3

Two Options for the Branching of Histories

3.1 Indeterminism as the branching of histories

The branching of histories in BST is intended to represent indeterminism as a feature of our world. Now, if a common BST structure contains just one history, then it is trivial from the perspective of indeterminism: all events are compatible, and the picture of a world with just one history is that of a deterministic world. Since there are multiple histories in any non-trivial BST structure, there are different ways in which these histories can interrelate. A strong intuitive principle is historical connection (Postulate 2.2): The idea is that any two histories should share some common past. As we said, we will show that historical connection is implied by stronger principles concerning the interrelation of histories. These so-called prior choice principles (Defs. 3.4 and 3.14) make specific demands on the way in which histories branch off from one another. The key decision is what the branching of histories looks like locally: What are the objects at which histories branch? BST_{92} decides for points: histories branch, or remain undivided, at points, which means that there is a maximal element, called a *choice point*, in the overlap of any two histories.

The existence of choice points has important implications for the topological properties of the resulting structures, a matter to which we will turn later in Chapter 4.4. But do choice points exist, or, more precisely, do the postulates of a common BST structure decide whether there are choice points? It turns out that the answer is in the negative: we can show that both the existence and the non-existence of choice points are live options for the branching of histories in common BST structures. While BST_{92} requires the existence of choice points, the "new foundations" theory BST_{NF} prohibits the existence of choice points and works with so-called *choice sets*. The difference is illustrated by the two common BST structures of Figure 3.1

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(a) and (b).[1] These structures illustrate the two possibilities for histories to branch in common BST structures.

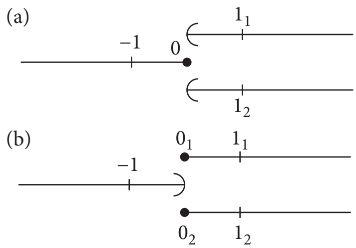

Figure 3.1 Two simple common BST structures with (a) and without (b) a choice point. Both (a) and (b) depict partial orderings in which there are two continuous histories branching at point 0. In (a), point 0 is the shared maximum in the intersection of the histories, so that 0 is a choice point. In (b), the intersection of the histories has no maximum, and points 0_1 and 0_2 are different history-relative suprema (minimal upper bounds) of the intersection, which together form a choice set.

We provide a formal definition of these structures, so that we do not rely on pictures alone. Both structures are defined as quotients of the double real line

$$L_2 =_{\text{df}} \{\langle x, i\rangle \mid x \in \mathbb{R}, i \in \{1,2\}\}$$

under the equivalence relations $\equiv_a$ and $\equiv_b$, which are defined, respectively, as

$$\langle x, i\rangle \equiv_a \langle x', i'\rangle \leftrightarrow (x = x' \wedge (i = i' \vee x \leqslant 0));$$
$$\langle x, i\rangle \equiv_b \langle x', i'\rangle \leftrightarrow (x = x' \wedge (i = i' \vee x < 0)).$$

These relations differ only in their handling of $x = 0$. The ordering on the quotient structures $M_a =_{\text{df}} L_2/\equiv_a$ and $M_b =_{\text{df}} L_2/\equiv_b$ is defined uniformly via

$$[\langle x, i\rangle] < [\langle x', i'\rangle] \Leftrightarrow_{\text{df}} (x < x' \wedge [\langle x, i\rangle] = [\langle x, i'\rangle]).$$

[1] These structures have just the bare minimum of complexity to fulfill the axioms of Def. 2.10 in a non-trivial way: they contain just two histories each. Furthermore, they do not include any spatial extension—so in fact they are branching time structures as well.

It is easy to check that these structures are non-empty partial orderings satisfying all the conditions of a common BST structure. The two histories h_1^a, h_2^a in M_a and h_1^b, h_2^b in M_b are, respectively (for γ one of a or b),

$$h_1^\gamma = \{[\langle x, 1\rangle] \in M_\gamma \mid x \in \mathbb{R}\}; \quad h_2^\gamma = \{[\langle x, 2\rangle] \in M_\gamma \mid x \in \mathbb{R}\}.$$

The intersections of these two histories are, respectively, the upper bounded chains

$$l_a =_{\text{df}} h_1^a \cap h_2^a = \{[\langle x, 1\rangle] \in M_a \mid x \leqslant 0\}; \; l_b =_{\text{df}} h_1^b \cap h_2^b = \{[\langle x, 1\rangle] \in M_b \mid x < 0\}.$$

The difference is the following: while the chain l_a in M_a has a maximal element, $[\langle 0, 1\rangle]$, the chain l_b in M_b has no maximal element. That latter chain instead has two different history-relative suprema:

$$\sup_{h_i^b} l_b = [\langle 0, i\rangle], \quad i = 1, 2.$$

3.2 On chains in common BST structures

As history-relative suprema of chains will play a crucial role in the chapters to come, we provide a number of pertinent definitions and facts.

Definition 3.1 (Chains and related sets)**.** Let $\mathscr{W}$ be a common BST structure. We define the following classes of chains and related sets in $\mathscr{W}$:

- $\mathscr{C}_e$: the set of chains ending in, but not containing, e. That is:
  $l \in \mathscr{C}_e$ iff l is an upper bounded chain and there is some $h \in \text{Hist}$ for which $l \subseteq h$ and $\sup_h l = e$, but $e \notin l$.
- $\mathscr{S}(l)$: the set of history-relative suprema for an upper bounded chain l:

$$\mathscr{S}(l) =_{\text{df}} \{s \in W \mid \exists h \in \text{Hist}\,[l \subseteq h \wedge s = \sup_h l]\}.$$

- For l a chain and $e \in W$, we define initial and final segments:

$$l^{\leqslant e} =_{\text{df}} \{e' \in l \mid e' \leqslant e\}; \quad l^{\geqslant e} =_{\text{df}} \{e' \in l \mid e' \geqslant e\};$$

- $\mathscr{P}_e$: the proper past of e:

$$\mathscr{P}_e =_{\mathrm{df}} \{e' \in W \mid e' < e\}.$$

We establish the following Facts, using the existence of history-relative suprema and Weiner's postulate (see Section 2.5):

Fact 3.1. *Let l be an upper-bounded chain in a common BST structure $\mathscr{W}$ and $l \subseteq h'$ for some $h' \in \mathrm{Hist}(\mathscr{W})$, and $s = \sup_{h'} l$. Then for all $h \in H_s$, we have $\sup_h l = s$.*

Proof. Assume that $s \in h$. Observe that $\{s\}$ is a (trivial) chain with $\sup_{h^*}\{s\} = s$ for any $h^* \in H_s$. We can use Weiner's postulate on the chains l and $\{s\}$. As $\sup_{h'}\{s\} = s = \sup_{h'} l$, we also have to have $\sup_h l = \sup_h\{s\} = s$. □

Alternatively, the result also follows from the definition of suprema and the downward closure of histories, so Weiner's postulate does not need to be used—see Exercise 3.7.

Fact 3.2. *Let l be an upper bounded chain and $h_1, h_2 \in H_{[l]}$, and let*

$$\sup_{h_1} l = s_1 \neq s_2 = \sup_{h_2} l.$$

Then there is no history h containing both s_1 and s_2.

Proof. Assume otherwise, and let $\{s_1, s_2\} \subseteq h$ for some $h \in \mathrm{Hist}$. We have $l \subseteq h$, since $s_1 \in h$ and $l \leqslant s_1$. By Fact 3.1, we have both $\sup_h l = s_1$ (as $s_1 \in h$) and $\sup_h l = s_2$ (as $s_2 \in h$), contradicting our assumption that $s_1 \neq s_2$. □

Here is another useful fact about the suprema of chains. If a chain l contains its history-relative supremum s, $s \in l$, then a chain obtained by removing s from l has the same history-relative supremum, s. An analogous fact holds for infima.

Fact 3.3. *The suprema and infima of maximal chains are unaffected by the removal of the supremum or infimum:*

1. *Let l be a maximal upper bounded chain, and let $h \in \mathrm{Hist}$ such that $l \subseteq h$. Let $s =_{\mathrm{df}} \sup_h l$. Then for $l' =_{\mathrm{df}} l \setminus \{s\}$, we also have $\sup_h l' = s$.*
2. *Let l be a maximal lower bounded chain, and let $e =_{\mathrm{df}} \inf l$. Then for $l' =_{\mathrm{df}} l \setminus \{e\}$, we also have $\inf l' = e$.*

Proof. (1) If $s \notin l$, we have $l' = l$, and there is nothing to prove. Otherwise, let $s' =_{\mathrm{df}} \sup_h l'$. Clearly, $l' \leqslant s$, so $s' \leqslant s$ (by the definition of suprema). Now assume for reductio that $s \neq s'$, i.e., $s' < s$. By the construction of l', we then have

$$(*) \qquad \forall x \in l\, [x \neq s \rightarrow x \leqslant s'].$$

By density, there is some $e \in W$ for which $s' < e < s$. By $(*)$, we have $e \notin l$. But then, again by $(*)$, we have that $l^* =_{\mathrm{df}} l \cup \{e\}$ is also a chain with $\sup_h l^* = s$, and $l^* \supsetneq l$. This contradicts the maximality of l. So, we have $s = s'$.

The proof for (2) is exactly parallel to that for (1). □

Our next Fact concerns the maximality of chains in a history $h \in \mathrm{Hist}(W)$ and in W:

Fact 3.4. *Let l be a maximal chain in $h \in \mathrm{Hist}(W)$. Then l is a maximal chain in W as well.*

Proof. For reductio, let us suppose that l is a maximal chain in $h \in \mathrm{Hist}(W)$, but not in W. Since l is not a maximal chain in W, there exists some e in $W \backslash l$ such that the set $l \cup \{e\}$ forms a chain. Observe that (†) $l < e$, since otherwise there would be some element $x \in l$ such that $e < x$, from which it follows by downward closure of h that l is not a maximal chain in h, which is a contradiction. Thus l is an upper-bounded chain in W, so we can apply the history-relative supremum postulate to conclude that l has an h-relative supremum $s = \sup_h l \in h$. By maximality of l, $s \in l$ and (by the same assumption) s is a maximal element of h. By Fact 2.1(9), s is then a maximal element of W, so there is no $e > l$, contradicting (†). Thus l is a maximal chain in W. □

The next Fact shows that the proper past of an event e consists of all the chains ending in, but not containing, e.

Fact 3.5. *For $e \in W$, we have*

$$\mathscr{P}_e = \bigcup_{l \in \mathscr{C}_e} l.$$

Proof. "$\supseteq$" Let $x \in \bigcup_{l \in \mathscr{C}_e} l$, i.e., $x \in l$ for some $l \in \mathscr{C}_e$. Since $l \in \mathscr{C}_e$, there is $h \in H_e$ such that $\sup_h l = e$ and $e \notin l$, we have $l < e$, and thus, $x < e$, i.e., $x \in \mathscr{P}_e$.

"$\subseteq$" Let $x \in \mathscr{P}_e$, i.e., $x < e$. Then $\{x, e\}$ is a chain, which by the Hausdorff maximal principle[2] can be extended to a maximal chain l ending in e. By Fact 3.1, for any history h in H_e we have that $\sup_h l = e$. By Fact 3.3, for $l' =_{\text{df}} l \setminus \{e\}$ we also have $\sup_h l' = e$, whereby we have some $l' \in \mathscr{C}_e$ for which $x \in l'$. □

3.3 Extending common BST: two options

As suggested by Figure 3.1, there are two options for fulfilling the common BST axioms in a simple case. Which one should we choose? We could now enter into a philosophical discussion as to the correct manner in which we are to proceed—but we refrain from attempting any a priori arguments here. One can give good reasons for each option. Thus, in favor of the existence of choice points, one can argue that a causal account of indeterministic choice requires a special final element of indecision and, therefore, a maximal element of any two branching histories. On the other hand, since choice points are distinguished as maximal elements of the overlap of histories, considerations of uniformity argue against them: as we will see, it is possible to have branching without maxima in the overlap of histories in a uniform way. The further topological considerations to be discussed in Chapter 4.4 also argue against the assumption of choice points. In our view, these controversial issues provide a good motivation for investigating common BST structures both with and without choice points.

As a matter of fact, the theory of Branching Space-Times was initially developed with the requirement of the existence of choice points, and the resulting axiomatic theory, BST_{92} (named after the year of publication of the original BST paper by Belnap (1992)), has proved to be fruitful for quite a number of applications, for example, to causation (Belnap, 2005b), to probability theory (Weiner and Belnap, 2006; Müller, 2005), and to physics (Placek, 2004, 2010). On the other hand, in applications of BST to physics in which an attempt is made to link BST histories to physical space-times, topological considerations argue against the existence of choice points, so that a slightly different axiomatic theory, BST_{NF} (with "NF" for "new foundations"), is to be preferred.

[2] See note 8 on p. 30.

We will proceed by introducing both developments of the common BST framework, BST_{92} and BST_{NF}. Each will be obtained by adding a (different) axiom, a prior choice principle, called PCP_{92} or PCP_{NF}, to the list of axioms given in Definition 2.10. We will investigate the consequences of the resulting theories, focusing in particular on how they define local possibilities and how histories are to branch in each of them. Finally, we will broach the larger question of whether the differences between the two kinds of structures really matter. While pointing to the significance of topological differences on the one hand, on the other hand we exhibit translatability results that lessen the importance of topology somewhat. In a nutshell, if we have a BST_{92} structure with topologically worrisome features, it can be translated into a BST_{NF} structure, in which the worrisome feature is absent; and a translation in the other direction is also available. The remainder of this chapter is divided into sections devoted to BST_{92}, to BST_{NF}, to issues of topology, and to the translatability of the two kinds of structures.

3.4 BST_{92}

BST_{92} is the original BST theory put forward by Belnap (1992). Here we approach it somewhat differently, namely by considering it as a development of the common BST framework. The development consists of the addition of just a single axiom, the Prior Choice Principle (PCP_{92}), to the set of axioms of Definition 2.10.

3.4.1 BST_{92} in formal detail

As we just said, the theory of BST_{92} posits the axioms of a common BST structure together with the so-called prior choice principle (PCP_{92}).[3] In order to motivate the idea of prior choice, we start with the notion of *undividedness*. Let two histories h_1, h_2 share some event $e \in h_1 \cap h_2$. Then they also may or may not share a later event (provided there is one at all). In the former case, we call the histories *undivided at* e:

[3] Belnap (1992) does not mention Weiner's postulate, which proved critical for some applications developed later, especially regarding probability theory. See note 15.

Definition 3.2 (Undividedness). Let $h_1, h_2 \in \text{Hist}$, and let $e \in h_1 \cap h_2$. We say that h_1 *and* h_2 *are undivided at* e ($h_1 \equiv_e h_2$) iff either there is no $e' \in W$ at all for which $e < e'$, or there is some $e' \in h_1 \cap h_2$ for which $e < e'$.

There is also a relation opposite to undividedness: in case two histories h_1, h_2 share an event e but no event later than e, that event e is a maximum in the intersection of the histories $h_1 \cap h_2$. In that case (provided that e is not maximal in W to begin with), we say that the histories *split at e*:

Definition 3.3 (Splitting at a point; choice point). Let $h_1, h_2 \in \text{Hist}$, and let $e \in h_1 \cap h_2$. We say that h_1 *and* h_2 *split at* e, and that e *is a choice point for histories* h_1 *and* h_2 ($h_1 \perp_e h_2$) iff it is not the case that $h_1 \equiv_e h_2$. We extend the "$\perp$" notation to sets, with the universal reading, i.e., for $H \subseteq \text{Hist}$, we write $h_1 \perp_e H$ iff for any $h \in H$, we have $h_1 \perp_e h$.

It immediately emerges that this definition agrees with the informal explanation of a choice point for h_1 and h_2 as a maximal element in the intersection $h_1 \cap h_2$.

Basically, the prior choice principle PCP_{92} requires that whenever an event e belongs to one history h_1 but not to another history h_2, these two histories split at a choice point c in the past of e:

$$e \in (h_1 \setminus h_2) \rightarrow \exists c\, [c \in h_1 \cap h_2 \wedge c < e \wedge h_1 \perp_c h_2].$$

A motivation for PCP_{92} is that there should be a reason (however minimally understood) in the past of each point event for its being in one history rather than another.

We have written the notion of undividedness in a way that suggests that it is an equivalence relation, which seems natural enough. It turns out, however, that in order to enforce the transitivity of the relation of undividedness, PCP_{92} needs to be formulated not for points, but for lower bounded chains in the difference of two histories, as follows.[4]

[4] Why not generalize PCP_{92} to other sets of events contained in a history? Such a set can spread through much space and lack homogeneity, and so, intuitively speaking, a PCP of so large a scope seems unreasonable. Chains, however, form a distinguished category as they naturally represent (parts of) world-lines of (spatially non-extended) objects. And it is a common practice to ask questions like "where did a given particle go up, instead of going down?" We owe our thanks to J. Luc for discussing various versions of PCP.

Definition 3.4 (BST$_{92}$ prior choice principle, PCP$_{92}$)**.** A common BST structure $\langle W, < \rangle$ fulfills the *BST$_{92}$ prior choice principle* iff it fulfills the following condition:

Let $h_1, h_2 \in \text{Hist}$ be two histories, and let $l \subseteq (h_1 \setminus h_2)$ be a lower-bounded chain that belongs fully to history h_1 but does not intersect history h_2. Then there is a choice point $c \in h_1 \cap h_2$ such that $c < l$ and $h_1 \perp_c h_2$, i.e., c lies properly below l and is a choice point for h_1 and h_2, which is maximal in the intersection of h_1 and h_2.

That definition obviously implies the point version described above, as any singleton $\{e\}$ is a lower-bounded chain.[5] It also ensures the property of historical connection independently of the explicit requirement of Def. 2.10(7): any two different histories have a non-empty difference (see Fact 2.1(7)), so that it follows that they have to share a choice point.

We can now enter the BST$_{92}$ prior choice principle in its official form as an additional item to our list of axioms for BST$_{92}$:

Definition 3.5 (BST$_{92}$ structure)**.** A *BST$_{92}$ structure* is a common BST structure $\langle W, < \rangle$ (Definition 2.10) that also fulfills the BST$_{92}$ prior choice principle, PCP$_{92}$ (Definition 3.4).

We can prove that in BST$_{92}$ structures, there is a choice point (i.e., a maximal element in their overlap) for any two histories.

Fact 3.6. *Let* $h_1, h_2 \in \text{Hist}$ *be histories of a BST$_{92}$ structure,* $h_1 \neq h_2$. *Then there is a maximal element in* $h_1 \cap h_2$.

Proof. Left as Exercise 3.1. Note that by Def. 3.3, any maximal element in $h_1 \cap h_2$ is a choice point for h_1 and h_2. □

3.4.2 Local possibilities

One important application of PCP$_{92}$ is the construction of the concept of possibilities open at an event. For any event e, the relation $\equiv_e$ among the set H_e of all the histories containing e is obviously symmetrical (by the form of the definition) and reflexive (note that the definition takes care of e being a maximal element in W as well). It is transitive as well, as proved below.

[5] See Belnap (1992) for an argument that the chain version properly strengthens the point version of PCP$_{92}$.

Fact 3.7. *Suppose that a poset $\langle W, < \rangle$ satisfies density, existence of infima and PCP$_{92}$. Then for every e in W, the relation $\equiv_e$ is transitive on the set H_e.*

Proof. Fix some $e \in W$, and let $h_1, h_2, h_3 \in H_e$. Suppose toward a contradiction that (†) $h_1 \equiv_e h_2$ and $h_2 \equiv_e h_3$, but that $h_1 \not\equiv_e h_3$. By the definition of undividedness $h_1 \not\equiv_e h_3$ implies that e is not maximal in W, and hence e is not maximal in any history containing e (by Fact 2.1(9)). Consider the subset of $h_1 \cap h_2$ that is properly above e. Since $h_1 \equiv_e h_2$, and e is not maximal in h_1 and h_2, this set is nonempty. So by the Hausdorff maximal principle, there is a maximal chain l of points above e in $h_1 \cap h_2$ (i.e. a maximal chain lower-bounded by e). By Postulate 2.4, $\inf(l)$ exists, and $e \leqslant \inf(l)$. Moreover, $\inf(l) = e$ since by maximality of l there are no points properly between e and l, as this would contradict density. Further $h_1 \not\equiv_e h_3$ implies that no point later than e belongs to both histories, so $l \subseteq h_1$ but $l \cap h_3 = \emptyset$. By the construction, $l \subseteq h_2$, so $l \subseteq h_2 \setminus h_3$. Thus by PCP$_{92}$ (Def. 3.4), there must be a choice point e_1 for h_2 and h_3, (‡) $h_2 \perp_{e_1} h_3$, strictly below l, $e_1 < l$. Since $e = \inf(l)$, by the definition of infima, $e_1 \leqslant e$. But if $e_1 < e$, we contradict (‡) as $e \in h_2 \cap h_3$. And, if $e_1 = e$, then $h_2 \perp_e h_3$, contradicting (†). □

Transitivity of $\equiv_e$ allows one to establish that a choice point posited by PCP splits not just two histories, but a history and a set of histories:

Fact 3.8. *Let $c < l$, $l \subseteq h_1 \setminus h_2$, and $h_1 \perp_c h_2$. Then $H_{[l]} \perp_c h_2$, where $H_{[l]} = \{h \in \text{Hist} \mid l \subseteq h\}$ is the set of histories containing all of l.*

Proof. Suppose toward a contradiction that there is some $h \in H_l$ such that $h \not\perp_c h_2$. The latter implies, since $c \in h$ (for $c < l$) and $c \in h_2$, that $h \equiv_c h_2$. Since $l \in h \cap h_1$ and $c < l$, $h \equiv_c h_1$. By the transitivity of $\equiv_c$ we get $h_1 \equiv_c h_2$, which contradicts our premise. □

Given transitivity, $\equiv_e$ is an equivalence relation on H_e. We use the notation Π_e to indicate the partition of histories from H_e into equivalence classes according to $\equiv_e$, i.e.,

for $H \subseteq H_e$ with $H \neq \emptyset$, we have $H \in \Pi_e \leftrightarrow H$ is maximal with respect to the property that $\forall h_1, h_2 \in H \, [h_1 \equiv_e h_2]$.

It may be that in fact *all* histories from H_e are undivided at e, i.e., e is not maximal in the intersection of any two histories from H_e. In that case, we have $\Pi_e = \{H_e\}$. On the other extreme, it may be that any two histories in H_e split at e, so we have $\{h\} \in \Pi_e$ for every $h \in H_e$.

The relation of undividedness at an event is significant, as it allows us to define an entirely objective concept of 'elementary possibility at e'.

Definition 3.6 (Elementary possibility at e)**.** Let Π_e be the partition of H_e induced by $\equiv_e$. By an *elementary possibility at e* (a *basic outcome of e*) we mean a member of Π_e.

The concept of elementary possibilities deserves a few comments.

1. An elementary possibility can be represented as a set of histories. This idea is copied from 'possible worlds' theories.
2. Which English phrase shall we use for Π_e, apart from 'the elementary possibilities (open) at e'? One might think of Π_e as representing what might happen at e, or the way things might go immediately after e, or as the possible outcomes (results) of e.
3. The range of elementary possibilities open at e is therefore not an extra primitive. It is definable from the very structure of *Our World* $\langle W, < \rangle$.
4. An elementary possibility at e is always a set of histories, all of which contain e. It may be typically or even always true in *Our World* that the unit set $\{h\}$ of a history from H_e is not an elementary possibility at any e. Thus, the competing definition of an elementary possibility as such a unit set of a history would be too fine-grained (though perhaps not too fine-grained for every purpose). There are also possibilities open at an event that are not elementary. At least any union of a set of elementary possibilities at e will need to be counted as itself a possibility at e. Some notions of less immediate possibilities, or outcomes, will be discussed in Chapter 4.1.

The uniquely determined partition Π_e is a proper locus for a ground-level theory of objective transition possibilities (or outcomes) in the single case. The significance is this: the finest partition is delivered by the causal structure of *Our World*, not by human interests, language, concepts, universals, other possible worlds, or evolutionary entrenchment. The possibility in question is conditional in form (the condition being that the point event occurs), but more than that, it has a concrete foothold in *Our World*.

With the concept of local possibilities, we are finally in a position to define a local, modal, and relativity friendly notion of determinism and indeterminism.

Definition 3.7 (Determinism and indeterminism)**.** A point event, e, is indeterministic (is a choice point) if Π_e has more than one member. Otherwise, it is deterministic.

As a rhetorical variant, we may say that *Our World* is indeterministic at e. Note that on this account it makes perfectly good sense to locate indeterminism not metaphorically in a theory, but literally in our world, something very much in contrast with the dominant views in the philosophy of science.[6] The concept is also local: it makes sense to say that *Our World* was indeterministic in Boston yesterday, but might not be so in Austin tomorrow.

3.4.3 The pattern of branching of BST_{92}

Given how frugal the axioms of BST_{92} are, there are many different kinds of structures of BST_{92}. Since our aim is that the theory can have applications in physics, it will be useful to single out BST_{92} structures in which histories are isomorphic to Minkowski space-time, or to some solutions of the field equations of General Relativity. Details of these will be discussed in Chapter 9. At this stage, we will explore the pattern of branching that the BST_{92} axioms impose: If two histories split at a single choice point c, at which regions they are identified, and what do the regions of difference look like? Understandably, the histories share the past light-cone of c, and have separate future light-cones of c, but are the events space-like separated from c (forming the so-called 'wings', or the 'elsewhere' of c) in the shared region, or not? As we shall see, PCP_{92} provides a principled answer to this query.

To explore the matter further, it would be helpful to introduce some of the steps in our construction of Minkowskian Branching Structures, to be

[6] The standard approach in the philosophy of science (e.g., Earman, 2006; Butterfield, 2005) is to take determinism and indeterminism to be properties of theories, not of our world. According to that approach, roughly, a theory is called deterministic iff all of its models that have isomorphic initial segments also have isomorphic final segments. The individual models are usually taken to be separate possible worlds, so there is no notion of branching histories involved in that approach to indeterminism.

provided in Chapter 9.1. Here we put down a preliminary target definition (for the precise formulation, see Definition 9.5):[7]

Definition 3.8 (Target notion of MBS)**.** A Minkowskian Branching Structure (MBS) is a BST structure in which each history is order-isomorphic to Minkowski space-time, with the pre-causal ordering generalizing the Minkowskian ordering $<_M$ of Definition 2.1.

To be more specific, let us set out the task of producing an MBS that models a measurement of spin with two possible outcomes (idealized as histories), measured spin up or measured spin down. How does this measurement affect the causal contemporaries of the measurement? Do they belong to the intersection of the two histories, or just to one or the other? Are they ontologically indefinite or ontologically definite (if that language helps), relatively or absolutely?

Given the problem, we can consider an MBS that satisfies the following stipulations:

1. There are exactly two histories.
2. Each history is order-isomorphic to Minkowski space-time.
3. There is precisely one choice point c.

The matter is resolved by the Prior Choice Principle of BST_{92}, as the following fact testifies:

Fact 3.9. *Let $\langle W, < \rangle$ be a BST_{92} structure that contains two histories, h_1 and h_2, and exactly one choice point, c, which is maximal in $h_1 \cap h_2$. It follows from the BST_{92} Prior Choice Principle, PCP_{92}, that the 'wings' of c, i.e. $X = \{e \in W \mid e\, SLR\, c\}$, must be in the intersection $h_1 \cap h_2$ of the two histories.*

Proof. By the definition of X, no element of X has a choice point in its past. Therefore, if an element of X failed to lie in both h_1 and h_2, PCP_{92} would be violated. □

Thus, given PCP, the true picture of two Minkowski histories with exactly one choice point must be as it is displayed in Figure 3.2. The intersection of

[7] In Chapter 9.1 we explore MBSs that are BST_{NF} structures. A series of papers listed in Footnote 4 of Chapter 9 (p. 307) develops MBSs that are BST_{92} structures.

the two histories is shaded, and the upper borders belong 'on the light side' in the respective differences.

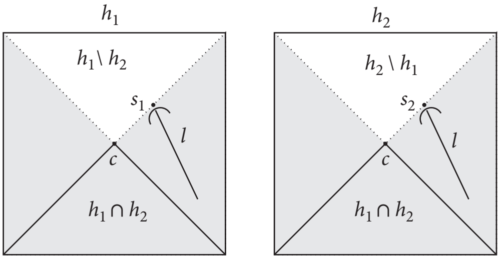

Figure 3.2 'Wings' ('elsewhere') belong to the intersection.

This formal result deserves additional comments:

(i) Observe that the difference made by the choice at c pertains only to the future of possibilities of c. It does not pertain to the causal contemporaries of c.

(ii) One might imagine that whenever there is a tiny indeterministic situation such as spin up/spin down, the entire causally unrelated universe simultaneously splits in twain. BST_{92} gives a sharp explanation of how and why this picture is generally wrong. It also offers a competing rigorous and positive theory of what is right: splitting in *Our World* fundamentally occurs at point events. A single splitting affects only its causal future, not everything above a simultaneity slice.

(iii) Thus, a single, local splitting (a single chancy event, idealized to be point-like) does not give rise to a simultaneity slice that divides h_1 into $h_1 \cap h_2$ and $h_1 \setminus h_2$. BST is thus in conflict with any interpretation of special relativity that assumes that a wave-function collapse occurring along a simultaneity surface can be effected by a single chancy event such as a measurement.[8]

A feature specific to the BST_{92} pattern of branching is the topological difference between what might be called "indeterminism without choice"

[8] In Chapter 10 we will discuss how the *additional* resources provided by BST—most prominently, modal correlations, to be discussed in Chapter 5—can be used to define different kinds of splitting. These, however, need coordinated choice points, and for each of them, the pattern of branching is as described, affecting only its causal future.

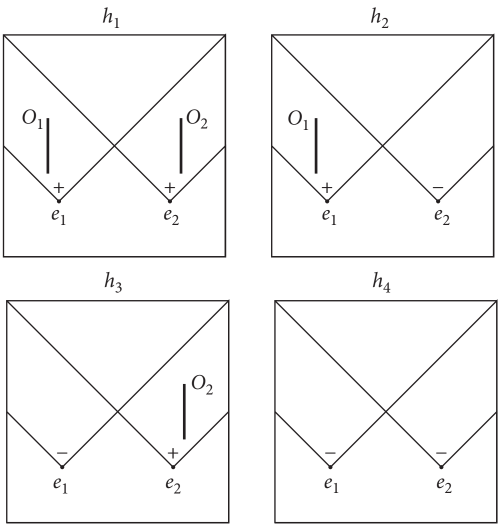

Figure 3.3 Two binary *SLR* choice points giving rise to four histories.

and "indeterminism with choice". To explain, consider again Figure 3.2 and a maximal chain, l, that traverses from the histories' overlap, $h_1 \cap h_2$, toward the histories's difference $h_1 \setminus h_2$, avoiding c. Does it have a last element in the intersection of $h_1 \cap h_2$? Note that the borders of a pair of histories that overlap do not belong to the overlap (apart from c), since they are above c. The chain l therefore has two minimal upper bounds, say, s_1 on the upper light cone of h_1 and s_2 on the upper light cone of h_2. If you are 'traveling along' this track, the situation as the track draws to a close is indeterministic: it is not determined whether you will wind up at s_1 or s_2. Still, there is no choice: the matter is entirely in the hands of your causal contemporary, c. Things are different, however, for a chain converging to a choice point, c, in the histories' overlap: this chain has a unique least upper bound (supremum), namely c. The difference between the two cases seems to be this. The only reason that l underdetermines whether s_1 or s_2 will occur is that it does not exhaust the entire past of either of these points: given the set of all proper predecessors of s_1, which includes a choice point for histories containing s_1 vs. s_2, the outcome, s_1, is uniquely determined (and analogously for s_2). In contrast, the entire past culminating in c does not suffice to decide what happens next.

Figure 3.3 illustrates the combination of choice points in the simplest case, in which two *SLR* choice points e_1 and e_2 have two outcomes each (denoted

"+" and "−"). The choices at e_1 and e_2 are uncorrelated, giving rise to a total of four histories. As we will show in Chapter 5, BST also allows for structures in which choice points are coordinated; compare Figure 5.1 on page 107.

3.4.4 Transitions

The notion of a transition is a powerful tool for discussing indeterminism. Belnap (1999) adopts the notion from von Wright (1963), who provides the basic idea of a transition as "first this, and then that", adding formal rigor. Generally, a transition is a pair $\langle I, O\rangle$, written $I \rightarrowtail O$, in which I is appropriately prior to O, and O is, in some appropriate sense, an outcome of I. There are several notions of transitions, which we discuss in Chapter 4.1. For our immediate purposes we focus here on the simplest notion of a transition, a *basic transition*, which in BST_{92} is from a point event e to one of the immediate possibilities open at e (i.e., from e to a member of the partition Π_e of the set H_e of histories containing e),

$$\tau = e \rightarrowtail H, \quad e \in W,\ H \in \Pi_e.$$

Observe that the kind of transitions that we consider are *modal* transitions. They are not merely *state* transitions. If at a certain moment, for example, there is a (real) possibility of motion, then 'remaining at rest' would cut off certain possibilities and thus be an (indeterministic) transition event of a kind that is the object of our investigation, even though there is no 'change of state'.

Basic transitions are divided up into those that witness local indeterminism, and those at which, so to speak, nothing happens. The formal distinction is provided by whether or not there are multiple immediate future possibilities open at e, or just one. Thus, at an indeterministic event e (at a choice point), the partition Π_e has more than one member (histories split at e; there are $h_1, h_2 \in H_e$ for which $h_1 \perp_e h_2$), whereas at a deterministic event e, there is only one immediate possibility for the future, whence for all $h_1, h_2 \in H_e$, we have $h_1 \equiv_e h_2$, and $\Pi_e = \{H_e\}$.

Definition 3.9 (Deterministic and indeterministic basic transitions). A *basic transition* is a pair $\langle e, H\rangle$, written $e \rightarrowtail H$, with $e \in W$ and $H \in \Pi_e$. For $h \in H_e$, we write $\Pi_e\langle h\rangle$ for the member of Π_e that contains h, so that the basic transition $e \rightarrowtail \Pi_e\langle h\rangle$ is from e to that (unique) basic outcome of

e that contains h. A basic transition is *indeterministic* iff Π_e has more than one member. On the other hand, if $\Pi_e = \{H_e\}$, then the transition $e \rightarrowtail H_e$ is called *deterministic* or *trivial*.

We denote the set of basic indeterministic transitions of a BST$_{92}$ structure $\langle W, < \rangle$ by $\mathrm{TR}(W)$, and the set of all basic transitions by $\mathrm{TR}_{\mathrm{full}}(W)$.

The set of basic transitions, whether deterministic or indeterministic, admits a natural ordering.

Definition 3.10 (Transition ordering). For $\tau_1 = e_1 \rightarrowtail H_1$, $\tau_2 = e_2 \rightarrowtail H_2$, we say that τ_1 *precedes* τ_2, written $\tau_1 \prec \tau_2$, iff ($e_1 < e_2$ and $H_2 \subseteq H_1$). The companion non-strict partial ordering is defined via $\tau_1 \preccurlyeq \tau_2 \Leftrightarrow_{\mathrm{df}} (\tau_1 \prec \tau_2 \vee \tau_1 = \tau_2)$.

We prove next that that $\prec$ is a partial ordering:

Fact 3.10. *Let* $\langle W', <' \rangle =_{\mathrm{df}} \langle \mathrm{TR}_{\mathrm{full}}(W), \prec \rangle$ *be the set of all basic transitions of a BST$_{92}$ structure* $\langle W, < \rangle$ *together with the transition ordering* $\prec$*. Then (1)* $\mathrm{TR}_{\mathrm{full}}(W)$ *is non-empty and (2)* $\prec$ *is a strict partial ordering on* $\mathrm{TR}_{\mathrm{full}}(W)$*.*

Proof. (1) Since W is non-empty, there is some $e \in W$ and hence $H_e \neq \emptyset$ (see Fact 2.1(4)), so there is a non-empty $H \in \Pi_e$, and hence there exists a transition $e \rightarrowtail H \in W'$.

(2) Since $<$ is irreflexive, $\prec$ is irreflexive as well. For transitivity, let $(e_1 \rightarrowtail H_1) \prec (e_2 \rightarrowtail H_2)$ and $(e_2 \rightarrowtail H_2) \prec (e_3 \rightarrowtail H_3)$. By transitivity of $<$ we have $e_1 < e_3$. Also, from $H_2 \subseteq H_1$ and $H_3 \subseteq H_2$ we have $H_3 \subseteq H_1$ by transitivity of $\subseteq$. Together this establishes $(e_1 \rightarrowtail H_1) \prec (e_3 \rightarrowtail H_3)$. □

We can also prove that on the assumption that $\langle W, < \rangle$ has no maxima nor minima, $\mathrm{TR}_{\mathrm{full}}(W)$ also has no maxima nor minima; see Exercise 3.2.

The following facts about alternatives to the definition of the transition ordering (Def. 3.10) will be helpful later on.

Fact 3.11. *Let* $\tau_1 = e_1 \rightarrowtail H_1$, $\tau_2 = e_2 \rightarrowtail H_2$ *be transitions in a BST$_{92}$ structure* $\langle W, < \rangle$*. (1) Generally,* $\tau_1 \prec \tau_2$ *iff* ($e_1 < e_2$ *and* $H_{e_2} \subseteq H_1$) *iff* ($e_1 < e_2$ *and for every* $h \in H_{e_2}$ *it is the case that* $H_1 = \Pi_{e_1}\langle h \rangle$)*. (2) If* τ_1 *is deterministic, then* $\tau_1 \prec \tau_2$ *iff* $e_1 < e_2$*. (3) For the non-strict companion order, we have* $\tau_1 \preccurlyeq \tau_2$ *iff* ($e_1 \leqslant e_2$ *and* $H_2 \subseteq H_1$)*.*

Proof. (1) We prove the first "iff", from which the second follows immediately (left as Exercise 3.3). Let $e_1 < e_2$. We have to show that $H_2 \subseteq H_1$ iff $H_{e_2} \subseteq H_1$.

As $H_2 \subseteq H_{e_2}$, the "$\Leftarrow$" direction is trivial. For "$\Rightarrow$", assume $H_2 \subseteq H_1$, and pick an arbitrary $h \in H_2$, so $h \in H_1$. Pick next some $h' \in H_{e_2}$. We have $e_2 \in h \cap h'$, which establishes $h \equiv_{e_1} h'$. It follows that $h' \in H_1 = \Pi_{e_1}\langle h \rangle$.

(2) The "$\Rightarrow$" direction is trivial. For "$\Leftarrow$", let $e_1 < e_2$, and assume that τ_1 is deterministic, so that $H_1 = H_{e_1}$. We have to show that $H_2 \subseteq H_1$. By downward closure of histories (Fact 2.1(5)), we have $H_{e_2} \subseteq H_{e_1}$, and $H_2 \subseteq H_{e_2}$ by definition, so that $H_2 \subseteq H_{e_2} \subseteq H_{e_1} = H_1$. The claim follows by transitivity of $\subseteq$.

(3) "$\Rightarrow$": Assume $\tau_1 \preccurlyeq \tau_2$ (i.e., either $\tau_1 \prec \tau_2$ or $\tau_1 = \tau_2$). In the first case, the claim follows from the definition of $\prec$, in the second case the claim is obvious as then, $e_1 = e_2$ and $H_1 = H_2$.

"$\Leftarrow$": Let $H_2 \subseteq H_1$, and let $e_1 \leqslant e_2$. Again there are two cases. If $e_1 = e_2$, then, as $H_1, H_2 \in \Pi_{e_1}$ and Π_{e_1} is a partition, $H_2 \subseteq H_1$ implies $H_2 = H_1$, whence $\tau_1 = \tau_2$, establishing the claim. The remaining case, $e_1 < e_2$ and $H_2 \subseteq H_1$, satisfies the definition of $\prec$ exactly as in Def. 3.10. □

Transitions figure prominently in our account of singular causation, propensities, and many other applications of BST. Importantly, it can be shown that the set of transition $\mathrm{TR}_{\mathrm{full}}(W)$ of a BST_{92} structure $\langle W, < \rangle$ together with the ordering $\prec$ of Definition 3.10 is a common BST structure. Under the further assumption that $\langle W, < \rangle$ has no minimal elements, $\langle \mathrm{TR}_{\mathrm{full}}(W), \prec \rangle$ is a $\mathrm{BST}_{\mathrm{NF}}$ structure—a notion that we now go on to define.

3.5 Introducing $\mathrm{BST}_{\mathrm{NF}}$

Similarly to our construction of BST_{92} structures, we begin with a common BST structure and then put forward a new prior choice principle, $\mathrm{PCP}_{\mathrm{NF}}$. With this principle in hand, we will define a $\mathrm{BST}_{\mathrm{NF}}$ structure as a common BST structure that satisfies $\mathrm{PCP}_{\mathrm{NF}}$. Having achieved this task in Chapter 3.5.1, we explore some of the basic features of the resulting theory in Chapters 3.5.2–3.5.3.

3.5.1 The new Prior Choice Principle and $\mathrm{BST}_{\mathrm{NF}}$ structures defined

Working with a common BST structure $\langle W, < \rangle$, we define a few concepts needed to formulate $\mathrm{PCP}_{\mathrm{NF}}$. The underlying idea of $\mathrm{BST}_{\mathrm{NF}}$ is that there should be no maximal elements in the overlap of histories. Thus, for any maximal chain l traversing from the histories' overlap $h_1 \cap h_2$ to one of its differences, say $h_1 \setminus h_2$, its segment in the overlap, $l' =_{\text{df}} l \cap h_1 \cap h_2$, has at least two history-relative suprema, $s_1 = \sup_{h_1}(l')$ and $s_2 = \sup_{h_2}(l')$. Since typically such pairs of distinct suprema abound for any two histories h_1, h_2, we should not say that the two histories split at all such pairs. We had better pick distinguished pairs, analogous to the topologically distinguished choice points of BST_{92}. A relevant observation is that a chain may be such that its pair of distinct suprema s_1, s_2 cannot be avoided, because *any* chain approaching *any one* of s_1, s_2 has (at least) two history-relative suprema, s_1 and s_2. In contrast, other chains with distinct history-relative suprema s_1, s_2 may be such that these distinct suprema can be avoided, that is, there is a different chain approaching one of s_1, s_2 that fails to have them both as history-relative suprema. In fact, given PCP_{92}, there will always be a chain that has only one history-relative supremum (see Exercise 3.4). But in common BST structures without PCP_{92}, the former case of unavoidable history-relative suprema is not excluded. Accordingly, we single out the first category of sets of history-relative suprema as choice sets, at which histories split. These observations translate into the following definitions:

Definition 3.11 (Choice set)**.** For $e \in W$, we define the *choice set based on* e, written $\ddot{e}$, to be the intersection of the sets of suprema of all chains ending in e (Def. 3.1).[9] In case e is a minimal element in W, we have $\mathscr{C}_e = \emptyset$, so we make sure that e belongs to its own choice set in this case as well.

$$\ddot{e} =_{\text{df}} \begin{cases} \{e\}, & \text{if} \quad \mathscr{C}_e = \emptyset, \\ \bigcap_{l \in \mathscr{C}_e} \mathscr{S}(l), & \text{otherwise.} \end{cases}$$

Fact 3.12. *For any $e \in W$, $e \in \ddot{e}$.*

Proof. If $\mathscr{C}_e = \emptyset$, we have $e \in \ddot{e}$ by definition. Otherwise, for any $l \in \mathscr{C}_e$, $e \in \mathscr{S}(l)$ by definition, and hence $e \in \ddot{e}$. □

We also call the choice set $\ddot{e}$ the set of *local point-wise alternatives* for e. Note that e then counts as an alternative to itself. The related notions of alternative histories and history-wise alternatives are defined via the point-wise alternatives:

[9] Our notation with the double dot over e is meant to be suggestive of a number of different history-relative suprema on top of a chain. Think of Figure 3.1(b) rotated counterclockwise by 90 degrees.

Definition 3.12 (Alternative histories and local history-wise alternatives)**.** We define the set of *alternative histories at* $\ddot{e}$, $H_{\ddot{e}}$, and the set of *local history-wise alternatives for* e, $\Pi_{\ddot{e}}$, to be

$$H_{\ddot{e}} = \{h \in \text{Hist} \mid h \cap \ddot{e} \neq \emptyset\}; \qquad \Pi_{\ddot{e}} =_{\text{df}} \{H_s \mid s \in \ddot{e}\}.$$

In order to spell out PCP_{NF}, we define two new relations between histories, splitting at a choice set and being undivided at a choice set, written $h_1 \perp_{\ddot{e}} h_2$ and $h_1 \equiv_{\ddot{e}} h_2$, respectively, in analogy to the respective BST_{92} notions.

Definition 3.13. Let h_1, h_2 be histories in $\text{Hist}(W)$, and let $e \in W$. We require as a presupposition for $h_1 \equiv_{\ddot{e}} h_2$ and $h_1 \perp_{\ddot{e}} h_2$ that $h_1, h_2 \in H_{\ddot{e}}$ (i.e., $h_1 \cap \ddot{e} \neq \emptyset$ and $h_2 \cap \ddot{e} \neq \emptyset$). Then the relations are defined as follows:

$$\begin{aligned} h_1 \equiv_{\ddot{e}} h_2 \quad &\Leftrightarrow_{\text{df}} \quad h_1 \cap \ddot{e} = h_2 \cap \ddot{e}; \\ h_1 \perp_{\ddot{e}} h_2 \quad &\Leftrightarrow_{\text{df}} \quad h_1 \cap \ddot{e} \neq h_2 \cap \ddot{e}. \end{aligned}$$

If $h_1 \perp_{\ddot{e}} h_2$, we say that the choice set $\ddot{e}$ is *a choice set for histories* h_1 *and* h_2.

With the required notions at hand, we put forward a new prior choice principle, PCP_{NF}:

Definition 3.14 (PCP_{NF})**.** Let $h_1, h_2 \in \text{Hist}(W)$, and let l be a lower bounded chain for which $l \subseteq h_1$ but $l \cap h_2 = \emptyset$. Then there is some $e \in W$ for which $e \leqslant l$ and for which the set $\ddot{e}$ of local alternatives to e satisfies $h_1 \perp_{\ddot{e}} h_2$.

Note the weak relation $e \leqslant l$ in the formulation of PCP_{NF}, in contradistinction to the strict relation in the formulation of PCP_{92} in Def. 3.4. For example, if l has just one element c (i.e., $l = \{c\}$) such that $c \in h_1 \setminus h_2$ and c is an element of a non-trivial choice set $\ddot{c} \neq \{c\}$, then the choice set for $l = \{c\}$ is just $\ddot{c}$ itself, and $h_1 \perp_{\ddot{c}} h_2$. In such a case we only have the weak ordering relation, $c \leqslant l$.

Having proposed a new prior choice principle, we can now give a full definition of the “new foundations” for BST, BST_{NF}:

Definition 3.15 (BST_{NF} structure)**.** A strict partial ordering $\langle W, < \rangle$ is *a structure of* BST_{NF} iff it is a common BST structure (Def. 2.10) for which PCP_{NF} (Def. 3.14) holds.

It can be shown that PCP_{NF} implies historical connection; see Exercise 3.5.

3.5.2 Local possibilities and the pattern of branching in $\mathrm{BST}_{\mathrm{NF}}$

In this section we ask how local possibilities are represented in $\mathrm{BST}_{\mathrm{NF}}$ and how histories branch according to this theory. To this end, we first investigate some features of the relations of splitting $\perp_{\ddot{e}}$ and being undivided $\equiv_{\ddot{e}}$ at a choice set $\ddot{e}$. A handy fact that we will use below says that a history and a choice set intersect at one point event at most:

Fact 3.13. *(1) For any $h \in \mathrm{Hist}(W)$ and for any $e \in W$, we either have $h \cap \ddot{e} = \emptyset$, or $h \cap \ddot{e} = \{e'\}$ for some $e' \in \ddot{e}$, i.e., a choice set has at most one element in common with any history. (2) The set of sets of histories $\Pi_{\ddot{e}}$ partitions $H_{\ddot{e}}$.*

Proof. (1) We have two cases here. Case 1: If e is a minimal element of W, then by Def. 3.11 we have $\ddot{e} = \{e\}$, and the result follows immediately.

Case 2: If e is not a minimal element of W, there exists some element $x < e$. We can then invoke the Hausdorff maximal principle to extend the chain $\{x, e\}$ to a maximal chain l and delete the segment of l that is above e. The remaining chain $l' =_{\mathrm{df}} l^{<e}$ is a maximal chain upper-bounded by e, i.e., $l' \in \mathscr{C}_e$. If the set $h \cap \ddot{e} \neq \emptyset$, then $l' \subseteq h$, and thus if $h \cap \ddot{e}$ were to contain more than one element, this would contradict Fact 3.2.

(2) Exhaustiveness is immediate: by definition, $\cup\Pi_{\ddot{e}} = H_{\ddot{e}}$. To prove disjointness of the elements of $\Pi_{\ddot{e}}$, let $H_1, H_2 \in \Pi_{\ddot{e}}$ such that $H_1 \neq H_2$. Then $H_1 = H_{s_1}$ and $H_2 = H_{s_2}$ for two distinct members $s_1, s_2 \in \ddot{e}$. Let $h \in H_1$; by (1), we then have $h \notin H_2$. □

The first part of this fact implies the following interrelation of the two relations, $\equiv_{\ddot{e}}$ and $\perp_{\ddot{e}}$:

Fact 3.14. *Let $e \in W$ and let $h_1, h_2 \in H_{\ddot{e}}$. Then $\equiv_{\ddot{e}}$ and $\perp_{\ddot{e}}$ are mutually exclusive and jointly exhaustive: we have $h_1 \equiv_{\ddot{e}} h_2$ iff not $h_1 \perp_{\ddot{e}} h_2$.*

Proof. Given the assumptions, we have $h_1 \cap \ddot{e} = \{s_1\}$ and $h_2 \cap \ddot{e} = \{s_2\}$ for some $s_1, s_2 \in \ddot{e}$. Now by our definitions, $h_1 \equiv_{\ddot{e}} h_2$ iff $s_1 = s_2$, and $h_1 \perp_{\ddot{e}} h_2$ iff $s_1 \neq s_2$. These are mutually exclusive and jointly exhaustive alternatives. □

Finally, we can prove a much desired consequence of the above definitions:

Fact 3.15. *The relation $\equiv_{\ddot{e}}$ is an equivalence relation on the set of alternative histories at $\ddot{e}$, $H_{\ddot{e}}$.*

Proof. Let $h_1, h_2, h_3 \in H_{\ddot{e}}$. We have to establish reflexivity, symmetry and transitivity. Reflexivity and symmetry are trivial. For transitivity, assume $h_1 \equiv_{\ddot{e}} h_2$ and $h_2 \equiv_{\ddot{e}} h_3$, so $h_1 \cap \ddot{e} = h_2 \cap \ddot{e}$ and $h_2 \cap \ddot{e} = h_3 \cap \ddot{e}$. By transitivity of identity $h_1 \cap \ddot{e} = h_3 \cap \ddot{e}$, which implies $h_1 \equiv_{\ddot{e}} h_3$. □

The immediate corollary of this Fact is that $H_{\ddot{e}}/\equiv_{\ddot{e}}$ is a partition of $H_{\ddot{e}}$. We thus define local possibilities as follows:

Definition 3.16 (Elementary possibility at $\ddot{e}$)**.** The set of elementary possibilities open at $\ddot{e}$ (or of possible outcomes of $\ddot{e}$) is the partition $H_{\ddot{e}}/\equiv_{\ddot{e}}$.

It is then immediately discernible that $H_{\ddot{e}}/\equiv_{\ddot{e}}$ is identical to the partition $\Pi_{\ddot{e}}$ of Def. 3.12:

Fact 3.16. $\Pi_{\ddot{e}} = H_{\ddot{e}}/\equiv_{\ddot{e}}$.

Proof. The claim is established by this chain of equivalences: $h_1, h_2 \in H \in (H_{\ddot{e}}/\equiv_{\ddot{e}}) \Leftrightarrow h_1 \equiv_{\ddot{e}} h_2 \Leftrightarrow \exists e \in \ddot{e}\, [e \in h_1 \cap h_2] \Leftrightarrow h_1, h_2 \in H_e \in \Pi_{\ddot{e}}$. □

In what follows, we will use the succinct notation $\Pi_{\ddot{e}}$ for the possibilities open at $\ddot{e}$. We turn next to facts shedding some light on the pattern of branching in BST_{NF} structures. Such structures, being common BST structures, satisfy historical connection just as BST_{92} structures do. The new PCP_{NF}, however, implies that the branching of histories in BST_{NF} looks different from the branching in terms of choice points in BST_{92}: there cannot be any maximal elements in the intersection of histories in a BST_{NF} structure.

Fact 3.17. *Let h_1, h_2 be two histories in a BST_{NF} structure $\langle W, < \rangle$, $h_1 \neq h_2$. Then $h_1 \cap h_2$, which is non-empty, contains no maximal elements. Accordingly, there are no choice points in a BST_{NF} structure.*

Proof. By historical connection (see Exercise 3.5), $h_1 \cap h_2 \neq \emptyset$. Assume for reductio that there is a point e that is maximal in $h_1 \cap h_2$. By assumption h_1 and h_2 are distinct histories. It follows from the contrapositive of Fact 2.1(10) that the element e is not maximal in W. By Fact 2.1(9) this means that in particular, e is not maximal in h_1. Therefore there is some element x in h_1 that lies strictly above e. By the Hausdorff maximal principle, there is a maximal lower bounded chain $l \subseteq \{x \in h_1 \mid e \leqslant x\}$. As $e \in l$, we have $\inf l = e$. Since e is not a maximal element of h_1, it follows that $l' =_{\text{df}} l \setminus \{e\} \neq \emptyset$. By Fact 3.3, e is also the infimum of l'. As we have $l' \subseteq h_1 \setminus h_2$, by PCP_{NF} there is $\ddot{c}$ with some $c_1, c_2 \in \ddot{c}$ such that $c_1 \in h_1$, $c_2 \in h_2$, $h_1 \cap \ddot{c} = c_1 \neq c_2 = h_2 \cap \ddot{c}$, and $c_1 \leqslant l'$. Then c_1 and c_2 cannot share a history (by Fact 3.13(1)).

However, since $e = \inf l'$, by the definition of inifima, $c_1 \leqslant e$, and as histories are downward closed, $c_1 \in h_2$. But we have $c_2 \in h_2$ as well, which again contradicts Fact 3.13(1). □

We also establish a fact about minimal points in BST_{NF} structures.

Fact 3.18 (Minimal elements in BST_{NF}). *A minimal point in a BST_{NF} structure belongs to all histories of that structure.*

Proof. For reductio, let e be a minimal point in a BST_{NF} structure $\langle W, < \rangle$ such that $e \in h_1 \setminus h_2$ for some histories $h_1, h_2 \in \text{Hist}(W)$. By PCP_{NF} there is a choice set $\ddot{c}$ for which $h_1 \perp_{\ddot{c}} h_2$, and so there are $c_1, c_2 \in \ddot{c}$ for which $c_i \in \ddot{c} \cap h_i$ $(i = 1, 2)$, $c_1 \leqslant e$, and $c_1 \neq c_2$. But two distinct $c_1, c_2 \in \ddot{c}$ can exist only if they are history-relative suprema of some chain $l \in \mathscr{C}_e$ (i.e., one that approaches them from below). Thus, $l < c_1 \leqslant e$, so e cannot be minimal, contradicting our assumption. □

As a consequence, in a BST_{NF} structure no two histories can branch (in the sense of Definition 3.13) at a set of minimal points. This illustrates a difference between BST_{92} structures and BST_{NF} structures: the former permit branching at minimal elements, but the latter do not.[10]

One might wonder if the BST_{NF} pattern of branching, which is different from one present in BST_{92}, affects the verdict about the set of events space-like related to a choice that was discussed above in terms of the problem of the wings. In the present context, the focus is on a single choice set $\ddot{c}$ such that two histories, h_1 and h_2 split at $\ddot{c}$, $h_1 \perp_{\ddot{c}} h_2$, and the question is whether the set of point events space-like related to an element of $\ddot{c}$, $c_i = \ddot{c} \cap h_i$, is in the histories' overlap, $h_1 \cap h_2$, or not. We invite the reader to show that in BST_{NF}, the wings are also in the overlap—see Exercise 3.6.

3.5.3 Facts about choice sets

The focus of this section are structures of BST_{NF}. We prove a few facts related to sets of local point-wise alternatives and sets of local history-wise alternatives, which will also justify our terminology. Our main result is Theorem 3.1, which states that choice sets fully capture the notion of a local alternative in BST_{NF}.

[10] As we will see, this difference implies a small limitation for the translatability of one kind of structure into the other kind; see Theorem 3.3.

Fact 3.19. *Let there be h_1, h_2 and $e \in W$ such that $h_1 \perp_{\ddot{e}} h_2$. Then there is no $c < e$ for which $h_1 \perp_{\ddot{c}} h_2$.*

Proof. Suppose that $e_i \in \ddot{e}$ are distinct, where $h_i \cap \ddot{e} = \{e_i\}$ $(i = 1, 2)$, and let $c \leqslant e$. Since e is a member of some history h_3, by the Hausdorff maximal principle we can extend the chain $\{c\}$ to a maximal chain l upper-bounded by e, which yields $\sup_{h_3} l = e$. This chain l is a member of $\mathscr{C}_e$, so by our supposition that $e_1 \in \ddot{e}$, it follows that there is some history h such that $\sup_h l = e_1$. By the definition of supremum, this means that $l \leqslant e_1$, and in particular, $c < e_1$. Since $e_1 \in h_1$, it follows from the downward closure of histories that $c \in h_1$, and thus we may apply Fact 3.13(1) to deduce that $h_1 \cap \ddot{c} = \{c\}$. A symmetrical argument can be made to deduce that $h_2 \cap \ddot{c} = \{c\}$, from which we observe that $h_1 \cap \ddot{c} = \{c\} = h_2 \cap \ddot{c}$, which is precisely the definition of undividedness at $\ddot{c}$. The result then follows immediately from Fact 3.14. □

Lemma 3.1. *Let $s \in \ddot{e}$ for some $e \in W$. Then we have $x < s$ iff $x < e$, i.e., $\mathscr{P}_e = \mathscr{P}_s$.*

Proof. If $e = s$, there is nothing to prove. Also, if e is a minimal element of W, then $s = e$, and there is nothing to prove either. We thus assume below that e is not a minimal element of W and $e \neq s$.

"$\Leftarrow$": Let $s \in \ddot{e}$, and let $x < e$. By the Hausdorff maximal principle, there is some chain $l \in \mathscr{C}_e$ for which $x \in l$. As $s \in \ddot{e}$, we know that there is some $h \in$ Hist for which $\sup_h l = s$. We cannot have $s \in l$: otherwise, for h' witnessing $\sup_{h'} l = e$, we would have $\{e, s\} \subseteq h'$, contradicting Fact 3.2. Thus, $l < s$, which implies $x < s$.

"$\Rightarrow$": Let $s \in \ddot{e}$, and let $x < s$. Assume for reductio that $x \not< e$. We first show that under this assumption, x and e cannot share a history. Assume otherwise, and let $h_1 \in H_e \cap H_x$. Let $h_2 \in H_s$. Take some $l \in \mathscr{C}_e$ (it exists since e is not a minimal element of W). We have $x \in h_1$, $x \in h_2$ (by downward closure of histories, as $x < s$), and $l \subseteq h_1$ (as $e \in h_1$). Now, as $s \in h_2$ and $s \in \ddot{e}$, we have $l < s$, so by downward closure of histories, $l \subseteq h_2$ as well. Noting that $\sup_{h_2} \{x\} = x < s = \sup_{h_2} l$, Weiner's postulate implies that $\sup_{h_1} \{x\} = x < e = \sup_{h_1} l$, contradicting the assumption that $x \not< e$.

Under our reductio assumption, we must thus have that x and e do not share a history. Choose some $h_1 \in H_e$ and some $h_2 \in H_s$. Since $s \in \ddot{e}$ and $e \neq s$, by Facts 3.13 and 3.14, $h_1 \perp_{\ddot{e}} h_2$. Moreover, by the downward closure of histories, we have $x \in h_2$, and as $e \in h_1$ and x and e do not share a history,

$x \notin h_1$. By PCP_{NF} applied to $x \in h_2 \setminus h_1$, there is $c \in W$ such that $h_1 \perp_{\ddot{c}} h_2$ and $c \leqslant x$, and hence $c < s$ and $c \in h_2$. And there is $c' \in \ddot{c}$ such that $c' \in h_1$. Picking $I \in \mathscr{C}_e$ and $J \in \mathscr{C}_c$, $I, J \subseteq h_1 \cap h_2$ and observing $c = \sup_{h_2} J < \sup_{h_2} I = s$, Weiner's postulate implies $c' = \sup_{h_1} J < \sup_{h_1} I = e$.

Fact 3.19 says that since $c' < e$ and $h_1 \perp_{\ddot{e}} h_2$, it cannot be the case that $h_1 \perp_{\ddot{c}'} h_2$. Thus h_1 and h_2 are undivided at c'. However, by definition of undividedness this means that $h_1 \cap \ddot{c}' = h_2 \cap \ddot{c}' = \{c'\}$, and thus $c' \in h_2$, which contradicts $h_2 \cap \ddot{c} = \{c\}$. □

With Lemma 3.1 to hand, it is easy to see that an element of a non-trivial choice set is always a minimal element in the difference of some two histories that split at this choice set.

Fact 3.20. *Let $h_1 \perp_{\ddot{c}} h_2$. Then c_1, the unique element of $h_1 \cap \ddot{c}$, is a minimal element in $h_1 \setminus h_2$, and c_2, the unique element of $h_2 \cap \ddot{c}$, is a minimal element in $h_2 \setminus h_1$.*

Proof. To prove that c_1 is a minimal element in $h_1 \setminus h_2$, let us assume for reductio that there is $e \in h_1 \setminus h_2$ such that $e < c_1$. By Lemma 3.1, $e < c_2$. But $c_2 \in h_2$, and by downward closure of histories, also $e \in h_2$, contradicting our assumption. The argument that c_2 is a minimal element in $h_2 \setminus h_1$ is exactly analogous. □

It follows, moreover, that for any two histories h_1, h_2 in a BST_{NF} structure, there is a minimal element in their difference, $h_1 \setminus h_2$.

Fact 3.21. *Let $h_1, h_2 \in \text{Hist}$, with $h_1 \neq h_2$. Then there is a minimal element in $h_1 \setminus h_2$.*

Proof. Since histories are maximal, there is $e \in h_1 \setminus h_2$. By PCP_{NF} there is $\ddot{c}$ such that $h_1 \perp_{\ddot{c}} h_2$. Then by Fact 3.20, $c_1 \in \ddot{c} \cap h_1$ is a minimal element in $h_1 \setminus h_2$. □

Another fact concerns maximal chains in the difference of two histories:

Fact 3.22. *Let $l \subseteq h \setminus h'$ be a maximal chain in the difference of histories h and h'. PCP_{NF} guarantees that there is a choice set $\ddot{c}$ such that $h \perp_{\ddot{c}} h'$, and for $c \in \ddot{c} \cap h$, we have $c \leqslant l$. We claim that in fact, $c = \inf l$.*

Proof. By PCP_{NF}, $c \leqslant l$, so by the infima postulate, $i =_{\text{df}} \inf l$ exists. As i is the greatest lower bound of l and c is a lower bound of l, we have $c \leqslant i$. We need to show that $c = i$. Assume otherwise, i.e., $c < i$. Note that this implies that $c \notin l$. We also have $c \in h$. Given that $c \in \ddot{c}$ and $h \perp_{\ddot{c}} h'$, we have that $c \notin h'$ (by Def. 3.13). But then $l \cup \{c\}$ is also a chain ($c < i \leqslant l$) that lies wholly in $h \setminus h'$, and that chain properly extends l. This contradicts the maximality of t. □

Given Lemma 3.1, it is also not difficult to see that for $s \in \ddot{e}$, a chain ends in e iff it ends in s:

Fact 3.23. *Let $s \in \ddot{e}$. Then we have $l \in \mathscr{C}_s$ iff $l \in \mathscr{C}_e$.*

Proof. If $e = s$, there is nothing to prove. Also, if e is a minimal element of W, then $s = e$, and there is nothing to prove either. We thus assume below that e is not a minimal element of W and $e \neq s$. The former implies $\mathscr{C}_e \neq \emptyset$.

"$\Leftarrow$": Given $s \in \ddot{e}$ and $l \in \mathscr{C}_e$, by the definition of $\ddot{e}$ there is some history h for which $\sup_h l = s$. We argue that $s \notin l$. Suppose otherwise. Since l is upper-bounded by e, this means that $s < e$. Then for any history h containing e, we would have that $\{s, e\} \subseteq \ddot{e} \cap h$, which contradicts Fact 3.13(1). Then it must be the case that $s \notin l$, which implies that $l \in \mathscr{C}_s$.

"$\Rightarrow$": Let $s \in \ddot{e}$, and let $l \in \mathscr{C}_s$, i.e., $l < s$ and for some $h \in H_s$, $\sup_h l = s$. By Lemma 3.1, $l < e$. Take some $h' \in H_e$, and pick some $k \in \mathscr{C}_e$, so $k < e$, and hence $k \subseteq h'$. By the same Lemma, since $s \in \ddot{e}$, we have $k < s$, which gives us $k \subseteq h$. We claim that $\sup_h k = s$. For, if there were some x in h such that $k \leqslant x < s$, then Lemma 3.1 implies that $x < e$. Then $k \leqslant x < e$ in h', which means that $e \neq \sup_{h'} k$, contradicting $k \in \mathscr{C}_e$. We thus have $\sup_h l = s = \sup_h k$. Hence by Weiner's postulate, we also have $\sup_{h'} l = \sup_{h'} k = e$, and therefore, $l \in \mathscr{C}_e$. □

Given the previous results, we see that the set $\ddot{e}$ is independent of the witness chosen:

Fact 3.24. *Consider a BST$_{NF}$ structure $\langle W, < \rangle$. Let $s \in \ddot{e}$. Then $e \in \ddot{s}$.*

Proof. If e is a minimal element of W, then $e = s$ and we are done. We thus assume that e is not minimal in W, and hence $\mathscr{C}_e \neq \emptyset$. By Lemma 3.1, $\mathscr{C}_s \neq \emptyset$ as well.

Let $s \in \ddot{e}$. We have to show that $e \in \ddot{s}$, i.e., $e \in \mathscr{S}(l)$ for all $l \in \mathscr{C}_s$. Thus consider an arbitrary $l \in \mathscr{C}_s$. By Fact 3.23, $l \in \mathscr{C}_e$. Now take some $h \in H_e$; we have $\sup_h l = e$, i.e., $e \in \mathscr{S}(l)$. As l was arbitrary, we have $e \in \ddot{s}$. □

Lemma 3.2. *We have $s \in \ddot{e}$ iff $\ddot{e} = \ddot{s}$.*

Proof. "$\Leftarrow$": Immediate, since $s \in \ddot{s}$ by Fact 3.12.

"$\Rightarrow$": Let $s \in \ddot{e}$. For $s = e$ there is nothing to prove, so suppose that $s \neq e$.

"$\subseteq$": Let $x \in \ddot{e}$. We have to show that $x \in \ddot{s}$, i.e., that $x \in \mathscr{S}(l)$ for all $l \in \mathscr{C}_s$. Thus, take some $l \in \mathscr{C}_s$. By Fact 3.23, $l \in \mathscr{C}_e$, and as $x \in \ddot{e}$, we have $x \in \mathscr{S}(l)$.

"$\supseteq$": Let $x \in \ddot{s}$. Take some $l \in \mathscr{C}_e$. As above, by Fact 3.23, $l \in \mathscr{C}_s$, and as $x \in \ddot{s}$, we have $x \in \mathscr{S}(l)$. □

Having prepared the groundwork, we can now finally fully justify calling the partition $\Pi_{\ddot{e}}$ the set of local history-wise alternatives: the set of sets of histories $\Pi_{\ddot{e}}$ partitions the set of histories containing $\mathscr{P}_e$. That is, any history containing the whole proper past of e ends up in exactly one of the elements of $\Pi_{\ddot{e}}$.

Theorem 3.1. *Let $e \in W$. Then $\Pi_{\ddot{e}}$ partitions $H_{[\mathscr{P}_e]}$, i.e.: (1) $\bigcup \Pi_{\ddot{e}} = H_{\ddot{e}} = H_{[\mathscr{P}_e]}$ and (2) for $H_1, H_2 \in \Pi_{\ddot{e}}$, if $H_1 \neq H_2$, then $H_1 \cap H_2 = \emptyset$.*

Proof. (1) We have to show that $\bigcup \Pi_{\ddot{e}} = H_{[\mathscr{P}_e]}$. Note that $\bigcup \Pi_{\ddot{e}} = H_{\ddot{e}}$ by Fact 3.13(2).

"$\subseteq$": Take $h \in \bigcup \Pi_{\ddot{e}}$, i.e., $h \in H_s$ for some $s \in \ddot{e}$. By the definition of $\mathscr{P}_s$, we have $\mathscr{P}_s \subseteq h$, and by Lemma 3.1, $\mathscr{P}_e \subseteq h$. Thus, $h \in H_{[\mathscr{P}_e]}$.

"$\supseteq$": We need to consider two cases.

Case 1: Event e is minimal in W. By definition this means that $\mathscr{P}_e = \emptyset$. Accordingly, $H_{[\mathscr{P}_e]} = \text{Hist}(W)$. By Fact 3.18 the minimal point e belongs to every history of W, so it is also the case that $H_{\ddot{e}} = H_e = \text{Hist}(W)$, which gives us the desired identity.

Case 2: Event e is not minimal in W, hence $\mathscr{C}_e \neq \emptyset$. Consider $h \in H_{[\mathscr{P}_e]}$, which implies $\mathscr{P}_e \subseteq h$. By Fact 3.5, for all $l \in \mathscr{C}_e$ we have $l \subseteq h$. Take some $l_0 \in \mathscr{C}_e$, and let $s =_{\text{df}} \sup_h l_0$. We show that s is the h-relative supremum of any chain from $\mathscr{C}_e$. Fix some $h' \in H_e$. Take any $l \in \mathscr{C}_e$. We have $\sup_{h'} l = e = \sup_{h'} l_0$, and thus by Weiner's postulate we also have $\sup_h l = \sup_h l_0 = s$. Thus we have $s \in \mathscr{S}(l)$ for any $l \in \mathscr{C}_e$, which implies $s \in \ddot{e}$. As $h \in H_s$, we have $h \in \bigcup \Pi_{\ddot{e}}$.

(2) This follows from Fact 3.13(2), as $H_{\ddot{e}} = H_{[\mathscr{P}_e]}$ (by item (1) of this Fact). □

The main message of the constructions studied in this section is that some $e \in W$ generate a non-trivial choice set $\ddot{e}$, in the sense that $\ddot{e} \neq \{e\}$. Such a set $\ddot{e}$ indeed consists of local point-wise alternatives to e. We can think of a choice set as a set of "indeterministic transitions", and each choice set induces a set of history-wise alternatives for e, namely $\Pi_{\ddot{e}}$. Finally, PCP_{NF}

requires that any two histories split at a choice set. So, in BST_{NF} the basic concepts of branching histories still apply, but in a slightly different way from BST_{92}. As we will later show (Chapter 4.4), this has some beneficial topological consequences.

3.6 BST_{92} or BST_{NF}: Does it matter?

The difference between BST_{92} and BST_{NF} amounts to the issue of whether there is a maximal element in the overlap of histories or a minimal element in the difference of histories (cf. Facts 3.6 and 3.21). This may seem to be a minor issue, and one may accordingly doubt whether the difference matters. Indeed, in many applications of BST to general philosophical problems, as well as to some problems in the philosophy of physics, the issue does not seem to have any bearing. It becomes important, however, when BST is used to model space-times of general relativity (GR), as then topological questions come to the fore. For a precise evaluation of whether BST_{92} or BST_{NF} can accommodate the topological requirements of GR space-times, we first need to describe BST structures topologically.[11] There is a natural topology on BST structures, both BST_{92} and BST_{NF}, the so-called diamond topology. We will describe this topology in technical detail when we return to topological issues in Chapter 4.4. In Chapter 3.6.1 which follows, we remain on an intuitive level, adding some promissory notes that will be substantiated later. After our overview of topology in BST, in Chapter 3.6.2 we then introduce wide-ranging translatability results for the two frameworks of BST_{92} and BST_{NF}. These can be read as showing that a choice between the two frameworks can be left a matter of pragmatic choice.

3.6.1 Topological issues: An overview

For representing space-times, physics uses so-called differential manifolds that have a number of defining topological features. In particular, differential manifolds have two properties that are hard to satisfy in branching structures.

[11] Topological worries about the appropriateness of BST were raised, e.g., by Earman (2008). Jeremy Butterfield asked about reasons to assume non-Hausdorff branching and its compatibility with space-time physics already in 2001 (Butterfield, personal communication with TP).

First, by definition, a differential manifold is locally Euclidean, which means that each point of the manifold has a neighborhood that can be mapped continuously onto an open subset of $\mathbb{R}^n$ (in realistic applications, $n = 4$; for the precise statement, see Def. 4.16). Local Euclidicity is standardly presupposed (often without explicitly mentioning the condition by name) when the notion of a space-time manifold is introduced. On such a manifold, local coordinates are defined via so-called charts (see, e.g., Wald, 1984, pp. 12f.): at each point of the manifold, it is possible to find a neighborhood that is homeomorphic to some open set of $\mathbb{R}^n$, and the respective mapping induces the coordinates. If a topological space is not locally Euclidean, it is not possible to assign coordinates in this way. It is hard to relax this requirement, as space-time points without some coordinates go against both common sense and the practice of physics.

Local Euclidicity is a challenge for BST. Given the frugality of the BST axioms, BST structures come in many varieties, so hoping that their topology will always be locally Euclidean is not realistic anyway. One can reasonably hope, however, that local Euclidicity should transfer from the individual histories to the whole global structure. More precisely, if for each history h of a BST_{92} structure the history-relative topology $\mathscr{T}_h$ is locally Euclidean, then the global topology $\mathscr{T}$ should be locally Euclidean as well. The underlying thought is that if we have a collection of physically reasonable space-times, each with an assignment of coordinates, then a BST analysis of indeterminism should not destroy the coordinate assignment.

Unfortunately, local Euclidicity is not preserved in BST_{92} as one moves from the history-relative topologies to the global topology. In fact, barring trivial one-history cases, BST_{92} structures are never locally Euclidean with respect to their natural topology. The reason is that a neighborhood of a maximal element in the intersection of two histories cannot be appropriately mapped onto $\mathbb{R}^n$. A case in point is the simple two-history model depicted in Figure 3.1(a) on p. 44. The two histories are h_1^a and h_2^a, and their overlap has a maximal element $[\langle 0,1\rangle] = [\langle 0,2\rangle]$. By the natural topology on the histories, which is the topology of the real line, the open sets of $\mathscr{T}_{h_i^a}$ $(i = 1,2)$ are either of the form $\{[\langle x,i\rangle] \mid x \in (c,d)\}$, for some open interval $(c,d) \subseteq \mathbb{R}$, or they are unions of such sets. As every element of the history h_i^a belongs to a set $\{[\langle x,i\rangle] \mid x \in (c,d)\}$, and such a set is trivially homeomorphic to an open interval of $\mathbb{R}$, $\mathscr{T}_{h_i^a}$ is locally Euclidean of dimension 1. Let us now turn to the global topology $\mathscr{T}$ on M_a and consider an open neighborhood (in $\mathscr{T}$) of the branching point $[\langle 0,1\rangle]$. Each neighborhood of that point must extend somewhat to the trunk and to both of the arms. Accordingly, it must contain subsets $\{[\langle x,1\rangle] \mid x \in (c,d)\}$ and $\{[\langle x,2\rangle] \mid x \in (c,d')\}$, with $c < 0$ and $d, d' > 0$. A fork of that sort, however, cannot be homeomorphically mapped onto an open interval of the real line. Thus, the global topology of M_a is not locally Euclidean, despite the fact that each history-relative topology is.

Note that no such problem arises for the structure M_b of Figure 3.1(b), which is a BST_{NF} structure in which the intersection of the two histories does not contain a maximum. In fact, as we will show in Chapter 4.4, BST_{NF} vindicates the idea that if one starts with locally Euclidean histories (space-times) that allow for the assignment of spatio-temporal coordinates, one does not destroy that feature by analyzing indeterminism within the framework of Branching Space-Times.

The second property that individual space-times satisfy, but which is typically violated both by BST_{92} and by BST_{NF} structures, is a topological separation property known as the Hausdorff property (see, e.g., Wald, 1984, p. 12). The property requires that any two points of a topology's base set have non-overlapping open neighborhoods. In contrast to the requirement of local Euclidicity, however, a failure of the Hausdorff property in BST structures is not a troubling one. After all, in BST, individual space-times are represented by single histories, and it can be proved under modest assumptions that histories are Hausdorff both in BST_{92} and BST_{NF} (see Chapter 4.4). The non-Hausdorffness of the global topology of a BST structure simply reflects the fact that such a structure typically brings together multiple space-times, as it represents a number of alternative spatio-temporal developments.

3.6.2 Translatability results: An overview

We now present a number of theorems that show that there is a systematic way of translating branching structures of one kind into the branching structures of the other kind, and vice versa. In this sense, we can leave the question open as to what branching is really like. A key motivation for working out an ecumenical position is that there does not seem to be a really convincing argument for preferring one of the two options mentioned earlier. In the original paper developing BST_{92} (Belnap, 1992), the decision in favor of maxima in the intersection of histories is commented as follows:

> Finally, let me explicitly note that on the present theory, and in the presence of the postulates of this section, a causal origin has always 'a last point of indeterminateness' (the choice point) and never 'a first point of

> determinateness'. I find the matter puzzling since it's neither clear to me how an alternate theory would work nor clear what difference it makes. (Belnap, 1992, p. 428)

This feeling of puzzlement also lies behind some of the objections to BST_{92}: the objectors ask about the reasons for assuming a specific pattern of branching, or they are skeptical whether that pattern is compatible with the physics of space-time. To such objections, the results given in this section answer that it is almost always possible to translate a branching structure with one pattern of branching into a structure with the other pattern of branching.[12] Thus, if we have a BST_{92} structure modeling some phenomena and are worried that it is non-Euclidean, we will dispense with the worry by translating it into a $\mathrm{BST}_{\mathrm{NF}}$ structure. And in the opposite direction, if we prefer for some reasons (most likely, for simplicity) the BST_{92} framework, we are always able to transform a $\mathrm{BST}_{\mathrm{NF}}$ structure into that framework. In order to keep our text concise, here we just give the statement of relevant theorems and facts, while putting all the required proofs in Appendix A.2.

The translatability results simplify the use of the BST structures in this book, as we need not develop the whole machinery for BST_{92} and for $\mathrm{BST}_{\mathrm{NF}}$ in parallel. If a topic at hand is not related to general relativity, we always use BST_{92} structures, as they are somewhat easier to handle and also because much of our earlier work on BST and its applications applied that framework. We will provide some hints to help the reader to connect to the other framework, however. The exception is Chapter 9 which deals with the space-times of General Relativity and a related topological issue: there we rely exclusively on $\mathrm{BST}_{\mathrm{NF}}$ structures.

Our first result concerns the set of transitions of a BST_{92} structure, which is needed to define a full transition structure; later we argue that it has the sought-for properties.

Definition 3.17 (The Υ transform as the full transition structure of a BST_{92} structure.). Let $\langle W, < \rangle$ be a BST_{92} structure. Then we define the transformed structure, $\Upsilon(\langle W, < \rangle)$, to be the full transition structure (including trivial transitions) together with the transition ordering $\prec$ from Def. 3.10, as follows:

$$\Upsilon(\langle W, < \rangle) =_{\mathrm{df}} \langle W', \prec \rangle, \text{ where } W' =_{\mathrm{df}} \mathrm{TR}_{\mathrm{full}}(W) = \{e \rightarrowtail H \mid e \in W, H \in \Pi_e\}.$$

[12] A small qualification applies if there are maxima or minima; see the Lemmas and Theorems below.

It turns out that the common BST properties of Def. 2.10 still hold for the Υ transform of a BST_{92} structure:

Lemma 3.3. *Let $\langle W, < \rangle$ be a BST_{92} structure without minima. Then its full transition structure $\Upsilon(\langle W, < \rangle)$ is still a common BST structure according to Definition 2.10.*

Further we have the already advertised results:

Lemma 3.4. *Let $\langle W, < \rangle$ be a BST_{92} structure without minima. Then that structure's full transition structure $\langle W', <' \rangle =_{df} \Upsilon(\langle W, < \rangle)$ satisfies the PCP_{NF} as in Definition 3.14.*

Taken together, these Lemmas yield our first translatability result (note that we need to restrict to BST_{92} structures without minimal elements).

Theorem 3.2. *Let $\langle W, < \rangle$ be a BST_{92} structure without minima. Then that structure's full transition structure $\Upsilon(\langle W, < \rangle)$ is a BST_{NF} structure.*

It turns out that, in the other direction, there is also a fairly simple translation, viz., lumping together all the elements of a choice set to form a single point.[13]

Definition 3.18 (The Λ transformation from BST_{NF} to BST_{92}.). Let $\langle W, < \rangle$ be a BST_{NF} structure. Then we define the companion Λ-transformed ("collapsed") structure as follows:

$$\Lambda(\langle W, < \rangle) =_{df} \langle W', <' \rangle, \quad \text{where}$$
$$W' =_{df} \{\ddot{e} \mid e \in W\};$$
$$\ddot{e}_1 <' \ddot{e}_2 \quad \text{iff} \quad e_1' < e_2' \quad \text{for some} \quad e_1' \in \ddot{e}_1, e_2' \in \ddot{e}_2.$$

With the Λ transform we can prove the desired translatability results in the other direction, from BST_{NF} to BST_{92}. Mirroring the situation for the Υ transform results that required no minimal elements, here we have to work under the provision that the given BST_{NF} model contains no maximal elements.

Lemma 3.5. *Let $\langle W, < \rangle$ be a BST_{NF} structure without maxima. Then its Λ-transform, $\langle W', <' \rangle =_{df} \Lambda(\langle W, < \rangle)$, is a common BST structure.*

[13] Graphically, we have chosen "Λ", which suggests pulling elements (of a choice set) together into one, as the reverse of "Υ", which suggests fanning out elements from a common base (viz., the choice point).

Lemma 3.6. *The* Λ*-transform* $\Lambda(\langle W, <\rangle)$ *of a BST*$_{NF}$ *structure without maxima* $\langle W, <\rangle$ *satisfies the BST*$_{92}$ *prior choice principle.*

Theorem 3.3. *The* Λ*-transform* $\Lambda(\langle W, <\rangle)$ *of a BST*$_{NF}$ *structure without maxima* $\langle W, <\rangle$ *is a BST*$_{92}$ *structure.*

We can even go full circle. As there is a way to get from BST$_{92}$ structures without minimal elements to BST$_{\mathrm{NF}}$ structures, and a way to get from BST$_{\mathrm{NF}}$ structures without maximal elements to BST$_{92}$ structures, the question arises as to where we end up when we concatenate these transformations. We can show that, as hoped, we return to where we started: the resulting structures are order-isomorphic to the ones we started with. For simplicity's sake, we work with structures without maximal or minimal elements.

Consider first the direction from BST$_{92}$ to BST$_{\mathrm{NF}}$ to BST$_{92}$.

Theorem 3.4. *The function* $\Lambda \circ \Upsilon$ *is an order isomorphism of BST*$_{92}$ *structures without maximal or minimal elements: Let* $\langle W_1, <_1\rangle$ *be a BST*$_{92}$ *structure without maximal or minimal elements, let* $\langle W_2, <_2\rangle =_{\mathrm{df}} \Upsilon(\langle W_1, <_1\rangle)$*, and let* $\langle W_3, <_3\rangle =_{\mathrm{df}} \Lambda(\langle W_2, <_2\rangle)$*. Then there is an order isomorphism* φ *between* $\langle W_1, <_1\rangle$ *and* $\langle W_3, <_3\rangle$*, i.e., a bijection between* W_1 *and* W_3 *that preserves the ordering. Accordingly,* $\langle W_3, <_3\rangle$ *has no minima and no maxima.*

The result in the opposite direction, that is, from BST$_{\mathrm{NF}}$ to BST$_{92}$ to BST$_{\mathrm{NF}}$, also holds:

Theorem 3.5. *The function* $\Upsilon \circ \Lambda$ *is an order isomorphism of BST*$_{NF}$ *structures without maximal or minimal elements: Let* $\langle W_1, <_1\rangle$ *be a BST*$_{NF}$ *structure without maximal or minimal elements, let* $\langle W_2, <_2\rangle =_{\mathrm{df}} \Lambda(\langle W_1, <_1\rangle)$*, and let* $\langle W_3, <_3\rangle =_{\mathrm{df}} \Upsilon(\langle W_2, <_2\rangle)$*. Then there is an order isomorphism* φ *between* $\langle W_1, <_1\rangle$ *and* $\langle W_3, <_3\rangle$*, i.e., a bijection between* W_1 *and* W_3 *that preserves the ordering. Accordingly,* $\langle W_3, <_3\rangle$ *has no minima and no maxima.*

3.7 Exercises to Chapter 3

Exercise 3.1. Prove that there is a choice point for any two histories in a BST$_{92}$ structure (i.e., a maximal element in their overlap).

Hint: Use the fact that histories are maximal and apply PCP$_{92}$.

Exercise 3.2. Prove the following extension of Fact 3.10: For a BST_{92} structure $\langle W, < \rangle$ that has neither maximal nor minimal elements, its full transition structure $\langle W', <' \rangle =_{df} \langle TR_{full}(W), \prec \rangle$ has no maxima nor minima either.

Hint: Take an appropriate initial from W and a fitting history to define the outcome of a witnessing transition. (A full proof is given in Appendix B.3.)

Exercise 3.3. Prove the second "iff" in Fact 3.11(1).

Exercise 3.4. Prove that there are no non-trivial choice sets in BST_{92} structures.

Hint: For reductio, assume $\ddot{e} \subseteq W$, with $e, e' \in \ddot{e}$ and $e \neq e'$. There are histories h, h' with $e \in h, e' \in h'$. Since $e \notin h'$ (why?), we have $e \in h \setminus h'$, to which we apply PCP of BST_{92} to get some $c < e$ such that $h \perp_c h'$. Pick next a chain $l \in \mathscr{C}_e$ such that $c \in l$. Observe then that $\sup_{h'}(l) \neq e'$ (why?). This proves $e' \notin \ddot{e}$ (why?) which contradicts the reductio assumption.

Exercise 3.5. Prove that the BST_{NF} prior choice principle (Def. 3.14) implies historical connection.

Hint: By maximality of histories, for any two histories h_1, h_2 there is e such that $e \in h_1 \setminus h_2$. Apply then PCP_{NF} to e to obtain a choice set $\ddot{c}$, at which h_1 and h_2 split. There must then be two different $c_1, c_2 \in \ddot{c}$. By definition, they are history h_1- and h_2-relative suprema of chains from $\mathscr{C}_{c_1}$, where these chains are contained in $h_1 \cap h_2$.

Exercise 3.6. Discuss the problem of wings in BST_{NF} structures (cf. Figure 3.2). Show that the wings are in the histories' overlap.

Hint: produce a proof analogous the proof of Fact 3.9, using PCP_{NF} at some appropriate stage.

Exercise 3.7. Let $\langle W, < \rangle$ satisfy Postulates 2.1–2.5. Let l be an upper-bounded chain, and let $e =_{df} \sup_{h'}(l)$. Then for every history h of W containing the chain l, if e lies in h, then $e = \sup_h(l)$.

Hint: Derive a contradiction from the assumption that there is an upper bound of l below e. (A full proof is given in Appendix B.3.)