rangle \in A'\} \cup \{x \in W \mid \langle x,n\rangle \in A'\}$ is directed. We now claim that there is $h \in$ Hist st $A = h$. Since A is a directed subset of W and histories are maximal directed subsets of W, there is an $h \in \mathrm{Hist}(W)$ st $A \subseteq h$. Suppose that $A \subsetneq h$. But then

$$A' \subsetneq \{\langle x,0\rangle \mid e^* \not< x \wedge x \in h\} \cup \{\langle x,n\rangle \mid e^* < x \wedge x \in h\},$$

As the set on the RHS is directed, A' is not maximally directed, which contradicts the premise. Hence $A = h$, which implies, given the form of A', that $A' = \varphi^n(h)$. □

Fact 8.15. *Let $\mathscr{W} = \langle W, <, \mu, e^*, E, C\rangle$ be a probabilistic BST_{92} surface structure and let $\mathscr{W}'$ be the N-multiplied structure corresponding to $\mathscr{W}$. Then*

(1) for every $n \in \{1,\dots,N\}$ and every $h_1, h_2 \in H_{e^*}$: $\varphi^n(h_1) \equiv_{\langle e^*,0\rangle} \varphi^n(h_2)$.

(2) for every $n, m \in \{1,\dots,N\}$ such that $n \neq m$ and every $h \in H_{e^*}$: $\varphi^n(h) \perp_{\langle e^*,0\rangle} \varphi^m(h)$.

(3) $\langle e^*, 0\rangle$ is a choice point with N outcomes $\Pi_{\langle e^*,0\rangle}\langle \varphi^n(h)\rangle$, where h is an arbitrary history from H_{e^*};

(4) for every $n \in \{1,\dots,N\}$, every $e \in W$, and every $h_1, h_2 \in \mathrm{Hist}(W)$: $h_1 \perp_e h_2$ iff $\varphi^n(h_1) \perp_{\langle e,l\rangle} \varphi^n(h_2)$, where $l = n$ iff $e^* < e$, and $l = 0$ otherwise;

(5) for every $m, n, l \in \{1,\dots,N\}$ with $m \neq n$, every $e \in W$ such that $e^* < e$, and every $h_1, h_2 \in \mathrm{Hist}(W)$: neither $\varphi^m(h_1) \equiv_{\langle e,l\rangle} \varphi^n(h_2)$, nor $\varphi^m(h_1) \perp_{\langle e,l\rangle} \varphi^n(h_2)$;

(6) for every $m, n \in \{1,\dots,N\}$ with $m \neq n$, every $e \not> e^*$ and every $h \in H_e$: $\varphi^m(h) \equiv_{\langle e,0\rangle} \varphi^n(h)$.

Proof. (1) Since e^* is deterministic, $h_1 \equiv_{e^*} h_2$, so there is $e > e^*$ such that $e \in h_1 \cap h_2$. Accordingly $\langle e,n\rangle >' \langle e^*,0\rangle$ and $\langle e,n\rangle \in \varphi^n(h_1) \cap \varphi^n(h_2)$, which proves $\varphi^n(h_1) \equiv_{\langle e,0\rangle} \varphi^n(h_2)$.

(2) Take any $h \in H_{e^*}$. Then $\langle e^*, 0\rangle \in \varphi^n(h) \cap \varphi^{n'}(h)$ for any $n, n' \in \{1, \ldots, N\}$. For any $e > e^*$ any m and any different n, n', event $\langle e, m\rangle$ cannot be shared by $\varphi^n(h)$ and $\varphi^{n'}(h)$. This proves that $\varphi^n(h) \perp_{\langle e^*,0\rangle} \varphi^{n'}(h)$, and hence that N histories split at $\langle e^*, 0\rangle$.

(3) By (1) and (2) above.

(4) Take $h_1, h_2 \in \text{Hist}(W)$ such that $h_1 \perp_e h_2$. It follows that $\langle e, l\rangle \in \varphi^n(h_1) \cap \varphi^n(h_2)$. Also, there is no $e' > e$ such that $e' \in h_1 \cap h_2$. Hence there is no $\langle e', l'\rangle$ with $\langle e', l'\rangle >' \langle e, l\rangle$ such that $\langle e', l'\rangle \in \varphi^n(h_1) \cap \varphi^n(h_2)$, which proves the $\Rightarrow$ direction. In the opposite direction, assume for reductio that $h_1 \not\perp_e h_2$. If $e \notin h_1 \cap h_2$, then $\langle e, l\rangle \notin \varphi^n(h_1) \cap \varphi^n(h_2)$, which contradicts $\varphi^n(h_1) \perp_{\langle e,l\rangle} \varphi^n(h_2)$. So let $h_1 \equiv_e h_2$. There is then $\langle e', l'\rangle >' \langle e, l\rangle$ such that $\langle e', l'\rangle \in \varphi^n(h_1) \cap \varphi^n(h_2)$, which contradicts $\varphi^n(h_1) \perp_{\langle e,l\rangle} \varphi^n(h_2)$.

(5) Since $e > e^*$, it can only be associated with some $l \in \{1, \ldots, N\}$. Furthermore, for different l, l', $\langle e, l\rangle$ and $\langle e, l'\rangle$ have no upper bound with respect to $<'$, and hence there is no history to which they belong. Thus, neither $\varphi^m(h_1) \equiv_{\langle e,l\rangle} \varphi^n(h_2)$, nor $\varphi^m(h_1) \perp_{\langle e,l\rangle} \varphi^n(h_2)$.

(6) Let $e \not> e^*$. Then $\langle e, 0\rangle \in \varphi^n(h)$ for every $n \in \{1, \ldots, N\}$ and every $h \in H_e$. If e is maximal in W, $\langle e, 0\rangle$ is maximal in W', and hence $\varphi^m(h) \equiv_{\langle e,0\rangle} \varphi^n(h)$ for every $m, n \in \{1, \ldots, N\}$ and every $h \in H_e$. If e is not maximal in W, pick an arbitrary $h \in H_e$; there is then $e' \in h$ such that $e' > e$ and $e' \not> e^*$ as well (by density). Each e and e' is associable with 0 only, $\langle e, 0\rangle <' \langle e', 0\rangle$ and $\langle e', 0\rangle \in \varphi^n(h)$ for any $n \in \{1, \ldots, N\}$. Hence $\varphi^m(h) \equiv_{\langle e,0\rangle} \varphi^n(h)$ for any $m, n \in \{1, \ldots, N\}$. □

Lemma 8.6. *Let $\mathscr{W} = \langle W, <, \mu, e^*, E, C\rangle$ be a probabilistic BST$_{92}$ surface structure in which transitions $\{I_1 \rightarrowtail \mathbf{1}_1, \ldots, I_K \rightarrowtail \mathbf{1}_K\}$ with random variables $X_1, \ldots, X_K$ exhibit PFB, where $\mathbf{1}_k = \{\hat{O}_{k,\gamma(k)} \mid \gamma(k) \in \Gamma(k)\}$, $\Gamma(k)$ are index sets and $1 \leqslant k \leqslant K$. Let $cll(I_k \rightarrowtail \hat{O}_{k,\gamma(k)}) \subseteq E$ for every $k \leqslant K$ and every $\gamma(k) \in \Gamma(k)$ and $C = \emptyset$. Then there exists a structure with a probabilistic hidden variable for this case of PFB. Moreover, the structure satisfies C/E propensity independence.*

Proof. Let us suppose that there is $e^* \in W$ that is below every I_k. We will explicitly exhibit a structure $\mathscr{W} = \langle W', <', \mu'\rangle$ with a probabilistic hidden variable, in which event $\langle e^*, 0\rangle$ has N outcomes $H^1, H^2, \ldots, H^N$ and there are N sets of uncorrelated random variables $\{X_1^n, \ldots, X_K^n\}$ $(1 \leqslant n \leqslant N)$ corresponding to $\{X_1, \ldots, X_K\}$, resp. Since the cardinality of S is N, we may number elements of S as $T_1, T_2, \ldots T_N$. Note that each $T_n \in S$ is determined as well by a sequence of values of random variables, $X_1(T_n) = \gamma(1), X_2(T_n) = \gamma(2), \ldots, X_K(T_n) = \gamma(K)$ with $\gamma(k) \in \Gamma(k)$, which is the observation we use below. By means of Outcome Independence we get

$$p^n(X_1^n = \gamma(1))p^n(X_2^n = \gamma(2)) \ldots p^n(X_K^n = \gamma(K)) =$$
$$p^n(X_1^n = \gamma(1) \wedge X_2^n = \gamma(2) \wedge \ldots \wedge X_K^n = \gamma(K)) = \mu'(T_l^n),$$

where T_l^n is a unique element of S^n st $X_k^n(T_l^n) = \gamma(k)$ for $k = 1, 2, \ldots K$ and $\gamma(k) \in \Gamma(k)$. By adequate probabilistic assignment we get

$$\mu(T_l) = \sum_{n=1}^{N} \mu'(\langle e^* 0\rangle \rightarrowtail H^n)\mu'(T_l^n), \text{ where } T_l \in S \text{ corresponds to } T_l^n \in S^n.$$

Given this correspondence, T_l is given by the same values of the corresponding random variables as T_l^n, so we have $\mu(T_l) = p(X_1 = \gamma(1) \wedge X_2 = \gamma(2) \wedge \ldots \wedge X_K = \gamma(K))$; then the above equations yield

$$p(X_1 = \gamma(1) \wedge X_2 = \gamma(2) \wedge \ldots \wedge X_K = \gamma(K)) =$$
$$\sum_{n=1}^{N} \mu'(\langle e^*0\rangle \rightarrowtail H^n) p^n(X_1^n = \gamma(1)) p^n(X_2^n = \gamma(2)) \ldots p^n(X_K^n = \gamma(K)).$$

Let us abbreviate the above formula as:

$$Z_{\gamma(1)\gamma(2)\ldots\gamma(K)} = \sum_{n=1}^{N} \alpha^n q^n_{1,\gamma(1)} q^n_{2,\gamma(2)} \cdots q^n_{K,\gamma(K)}, \tag{A.1}$$

where $Z_{\gamma(1)\gamma(2)\ldots\gamma(K)} = p(X_1 = \gamma(1) \wedge X_2 = \gamma(2) \wedge \ldots \wedge X_K = \gamma(K))$, $q^n_{k,\gamma(k)} = p^n(X_k^n = \gamma(k))$, and $\alpha^n = \mu'(\langle e^*0\rangle \rightarrowtail H^n)$. Formula A.1 encapsulates $|\Gamma(1)| \times \cdots \times |\Gamma(K)|$ equations. To construct a sought-after structure means to solve these equations for N unknown variables α^n and $N(|\Gamma(1)| + |\Gamma(2)| + \ldots + |\Gamma(K)|)$ unknown variables $q^n_{k,\gamma(k)}$. Here is a simple set of solutions (there are other sets of solutions):

$$\alpha^n = p(T_n)$$
$$q^n_{k,\gamma(k)} = \begin{cases} 1 \text{ iff } X_k(T_n) = \gamma(k) \\ 0 \text{ otherwise.} \end{cases}$$

It is easy to calculate that this ascription of values satisfies the above equations. The LHS is equal to: $p(T_l)$, where $T_l \in S$ is such that for every $k \in \{1, \ldots, K\}$: $X_k(T_l) = \gamma(k)$. The RHS is $\alpha^{l'} q^{l'}_{1,\gamma(1)} q^{l'}_{2,\gamma(2)} \cdots q^{l'}_{K,\gamma(K)} = \alpha^{l'} = p(T_{l'})$, where $T_{l'}$ is such that for every $k \in \{1, \ldots, K\}$: $X_k(T_{l'}) = \gamma(k)$. Thus, $T_l = T_{l'}$ and so the equation holds. □

Lemma 8.7. *Let $\mathscr{W} = \langle W, <, \mu, e^*, E, C\rangle$ be a probabilistic BST$_{92}$ surface structure harboring multiple cases of PFB for which $C = \emptyset$. Let μ be defined on every subset of* TR(W). *Then there is an N-multiplied probabilistic BST$_{92}$ structure corresponding to $\mathscr{W}$ that provides a hidden variable for every case of PFB in $\mathscr{W}$. Moreover, that extended structure satisfies C/E propensity independence.*

Proof. We first decide how large a multiplication we will introduce. We are after a quasi-deterministic hidden variable. By the finitistic assumptions, each $\tilde{T}_E$ and S_E is finite, and they contain all basic transitions in $\mathscr{W}$, and all maximal consistent subsets of basic transitions in $\mathscr{W}$, respectively. For N, the size of the multiplication, we take the cardinality of S_E. By the above there are finitely many cases of PFB as well. Note also that e^* is below the initial of every transition in $\tilde{T}_E$. In what follows we will need some bijection $f : S_E \mapsto \{1, 2, \ldots, N\}$.

We consider next an N-multiplied BST$_{92}$ structure $\mathscr{W}' = \langle W', <'\rangle$ corresponding to $\mathscr{W}$. We construct μ', which is intended to be adequate for $\mathscr{W}$ and N. In accord with Def. 8.20, it is enough to specify what μ' yields for every element of S_E and new transitions. We define μ' as follows:[7]

1. For every basic transition $\langle e^*, 0\rangle \rightarrowtail H^n$: $\mu'(\{\langle e^*, 0\rangle \rightarrowtail H^n\}) = \mu(f^{-1}(n))$, where $H^n \in \Pi_{\langle e^*,0\rangle}$;
2. For every $T \in S_E$ and $n \in \{1, \ldots, N\}$: $\mu'(T^n) = \delta_n^{f(T)}$ (where δ is Kronecker's delta).

[7] Note the simplification below due to the fact that every $e \in E_T$ is above e^*.

It is immediately discernible that μ' is adequate for $\mathscr{W}$ and N. To calculate a single non-trivial clause of Def. 8.20: $\sum_{n\leqslant N}\mu'(\langle e^*,0\rangle \rightarrowtail H^n)\cdot\mu'(T^n) = \sum_{n\leqslant N}\mu(f^{-1}(n))\delta_n^{f(T)} = \mu(T)$. Note that the rules above induce a zero-one assignment to sets corresponding to subsets of $\tilde{T}_E$: for any $Y \subseteq \tilde{T}_E$ and any $n \in \{1,\dots,N\}$,

$$\mu'(Y^n) = \begin{cases} 1 & \text{if there is } T \in S_E \text{ such that } Y \subseteq T \text{ and } \mu'(T^n) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

To check if Outcome Independence is satisfied, pick an arbitrary case of PFB in $\mathscr{W}$. This is given by a set of transitions $\{I_1 \rightarrowtail \mathbf{1}_1,\dots,I_K \rightarrowtail \mathbf{1}_K\}$ with SLR initials, each transition being associated with a random variable, $X_1,\dots,X_K$, respectively, and where $\mathbf{1}_k = \{\hat{O}_{k,\gamma(k)} \mid \gamma(k) \in \Gamma(k)\}$, with finite index sets $\Gamma(k)$, and $1 \leqslant k \leqslant K$. Then these random variables $X_1,\dots,X_K$ exhibit PFB, We abbreviate: $Y_{\langle\gamma(1),\dots,\gamma(K)\rangle} =_{\text{df}} CC(G \rightarrowtail \hat{O}_{1,\gamma(1)} \cup \dots \cup \hat{O}_{K,\gamma(K)})$ and $Y_{\langle k,\gamma(k)\rangle} =_{\text{df}} CC(I_k \rightarrowtail \hat{O}_{k,\gamma(k)})$, where $G = \bigcup_{k\leqslant K} I_k$. Our random variables are defined on CPS, for which the base set is: $S = \{Y_{\langle\gamma(1),\dots,\gamma(K)\rangle} \mid \gamma(1) \in \Gamma(1),\dots,\gamma(K) \in \Gamma(K)\}$. We write $Y^n_{\langle\gamma(1),\dots,\gamma(K)\rangle}$ and $Y^n_{\langle k,\gamma(k)\rangle}$ for the sets corresponding to $Y_{\langle\gamma(1),\dots,\gamma(K)\rangle}$ and $Y_{\langle k,\gamma(k)\rangle}$, respectively.

Consider now a corresponding CPS, $\langle S^n, \mathscr{A}^n, p^n\rangle$ with an arbitrary $n \leqslant N$. We need to check if the following identity is satisfied:

$$p^n(X_1^n = \gamma(1) \wedge \dots \wedge X_K^n = \gamma(K)) = p^n(X_1^n = \gamma(1)) \cdot \ldots \cdot p^n(X_K^n = \gamma(K)), \qquad (\dagger)$$

which is equivalent to

$$\mu'(Y^n_{\langle\gamma(1)\dots\gamma(K)\rangle}) = \mu'(Y^n_{\langle 1,\gamma(1)\rangle}) \cdot \ldots \cdot \mu'(Y^n_{\langle K,\gamma(K)\rangle}).$$

By the definition of μ', each value of μ' above must be either 0 or 1 . Then the argument that this identity (†) holds is exactly the same as in the proof of Lemma 8.6. This means that Outcome Independence is satisfied. As the propensity assignment μ' is adequate as well, $\langle W', <', \mu'\rangle$ is an N-multiplied BST$_{92}$ probabilistic structure corresponding to $\mathscr{W}$ that provides a hidden variable for every case of PFB in $\mathscr{W}$. Furthermore, the N-multiplied structure satisfies independence with respect to images of C and E, since C is assumed to be empty. □