7
Probabilities

In the preceding chapters we laid down a comprehensive formal framework for describing concrete spatio-temporal events in an indeterministic setting. We defined a number of different types of events and introduced their occurrence propositions. These, in turn, allowed us to define algebraic operations on events, such as union, intersection, or complement. We showed how transition-events can be defined on the basis of initial and outcome events of different types. Transitions are a handy concept to discuss indeterminism, as local alternatives to a transition represent indeterminism in a concrete way. There is a particularly simple type of transition, the so-called basic transitions. These are the basic indeterministic building blocks of BST structures. For a transition $I \rightarrowtail \mathcal{O}^*$ to a singular, non-disjunctive outcome, we defined the *causae causantes* to be a certain set of basic transitions, $CC(I \rightarrowtail \mathcal{O}^*)$, and we argued that these *causae causantes* play the causal role of bringing about the transition in question.[1] To support this claim, we showed that the *causae causantes* for a given transition satisfy Mackie's inus (or some inus-like) conditions: At a junction at which an outcome could be rendered impossible, a *causa causans* keeps the occurrence of the outcome possible, but the *causa causans* need not necessitate the outcome. *Causae causantes* thus represent an objective notion of causation under indeterminism

Given these features and defined notions, in this chapter we will show that BST also provides a promising background for a theory of propensities (i.e., of objective single-case probabilities).

7.1 Two conditions of adequacy and two crucial questions

The key ingredients for a theory of objective single-case probabilities in BST have already been supplied in the previous chapters: BST combines

[1] As detailed in Chapter 6.3 (see Def. 6.2), a transition to a disjunctive outcome calls for an additional set-theoretical layer: $CC(I \rightarrowtail \check{\mathbf{O}})$ is identified with the family of the reduced sets of *causae causantes* to the "ingredient" transitions, $I \rightarrowtail \hat{O}_\gamma$, where $\check{\mathbf{O}} = \{\hat{O}_\gamma \mid \gamma \in \Gamma\}$.

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possibilities with space and time, it provides a way of defining algebraic operations on concrete events, and it allows for a formal analysis of causation in indeterministic settings. Our aim in this chapter is to use these resources to lay down a general framework for probabilities that does justice to the indeterministic and spatio-temporal features of our world. We will propose a rigorous formal theory of objective single-case probabilities (propensities), in which we define probabilities as graded possibilities. Probability, in other words, codes the degree to which a given event is possible.[2]

In parallel to the development of our theory of causation in Chapter 6, we motivate the overall features of our theory of probabilities through considerations of conditions of adequacy and via the answer to two crucial questions about the representation of probability structures in BST.

7.1.1 Two conditions of adequacy

We impose two conditions of adequacy for our approach. The first is a purely formal condition, one which will ensure that our approach stays within a strictly mainstream notion of probability theory: we require that our theory should follow the axioms of standard (Kolmogorovian) probability theory. We provide the standard definition of a probability space here for later reference.

Definition 7.1 (Probability space). A *probability space* is a triple $\langle S, \mathscr{A}, p \rangle$, where S is the countable base set, or sample space, $\mathscr{A}$ is a σ-algebra of subsets of S (i.e., a set of subsets of S with the operations of union, intersection, and complement defined on $\mathscr{A}$, and which is closed under countable union), and p is a normalized, σ-additive measure; that is, a function $p : \mathscr{A} \mapsto [0,1]$ such that $p(S) = 1$ and for a family $\{a_i \mid i \in \Gamma\}$ of disjoint elements of $\mathscr{A}$, we have $p(\cup_{i\in\Gamma} a_i) = \sum_{i\in\Gamma} p(a_i)$.

In what follows, we will mostly focus on *finite probability spaces*, in which S is a finite base set. In that case, $\mathscr{A}$ can be taken to be the power-set (the set of *all* subsets) of S, and the requirement of σ-additivity boils down to finite

[2] This idea is advocated, e.g., by Van Fraassen (1980, p. 180), who says that "probability is a modality, it is a kind of graded possibility", or by Popper (1982, p. 70). See Gigerenzer et al. (1989, pp. 7f.) for some historical material on early 18th-century references to graded possibilities (e.g., in Leibniz).

additivity, which already holds iff for any $a_1, a_2 \in \mathscr{A}$ for which $a_1 \cap a_2 = \emptyset$, we have $p(a_1 \cup a_2) = p(a_1) + p(a_2)$.

Our second condition of adequacy is not formal but rather philosophical, although it has formal aspects: we require that our theory be able to make good sense of objective single-case probabilities (propensities) vis-à-vis known objections. We by no means argue that *all* uses of probability theory are concerned with objective single-case probabilities. There are, for example, completely adequate uses of subjective probabilities. But we aim at providing a BST analysis of the notion of ontologically basic, objective single-case probabilities, and so we are affirming the sensibility of the notion of propensities as probabilities. That idea has, however, been attacked as both philosophically ill-motivated and, perhaps more alarmingly, as formally untenable. The latter charge, which is often raised via an argument known as Humphreys's paradox, is connected with the alleged inapplicability of Bayes's theorem, a simple basic result of standard (Kolmogorovian) probability theory, to any notion of propensities. As we aim at a formally tenable and well-motivated theory, our second condition of adequacy is that our approach has to provide an answer to the challenge posed by Humphreys's paradox.

We take on our first condition of adequacy immediately, in Chapter 7.2, where we will work out a BST-based causal probability theory whose probability structures are probability spaces fulfilling Def. 7.1. A discussion of the second condition of adequacy, in the form of a defense of BST probabilities as propensities and of a thorough analysis of Humphreys's paradox, we defer to Chapter 7.3.

7.1.2 Two crucial questions

The following two questions are fundamental for any approach to probability, and they guide the development of our formal theory.

Question 7.1. What are the entities to which probabilities are assigned?

Question 7.2. What are the formal structures in which these entities are assigned their probabilities? In other words, what are the probability spaces?

There are several options for answering Question 7.1 in BST. For example, one could assign probabilities fundamentally to the individual histories in a BST structure, perhaps even in such a way that each history is assigned the same probability. One could then derive probabilities for other entities such as events, which typically occur in many histories, by measuring the respective set of histories, which might amount to simply counting their number. The idea of equiprobable histories and history counting has a certain intuitive appeal, but it faces severe technical challenges in the general case.[3] In what follows, we will sometimes appeal to the idea of probabilities attaching to histories for illustration, and in fact, our theory allows one to define the causal probability of a history as a derivative concept.[4] We will, however, assign probabilities fundamentally to other, more local objects. In parallel with our approach to objective causation, our answer to the question of which entities are assigned objective probabilities refers to the notion of a transition in BST.

Answer to Question 7.1: The entities to which probabilities are assigned are *indeterministic transitions* of any of the types definable in BST (Def. 4.4).

To motivate this answer, note that indeterministic transitions are a convenient, albeit abstract, means of representing various "chance set-ups", and objective probabilities arguably attach fundamentally to concrete chance set-ups such as a concrete toss of a concrete die. A chance set-up, understood as a singular entity, involves an initial (for example, a concrete measurement process) and a collection of possible outcomes together with a probability distribution on those outcomes conditional on the initial. Clearly, a chance set-up need not be human-made. A distant collision of two asteroids, together with a number of different possible outcomes of the collision and with a probability distribution on these outcomes conditional on the collision, counts as a chance set-up as well.

We do not claim that *every* BST transition can be assigned a probability. There are technical reasons for leaving the (anyway, uninteresting) case of trivial deterministic transitions out of the picture (see footnote 9), and there may be philosophical reasons in non-trivial cases as well—we believe that that issue is best left to further metaphysical investigation.[5] But if a

[3] McCall (1994, Ch. 5) bases his probabilistic theory of Branching Space-Times on counting histories that are taken to be equiprobable, providing a number of suggestive illustrations in support of his idea. McCall clearly recognizes the technical challenges of general real-valued probabilities, but in our view, his approach does not address these challenges in a fully satisfactory way, so that it remains open to formal and philosophical criticism (see, e.g. Briggs and Forbes, 2019).

[4] In the manner described in Section 7.2.4, one can define a causal probability space based on $\mathrm{TR}(h)$, the set of transitions fully characterizing the history h (see Def. 4.11), and then look at $p(\mathrm{TR}(h))$, the probability of that set of transitions, in that space.

[5] One might, for example, be hesitant to assign a probability to a particular free action of an agent. We do not take a stance on this matter, we merely flag it as providing a reason for perhaps not requiring *all* transitions to have a probability.

probability is assigned to some entity in BST, then that entity has to be a transition involving indeterminism, or a set of such transitions.

In what follows, we will make a terminological distinction between two notions of probability: causal probabilities or propensities defined on BST transitions, which we will denote by μ and which will be given directly via probabilistic BST structures (Def. 7.3), and measures p on mathematical probability spaces, which will be defined in accordance with Def. 7.1. It is important to keep these two notions separate, as they play different roles in our theory. Our discussion below can be viewed as providing an interface between them.

7.1.3 Propensities μ and probability measures p

Our aim is to develop the notion of the causal probability, or propensity, of a concrete BST transition $I \rightarrowtail \mathcal{O}^*$. We will follow our analysis of causation in Chapter 6, which has highlighted the crucial role of a transition's *causae causantes*, $CC(I \rightarrowtail \mathcal{O}^*)$. We will assume that the causal probability of an arbitrary transition should be provided via the causal probability of its set of *causae causantes*. As the set of $CC(I \rightarrowtail \mathcal{O}^*)$ is a subset of $\mathrm{TR}(W)$ (or, for disjunctive outcomes, a family of such sets), it makes sense to assign propensities to the relevant subsets of $\mathrm{TR}(W)$. For non-disjunctive outcomes, we will thus write the propensity as $\mu(CC(I \rightarrowtail \mathcal{O}^*))$, and we will interpret it as a measure of the causal strength—as the grade of the possibility—of the *causae causantes* in question. For some transitions, however, a propensity might not be defined. As a consequence, μ should be a partial function on the powerset of $\mathrm{TR}(W)$. Its range is the unit interval $[0, 1]$ of real numbers.

Having decided to let μ be a partial function on the powerset of $\mathrm{TR}(W)$, the identification of propensities of transitions with propensities of sets of *causae causantes* for these transitions could be handled, formally speaking, by introducing another function, say μ', defined generally on transition events rather than on subsets of $\mathrm{TR}(W)$. Our identification thesis would then be rendered as $\mu(CC(I \rightarrowtail \mathcal{O}^*)) = \mu'(I \rightarrowtail \mathcal{O}^*)$. However, in order not to introduce too much symbolism, we will use the same symbol, μ, and write $\mu(CC(I \rightarrowtail \mathcal{O}^*)) = \mu(I \rightarrowtail \mathcal{O}^*)$, as the domain of μ will always be clear from context.

In what follows, we first discuss a few metaphysical constraints on μ that follow from the fact that $\mu(I \rightarrowtail \mathcal{O}^*)$ is a propensity, that is, a measure of the grade of possibility of a concrete transition in BST. Then we discuss how propensities μ can be reflected via mathematically well-defined probability measures p. The first metaphysical constraint concerns the propensity μ of transitions to disjunctive outcomes. To begin, consider a non-trivial basic disjunctive transition $\tau =_{\mathrm{df}} e \rightarrowtail \check{\mathrm{I}}_e$ from a choice point e to the exhaustive disjunctive outcome $\check{\mathrm{I}}_e =_{\mathrm{df}} \{H \mid H \in \Pi_e\} = \Pi_e$ that collects together all of its (two or more) possible immediate outcomes.[6] The transition τ is deterministic in the sense that it happens anyway: its occurrence proposition is universal, $H_{e \rightarrowtail \check{\mathrm{I}}_e} = \mathrm{Hist}$. So τ has to have propensity one:

$$\mu(\tau) = \mu(e \rightarrowtail \check{\mathrm{I}}_e) = 1. \tag{7.1}$$

Given that the initial e is indeterministic, the transition $e \rightarrowtail H$ to any individual possible outcome $H \in \Pi_e$ of e has a cause-like locus at which it could be prevented from occurring, namely, e, and therefore, the grade of possibility of $e \rightarrowtail H$ will normally be strictly less than one. The complete disjunctive outcome $\check{\mathrm{I}}_e$, however, exhausts all the possibilities. Given that e occurs, one of the outcomes of e has to be realized, and this is what is expressed by Eq. (7.1).[7]

This constraint on the propensity function μ is of a logico-metaphysical nature: it follows from the way in which concrete transitions are defined in BST. The constraint will become important when we discuss marginal probabilities below. In a similar vein, we also assume a logico-metaphysically motivated constraint for other disjunctive outcomes of a point event: for a disjunctive outcome $\check{\mathbf{H}} \subseteq \Pi_e$ (i.e., for $\check{\mathbf{H}}$ a set of immediate basic outcomes of a choice point e, which are mutually incompatible by definition) we postulate

$$\mu(e \rightarrowtail \check{\mathbf{H}}) = \sum_{H \in \check{\mathbf{H}}} \mu(e \rightarrowtail H), \tag{7.2}$$

[6] In this chapter, we rewrite the definition of $\check{\mathrm{I}}_e$ (Def. 4.6) in terms of propositional outcomes rather than in terms of scattered outcomes. As shown in our discussion in Chapter 4.2, the two representations are equivalent for initials consisting of single events e.

[7] In a mathematical probability space that represents $\tau = e \rightarrowtail \check{\mathrm{I}}_e$, the full disjunction of all possible outcomes of e is represented via the unit element of the algebra, which has probability 1 by normalization. And the zero element has probability 0, of course. We remain impartial on the issue of so-called faithfulness, i.e., on the question of whether *only* the zero element of the algebra can have probability zero. By allowing for the assignment of zero probability to other elements of the algebra, we can, for example, simulate modal funny business in a probabilistic BST structure that harbors no real modal funny business. We will make use of this option in Chapter 8.4.

that is, the propensity of the transition from e to its disjunctive outcome $\check{\mathbf{H}}$ is the sum of the propensities of these basic outcomes taken individually. For example, considering a concrete throw of a fair die, for which all immediate outcomes $\boxed{1}, \ldots, \boxed{6}$ have propensity $1/6$, the disjunctive outcome "even number" (immediate outcome $\boxed{2}$, $\boxed{4}$, or $\boxed{6}$) has propensity $1/6 + 1/6 + 1/6 = 1/2$. In fact, Eq. (7.1) can be viewed as a special case of this additivity constraint, given the logico-metaphysical rule that unavoidable transitions have propensity one.

These considerations show that the set of all basic transitions from a given choice point naturally gives rise to a probability space. In the finite case on which we focus here, that is, when e has finitely many immediate basic outcomes, one can simply use the Boolean algebra $\mathscr{A}$ of the power-set of all the basic transitions, and a measure on the algebra can be induced by the propensities of the individual basic transitions.

At this juncture, our second notion of probability, the mathematical probability measure p, becomes important. Even if we can introduce some more constraints on the propensity function μ, we are not working toward a theory in which the propensity function μ itself fulfills the axioms of probability theory of Def. 7.1. On the one hand, this is due to our choice of answer to Question 7.1: We are working toward a theory in which concrete local BST transitions, which can represent concrete chance set-ups, have causal probabilities assigned, and therefore, we target the definition of local probability spaces with local measures p, whereas μ is defined globally. In addition, there are several technical hurdles that stand in the way of interpreting μ as a probability measure. For starters, note that we have allowed μ to be a partial function, which directly contradicts the assumption that a probability measure be defined for all elements of the algebra, (i.e., for all possible events). Second, and more alarmingly, consider the fact that, as we just said, all unavoidable transitions have propensity one. How should this be accommodated in a probability space defined on all transitions? Consider two transitions from choice points e_i to their exhaustive disjunctive outcome, $\tau_i =_{\text{df}} e_i \rightarrowtail \check{1}_{e_i}$ $(i = 1, 2)$, in incompatible possible futures (basic scattered outcomes) of another choice point e, so that $e < e_1$ and $e < e_2$. The two unavoidable transitions τ_1 and τ_2 both have propensity one, but their outcome-parts are incompatible. So it seems that in a probability space representation, τ_1 and τ_2 have to belong to disjoint elements of the algebra. But then we have to have two disjoint elements of the algebra that each have probability 1, and by the additivity of the measure, their union has to have probability 2, violating the normalization of the measure. Something is badly amiss here. Apart from extremely simple cases such as BST structures with just one choice point, it seems hopeless to try and define a *global* probability space in which the propensity function μ is really a probability measure.[8,9]

We do not claim that the issue cannot be resolved by some clever way of defining a global probability space based on a whole BST structure and deriving propensities of transitions in some other way. The propensities of concrete transitions in the BST structure would most likely have to be defined as conditional probabilities of some sort (something like the probability of the occurrence of the outcome conditional on the occurrence of the initial). We will not go down that route, however, as on the one hand it leads to unmanageably large probability spaces while, on the other, it goes against a basic principle of our approach to possibilities: a main advantage of BST is that we work with local notions of possibilities and transitions.[10] We look for a corresponding, local way of handling propensities as well.

So, given that we want to talk about the propensity $\mu(I \rightarrowtail \mathscr{O}^*)$ of some concrete transition $I \rightarrowtail \mathscr{O}^*$ *as a probability*, and a globally defined probability space is not promising in this respect, we have to find some locally defined mathematical probability space $\langle S, \mathscr{A}, p\rangle$ in which the given transition (together with some other transitions) is represented and in which it is assigned a probability that can be read as a propensity. This is the question of providing an interface between the causal BST notion μ and mathematical probability theory. As we will see, logico-metaphysical and causal constraints on μ (such as the one about unavoidable transitions having propensity one) provide useful guidance as to which mathematical structures are appropriate. To provide an indication of our approach: for a given indeterministic transition $I \rightarrowtail \mathscr{O}^*$, an adequate causal probability

[8] In this sense, therefore, propensities and probabilities come apart—but they are intimately linked. Our answer to Humphreys's challenge that "propensities cannot be probabilities" will stress that link; see Chapter 7.3.3.

[9] A further problem is how to handle trivial deterministic transitions, such as a transition from a deterministic event e to its only outcome H_e. For such transitions, the set of *causae causantes* is empty. The only sensible propensity one can assign for $\mu(\emptyset)$ is one, as such a transition is inevitable, having the universal occurrence proposition. But it is not possible to build a probability space in which the event algebra has only one element, the empty set. (The problem for fulfilling Def. 7.1 is with the measure, not with the algebra.) This is our technical reason for not considering causal probability spaces (see Def. 7.4) for transitions that have no indeterministic causes: there are no causal probability spaces without causation.

[10] Defining probabilities on the histories, as discussed in note 3, would be one such global approach; another would be to define a joint probability space for *all* indeterministic transitions in a given BST structure $\langle W, <\rangle$, i.e., on $\mathrm{TR}(W)$. Our approach naturally extends to accommodate these two ideas, but it does not presuppose them. It is, therefore, both more local and more general.

space, which is able to represent that transition and its propensity, will contain $CC(I \rightarrowtail \mathscr{O}^*)$ together with some appropriate representation of causal alternatives to $I \rightarrowtail \mathscr{O}^*$ (see Defs. 7.2 and 7.4). As we said, we will make use of the assumption that the propensity of a transition may only depend on the propensities of its set of *causae causantes*. The latter set is consistent, and accordingly, a causal probability space will have a sample space consisting of consistent sets of transitions.[11] Once the details are in place, a number of mathematical constraints on the measure in a given causal probability space in relation to measures in other causal probability spaces can be motivated as either logico-metaphysical postulates (e.g., involving marginal probabilities), or as causal postulates (e.g., involving a form of the Markov property).

We sum up the partial answer to our Question 7.2 which is implicit in our discussion above.[12]

Partial answer to Question 7.2: When $\mu(I \rightarrowtail \mathscr{O}^*)$ is defined, its value must be fully determined via the propensities of the basic transitions from the set $CC(I \rightarrowtail \mathscr{O}^*)$. This emphatically includes the possibility that one may need to take into account not only the propensities of the individual *causae causantes*, but also propensities of certain sets of them, taken as operating jointly.

This answer guarantees that for BST transitions that have causal probabilities assigned, if they have the same set of *causae causantes*, then their causal probability also has to be the same. Note that we made a caveat about sets of *causae causantes* possibly working together. This option is needed in order to make room for probabilistic correlations. Correlations are often scientifically important. In Chapter 5.1 we motivated our account of modal funny business via the phenomenon of modal correlations, and here, similarly, we also have to make room for the phenomenon of probabilistic correlations. We work toward the notion of probabilistic correlations, to be provided in Chapter 7.2.6, by considering a number of scenarios that will help to anchor our general ideas about representing propensities μ as Kolmogorovian probabilities p.

[11] Note that by Def. 6.4 the set of *causae causantes* for a BST transition to an outcome chain or to a scattered outcome is consistent, and that the *causae causantes* for a transition to a disjunctive outcome are a family of consistent sets of transitions.

[12] A full answer to Question 7.2 will be provided via our definition of causal probability spaces in Chapter 7.2.4; see p. 193.

7.2 Causal probability spaces in BST

With a view to our first condition of adequacy, we will start by developing our theory of BST probabilities for indeterministic transitions, which will lead to the definition of a *causal probability space*. This will help provide a full answer to Question 7.2 about the formal structures in which transitions are assigned probabilities that can be read as propensities.

7.2.1 Probabilities for transitions: The simplest case

We illustrate the idea of defining probabilities for transitions by starting with the simplest case, which we have already outlined above. The simplest case of a transition is a basic indeterministic transition, a notion that we discussed in Chapter 4.2. Consider Alice's throwing of a fair die, and assume for simplicity that this die-throwing is a localized indeterministic event that has exactly six possible immediate outcomes, namely, the numbers 1 through 6.[13] We thus represent the die-throwing by a set of six mutually incompatible basic transitions $\tau_1 = e \rightarrowtail \boxed{1}, \ldots, \tau_6 = e \rightarrowtail \boxed{6}$ with the same initial e, whose set of basic outcomes we write as $\Pi_e = \{\boxed{1}, \ldots, \boxed{6}\}$. Given that the die is fair, each of the concrete transitions τ_i has the same propensity, $\mu(\tau_i) = 1/6$ $(i = 1, \ldots, 6)$. In this case, it is perfectly natural to take the six basic transitions as the elementary events constituting a sample space $S = \{\tau_1, \ldots, \tau_6\} = \{e \rightarrowtail H \mid H \in \Pi_e\}$. We can then define a finite probability space $\langle S, \mathscr{A}, p\rangle$ using the Boolean algebra $\mathscr{A}$ of subsets of S, and assigning the fully symmetrical probability measure p in accord with the transitions' propensities: the value of p on the elements of the sample

[13] It is not easy to say whether a concrete transition of Alice's throwing a die, with the initial of Alice prepared to throw in a concrete situation and with the outcome of the die showing, for example, outcome 1, is indeed indeterministic. The issue may in fact depend on details of Alice's current physiological state. As to the mechanics of throwing leading to a specific number, given a concrete die thrown in a concrete way (with concrete speed and angular momentum etc.), the transition to the outcome is perhaps even deterministic. (See Diaconis et al., 2007, for a study of the related problem of coin-tossing.) In what follows, we idealize Alice's die-throwing, as well as other chance set-ups (Bob's tossing a coin and Eve's throwing an octahedron), to be indeterministic. The most realistic examples of truly indeterministic transitions with precisely specified probabilities are, for all we know, quantum experiments, which will be discussed in Chapter 8. We stick to everyday examples in this chapter so as not to bring up too many complications at once. And, historically, probability theory was in fact developed initially for use in the context of simple games of chance such as games with dice (see Hacking, 2006).

space (the atoms of the algebra) is $p(\tau_i) = \mu(\tau_i) = 1/6$ ($i = 1, \ldots, 6$).[14] Apart from the BST notion of a transition, this is all perfectly standard. Note that non-atomic elements of the Boolean algebra can be viewed as disjunctive outcomes of the initial e, in full accordance with Def. 4.4, as indicated above. To repeat, the event "throwing an even number", which is represented as the set $\{\tau_2, \tau_4, \tau_6\} \in \mathscr{A}$, corresponds to the transition from e to the disjunctive outcome $\{\boxed{2}, \boxed{4}, \boxed{6}\}$ of e. This will become important below in connection with marginal probabilities.

As transitions represent causal happenings in BST, our mathematically defined probabilities p for transitions represent causal probabilities, or propensities, μ. The causal probability $\mu(e \rightarrowtail \boxed{1})$ of $\tau_1 = e \rightarrowtail \boxed{1}$ can be viewed as the grade of possibility of the concrete possible causal process that leads from the initial event e to the concrete outcome-event $\boxed{1}$. In our example, using our probability space $\langle S, \mathscr{A}, p\rangle$ as the mathematical background representation, that probability is given as $\mu(e \rightarrowtail \boxed{1}) = p(\tau_1) = 1/6$. So, in this simple case, there is an immediate correspondence between the causal probabilities μ and the mathematical probabilities p.

Earlier, we discussed a logico-metaphysical constraint on causal probabilities: the causal probabilities for incompatible basic outcomes of the same initial event have to add up. Given the probability space of our example, the additivity of the measure secures that this constraint is satisfied. Take two basic outcomes $\boxed{i}$ and $\boxed{j}$ of e ($i \neq j$). The concrete transition $e \rightarrowtail \{\boxed{i}, \boxed{j}\}$ to the disjunctive outcome $\{\boxed{i}, \boxed{j}\}$ corresponds to the element $\{\tau_i, \tau_j\} \in \mathscr{A}$, and as $\{\tau_i\} \cap \{\tau_j\} = \emptyset$, by additivity we have

$$\mu(e \rightarrowtail \{\boxed{i}, \boxed{j}\}) = p(\{\tau_i, \tau_j\}) = p(\tau_i) + p(\tau_j) = \mu(e \rightarrowtail \boxed{i}) + \mu(e \rightarrowtail \boxed{j}).$$

Similarly, it follows by normalization that

$$\mu(e \rightarrowtail \check{1}_e) = p(\mathbf{1}_{\mathscr{A}}) = p(S) = 1.$$

Note that the probability space in which $p(\tau_1)$ is assigned a probability contains all the immediate causal alternatives to τ_1, namely, all the other transitions from e to one of its immediate outcomes, as well. This idea

[14] We follow the standard practice in abusing the notation slightly: given an elementary outcome $a \in S$, the probability is officially defined only for the corresponding element of the algebra, $\{a\} \in \mathscr{A}$, but we allow ourselves to write $p(a)$ instead of the more cumbersome $p(\{a\})$. We take the same liberty for μ.

will guide our general construction below. We work toward that general construction by considering a number of further simple stories.

7.2.2 Two BST transitions, one basic transition

Recall the story about Alice sending a letter to Bob from Chapter 6.3.1, which we used to make a distinction between an active and a passive causal contribution. In order to stay close to our story, assume that Alice is undecided as to whether she wants to send a letter to Bob (initial A-undec), and has chosen to leave the matter to chance. She will throw a fair die, and she will send a letter (outcome A-sends) exactly if the die shows 1 (outcome $\boxed{1^A}$ of die-throwing event e^A). This chance set-up is mathematically represented by the probability space for the die-throwing discussed above, which we now write $\langle S^A, \mathscr{A}^A, p^A \rangle$, where the sample space $S^A = \{\tau_1^A, \ldots, \tau_6^A\}$. Bob, on the other hand, is far away, and he is facing a (passive) transition from having no information in the morning as to whether he will receive a letter (initial B-noinf) to either having personally received Alice's letter (outcome B-receives) or not having received a letter (outcome B-noletter). Given our minimal story, we have

$$CC(\text{A-undec} \rightarrowtail \text{A-sends}) = \{e^A \rightarrowtail \boxed{1^A}\} = CC(\text{B-noinf} \rightarrowtail \text{B-receives}).$$

The scenario is pictured in Figure 7.1. Given our reliance on *causae causantes*, we already have

$$\mu(\text{A-undec} \rightarrowtail \text{A-sends}) = \mu(e^A \rightarrowtail \boxed{1^A}) = \mu(\text{B-noinf} \rightarrowtail \text{B-receives}),$$

and by our representation of Alice's die-throwing event via the probability space $\langle S^A, \mathscr{A}^A, p^A \rangle$, we have

$$\mu(e^A \rightarrowtail \boxed{1^A}) = p^A(\tau_1^A) = 1/6.$$

That is, our decision to make single-case probabilities depend exclusively on the *causae causantes* of a transition already has a substantial consequence for our simple example: given that there is only one indeterministic transition doing any causal work, it must be that all concrete transitions that have the

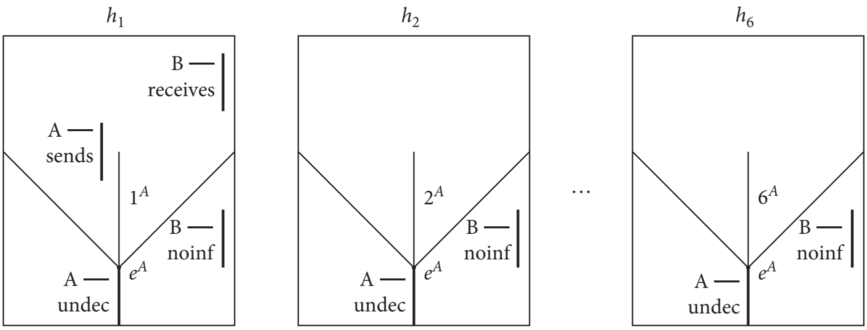

Figure 7.1 Alice throws a fair die to decide whether or not to send a letter to Bob. The BST structure contains six histories, one for each outcome of the throw. Alice sends the letter, and Bob receives it, exactly in history h_1.

same *causae causantes* also have the same causal probability. That causal probability is represented in a suitable probability space.

It is crucial that here we are dealing with causal probabilities of transitions, rather than the causal probabilities of outcomes in isolation. It arguably makes little sense to ask about the objective probability of the occurrence of an outcome event such as B-receives alone. The event of Bob's receiving the letter may be taken to have probability $1/6$, given the background of our story, but it may also be taken to be exceedingly improbable, considering, for example, the fact that Bob's parents only met via an unlikely coincidence and that it is therefore highly improbable that Bob was even ever born. A causal probability always depends on how far back in time one looks for alternatives.[15] As discussed in Chapter 6, a BST transition supplies all the details of the causal background before which the causal influences bringing about an event are assessed. Likewise, a transition supplies all the details of the causal background (the initial) before which the probability of an outcome is determined.

[15] It is possible to look back all the way, to the beginning of time, in which case no initial has to be given. Before we introduced the notion of *causae causantes* for a transition $I \rightarrowtail \mathcal{O}^*$ via its set of cause-like loci $cll(I \rightarrowtail \mathcal{O}^*)$ (Defs. 6.2 and 6.4), we in fact defined the set of cause-like loci $cll(O)$ of an outcome chain (Def. 5.10). This set includes *all* the risky junctions at which the occurrence of O could ever have been prevented. That set will normally be far too large to be of interest, and the objective probability of the occurrence event given no initial will normally be vanishingly small. (If we had not ruled this out by requiring an initial event to be non-empty (Def. 4.3), we could indicate the occurrence of the outcome $\mathcal{O}^*$ alone via the transition $\emptyset \rightarrowtail \mathcal{O}^*$.) To repeat, we aim at providing a *locally* anchored BST probability theory, and so we avoid the idea of looking back all the way.

7.2.3 Two or more transitions and some complications

On a more realistic account of the letter sending and receiving example, both concrete transitions in question—Alice's transition from being undecided to having sent the letter and Bob's transition from having no information to having personally received the letter—of course have many more *causae causantes*.

We first provide two different variants of our simplest story in which we add one extra indeterministic event, first on Bob's side and then on Alice's. We later combine the two variants into a more complex story, which already contains most of the ingredients needed for motivating our general account.

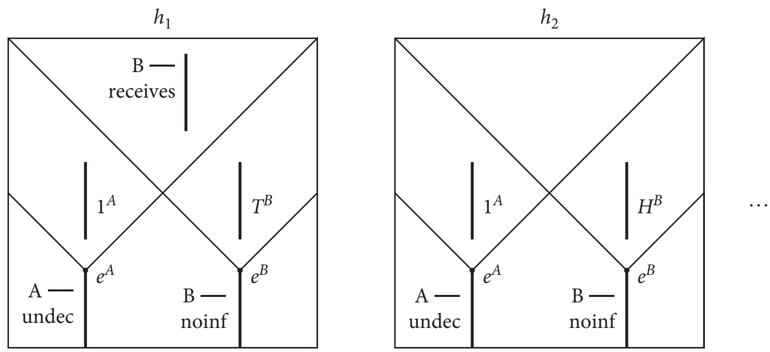

Figure 7.2 Alice throws a fair die to decide whether or not to send a letter to Bob, and Bob throws a fair coin to decide whether or not to go out. Bob receives the letter if and only if Alice's die has outcome 1 and his coin lands tails. The BST structure contains twelve histories, one for each pairing of an outcome of the throw of the die and the toss of the coin. Only histories h_1 and h_2 are shown.

Variant 1: Bob goes out. Consider, first, that Bob's personally receiving the letter depends causally on his being at home. In the morning, Bob is undecided whether to stay at home or to go out, and similar to Alice, he leaves the matter to chance: he tosses a fair coin, and if the coin shows heads (H^B), he stays at home; if not, if the coin shows tails (T^B), he takes a long walk (see Figure 7.2). Bob's coin-tossing can be represented in exact analogy to the formal account of Alice's die-throwing: the relevant mathematical probability space is based, in this case, on an underlying set of two mutually incompatible basic transitions leading from the choice point e^B to the outcomes T^B ($\tau_1^B = e^B \rightarrowtail T^B$) or H^B ($\tau_2^B = e^B \rightarrowtail H^B$). The symmetrical probability measure p^B for Bob's coin-toss assigns the values that we know to be the causal probabilities of the chance set-up of the concrete tossing of the fair coin, one half:

$$p^B(\tau_1^B) = \mu(e^B \rightarrowtail T^B) = p^B(\tau_2^B) = \mu(e^B \rightarrowtail H^B) = 1/2.$$

Let us assume that, as Alice and Bob are far away, their respective chance events are space-like related (for the initials, $e^A\, SLR\, e^B$): these events cannot causally influence one another. The transition A-undec $\rightarrowtail$ A-sends from Alice's being undecided to her having sent the letter, due to her die having shown outcome 1, still has a single *causa causans*, namely, the transition $\tau_1^A = e^A \rightarrowtail \boxed{1^A}$. The transition B-noinf $\rightarrowtail$ B-receives from Bob's having no information about the letter and being undecided whether to go out to his personally receiving the letter, on the other hand, now has two *causae causantes*: The originating causes of that transition are the two basic transitions $e^A \rightarrowtail \boxed{1^A}$ (Alice's die showing 1), which triggers Alice's sending the letter, and $e^B \rightarrowtail H^B$ (Bob's coin showing heads), which triggers Bob's staying at home.

This causal story, together with our decision that causal probabilities have to be based exclusively on *causae causantes*, affirms our earlier verdict about the probability of Alice's sending-the-letter-transition: μ(A-undec $\rightarrowtail$ A-sends) $= 1/6$. For Bob's transition from no information to having received the letter personally, however, our account gives only a partial answer:

$$\mu(\text{B-noinf} \rightarrowtail \text{B-receives}) = \mu(\{e^A \rightarrowtail \boxed{1_A}, e^B \rightarrowtail T^B\}) = ?$$

The causal probability of Bob's transition has to be the causal probability of its two-transition set of *causae causantes*, but that latter probability is not necessarily determined by the individual causal probabilities of the basic transitions alone. And so far it is also not clear in which mathematical probability space that causal probability can be adequately represented.

To make some headway, we provide some relevant logico-metaphysical observations about causal probabilities. Given that in our structure, e^A and e^B are the only choice points and these choice points are SLR, it follows that any history $h \in$ Hist contains both e^A and e^B. (For our reasoning below it is enough that for any history h, $e^A \in h$ iff $e^B \in h$.) The set $I = \{e^A, e^B\}$ makes sense as a BST initial event, as it is consistent. Now we can look at its outcomes. Assuming that there is no modal funny business (see p. 175 for a brief discussion), any basic outcome of e^A is compatible with any basic outcome of e^B, so that there are $6 \cdot 2 = 12$ possible basic (non-disjunctive) joint outcomes, and $2^{12} - 1 = 4095$ disjunctive outcomes (sets of basic outcomes; we have to disregard the empty set). We represent the elementary joint outcomes via outcome chains O_{ij}, where $i \in \{1, \ldots, 6\}$ and $j \in \{1, 2\}$. Spatio-temporally, the outcome chains O_{ij} begin in the joint future of possibilities of e^A and e^B. Epistemologically speaking, if you are situated at one of these chains, you have information about the outcome of both e^A and of e^B. Each joint outcome corresponds to one of the 12 histories in the structure, which are accordingly denoted as h_{ij}, so that $O_{ij} \subseteq h_{ij}$.

With a view to calculating marginal probabilities, consider the disjunctive outcome $\breve{\mathbf{O}}_i =_{\mathrm{df}} \{O_{i1}, O_{i2}\}$, so $H_{\langle \breve{\mathbf{O}}_i \rangle} = \{h_{i1}, h_{i2}\}$ ($i \in \{1, \ldots, 6\}$). The transition $I \rightarrowtail \breve{\mathbf{O}}_i$ fulfills the requirement of the appropriate spatio-temporal location of I and $\breve{\mathbf{O}}_i$ of Def. 4.4, as is easy to check. We then calculate

$$CC(I \rightarrowtail \breve{\mathbf{O}}_i) = \{\{e^A \rightarrowtail \boxed{i}, e^B \rightarrowtail H^B\}, \{e^A \rightarrowtail \boxed{i}, e^B \rightarrowtail T^B\}\}. \qquad (7.3)$$

We can now employ the logico-metaphysical principles about causal probabilities being based solely on *causae causantes* and about the probabilities of disjoint outcomes adding up. Since the elements of $CC(I \rightarrowtail \breve{\mathbf{O}}_i)$ are incompatible, the probability of $CC(I \rightarrowtail \breve{\mathbf{O}}_i)$ should be the sum of the probabilities of these elements. Further, since every history passing through $\boxed{i}$ contains either H^B or T^B, for our particular setup, in which e^A and e^B occur in each history, we have that:

$$H_{\{e^A \rightarrowtail \boxed{i}, e^B \rightarrowtail H^B\}} \cup H_{\{e^A \rightarrowtail \boxed{i}, e^B \rightarrowtail T^B\}} = H_{\{e^A \rightarrowtail \boxed{i}\}}. \qquad (7.4)$$

These observations lie behind the following summation formula, where we use O_i for an outcome chain witnessing the outcome $\boxed{i}$ of Alice's die throwing:

$$\begin{gathered} \mu(I \rightarrowtail \breve{\mathbf{O}}_i) = \mu(CC(I \rightarrowtail \breve{\mathbf{O}}_i)) = \\ \mu(\{\{e^A \rightarrowtail \boxed{i}, e^B \rightarrowtail H^B\}, \{e^A \rightarrowtail \boxed{i}, e^B \rightarrowtail T^B\}\}) = \\ \mu(\{e^A \rightarrowtail \boxed{i}, e^B \rightarrowtail H^B\}) + \mu(\{e^A \rightarrowtail \boxed{i}, e^B \rightarrowtail T^B\}) = \\ \mu(\{e^A \rightarrowtail \boxed{i}\}) = \mu(CC(I \rightarrowtail O_i)) = \mu(I \rightarrowtail O_i). \end{gathered}$$

The first transformation comes from Eq. 7.3, the second results from the summation of probabilities of incompatible elements of *causae causantes*, the third is based on Eq. 7.4, and the remaining transformations invoke our convention about the domain of μ. This calculation establishes that the causal probabilities yield sensible marginal probabilities: The causal probability of Alice's die showing outcome i and Bob's coin turning up either way is just the causal probability of Alice's die showing outcome i, full stop.

Given these facts about the causal probabilities, it is natural to try to represent them by combining the two given probability spaces for Alice's and for Bob's chance set-ups via their Cartesian product. That operation leads to the sample space $S^C = S^A \times S^B$. That is, a basic outcome $\langle \tau_i^A, \tau_j^B \rangle \in S^C$ in that probability space specifies one transition τ_i^A of Alice's die-throwing and one transition τ_j^B of Bob's coin-tossing, in natural correspondence to the outcome events O_{ij}. The algebra $\mathscr{A}^C$ will be the algebra of subsets of S^C, as usual. Furthermore, given the *SLR* layout, it seems natural to require that the distant outcomes be probabilistically uncorrelated. In that case, the joint measure p^C is the product measure, for which $p^C(\langle \tau_i^A, \tau_j^B \rangle) = p^A(\tau_i^A) \cdot p^B(\tau_j^B)$. In particular, we then have

$$p^C(\langle \tau_1^A, \tau_2^B \rangle) = p^A(\tau_1^A) \cdot p^B(\tau_2^B) = 1/6 \cdot 1/2 = 1/12.$$

If we assume that the probability space $\langle S^C, \mathscr{A}^C, p^C \rangle$ provides an adequate representation of the causal probabilities, we thus have $\mu(\text{B-noinf} \rightarrowtail \text{B-receives}) = p^C(\langle \tau_1^A, \tau_2^B \rangle) = 1/12$.

The just mentioned construction proceeds in two steps: first, combine the small (local) probability spaces by taking their Cartesian product, and then define a measure on the joint space via the product of the local measures. This construction is quite useful in the case at hand but, emphatically, we do *not* propose any of these two steps as a general basic recipe: we do not require in general that causal probability spaces combine via Cartesian products, and we do not require that in all cases, *SLR* initials give rise to probabilistically uncorrelated transitions.

We will soon turn to a case in which the Cartesian product construction is inappropriate; see our next scenario, Variant 2 (p. 177). There are, however, a number of important issues that we can discuss in relation to the present case (Variant 1). First, in the case of *SLR* initials, the question of whether the individual local possibilities combine so as to yield the Cartesian product is exactly the question of whether there is modal funny business (MFB) as discussed in Chapter 5; see clause 4 of Def. 5.5 of combinatorial consistency and the following Def. 5.6 of combinatorial funny business. Given the simplifying assumption of no MFB, the sample space for the combination of the possibilities in Alice's and Bob's chance set-ups is indeed adequately represented by the Cartesian product $S^C = S^A \times S^B$ as defined above.

Second, the Cartesian product construction does not uniquely determine the joint probability measure p^C, but it constrains the options. That is, again, in full accord with the situation regarding the causal probabilities. It might be, for all we know, that the transitions are fully correlated, so that in any case in which Alice's die shows 1, Bob's coin shows tails. In that case,

$$\begin{aligned}\mu(\text{B-noinf} \rightarrowtail \text{B-receives}) &= \mu\left(\{e^A \rightarrowtail \boxed{1^A}, e^B \rightarrowtail T^B\}\right)\\ &= \mu(e^A \rightarrowtail \boxed{1^A}) = 1/6,\end{aligned}$$

which in terms of the joint measure would correspond to $p^C(\langle \tau_1^A, \tau_2^B\rangle) = p^A(\tau_1^A) = 1/6$, rather than $1/12$ on the product measure. On the other hand, it might also be that the transitions are fully anti-correlated (either one can occur only if the other does not occur), so that

$$\mu(\text{B-noinf} \rightarrowtail \text{B-receives}) = \mu(\{e^A \rightarrowtail \boxed{1^A}, e^B \rightarrowtail T^B\}) = 0,$$

corresponding to $p^C(\langle \tau_1^A, \tau_2^B\rangle) = 0$. In terms of possible probability measures p^C based on p^A and p^B, all these numbers make sense, and empirically, in terms of observed causal probabilities, the well-confirmed phenomenon of EPR-like probabilistic quantum correlations arguably shows that the full range of these options can in fact be produced.[16] This means that given our causal set-up of $e^A\, SLR\, e^B$, there is quite some freedom for assigning a numerical value to the joint probability $p^C(\langle \tau_1^A, \tau_2^B\rangle)$.

Even so, there are still some constraints—one cannot just assign *any* number between 0 and 1 to $p^C(\langle \tau_1^A, \tau_2^B\rangle)$. Technically, we require that the probability of the single-transition events can be recovered as marginal probabilities from the joint probability measure. That is, $p^A(\tau_1^A)$ has to be recovered from the joint probabilities of all two-transition pairs including τ_1^A and some τ_j^B, and similarly for $p^B(\tau_2^B)$:

$$p^A(\tau_1^A) = \sum_{j=1}^{2} p^C(\langle \tau_1^A, \tau_j^B\rangle); \quad p^B(\tau_2^B) = \sum_{i=1}^{6} p^C(\langle \tau_i^A, \tau_2^B\}. \qquad (MP)$$

[16] We will discuss these issues in more detail in Chapter 8.4.

These equations simply follow from representing probabilities of transitions via the causal probabilities of the sets of their *causae causantes* and from our account of *causae causantes*.

Given the Cartesian product construction of joint probability spaces, the marginal property follows as a mathematical consequence. Our reason for requiring the marginal property is, however, not purely mathematical, but properly causal. Consider any history h in which e^A occurs, so that by our assumptions $e^B \in h$ as well, and $e^A\, SLR\, e^B$. As e^B is the initial of a set of incompatible basic transitions, it follows that on h, exactly one of the possible outcomes occurs, viz., $\Pi_{e^B}\langle h\rangle$. So, any history on which τ_1^A occurs must be a history on which one of the τ_j^B occurs as well: for τ_1^A to occur *is* for τ_1^A to occur and for some τ_j^B to occur. Thus, the probability for τ_1^A to occur must be divided up, so to speak, between the different ways for τ_1^A to occur and some τ_j^B to occur. Each of the pairs $\langle \tau_1^A, \tau_j^B\rangle$ has to have a non-negative probability with respect to the measure p^C, and together, these probabilities have to add up to the probability $p^A(\tau_1^A)$, as Eq. (MP) specifies.

To put this matter in explicitly causal terms, we can say that given our scenario, the transition from Alice's initial e^A to the outcome $\boxed{1^A}$ is causally equivalent to (has the same *causae causantes* as) the transition from the initial $\{e^A, e^B\}$ to the disjunctive outcome $\{\{\boxed{1^A}, T^B\}, \{\boxed{1^A}, H^B\}\}$, and therefore, as a matter of causal probability irrespective of the representation of these probabilities in any mathematical structures, we have to have

$$\mu(\tau_1^A) = \mu(\{\tau_1^A, \tau_1^B\}) + \mu(\{\tau_1^A, \tau_2^B\}).$$

So the marginal property has a proper metaphysical foundation. It follows in particular that, as probabilities are non-negative, for $j \in \{1,2\}$,

$$0 \leqslant \mu(\{\tau_1^A, \tau_j^B\}) \leqslant \mu(\tau_1^A) = 1/6,$$

and a parallel consideration establishes the bound that for $i \in \{1,\ldots,6\}$,

$$0 \leqslant \mu(\{\tau_i^A, \tau_2^B\}) \leqslant \mu(\tau_2^B) = 1/2.$$

So our requirement of basing probabilities solely on probabilities of sets of *causae causantes* (possibly working together), supplemented by considerations of the causal consequences of the SLR relation of the initials e^A and e^B, yields the following constraint:

$$0 \leqslant \mu(\text{B-noinf} \rightarrowtail \text{B-receives}) = \mu(\{\tau_1^A, \tau_2^B\}) \leqslant \min(\mu(\tau_1^A), \mu(\tau_2^B)) = 1/6.$$

And, to repeat, in the case of no correlations, which for our example is exceedingly plausible, we have the simple "just multiply" result,

$$\mu(\text{B-noinf} \rightarrowtail \text{B-receives}) = \mu(e^A \rightarrowtail \boxed{1}) \cdot \mu(e^B \rightarrowtail H^B) = 1/6 \cdot 1/2 = 1/12.$$

Variant 2: Mail gets lost. We return to our basic story of Alice throwing a die to determine whether or not to send a letter to Bob. This time round, we let Bob stay at home deterministically, canceling the indeterministic transition event on his side. However, we introduce a different complication, namely, there is an eavesdropper, Eve, who indeterministically blocks some letters from being delivered. If Alice's die has shown outcome 1 and she has sent her letter, there is a further indeterministic transition, from Eve seeing the letter (E-sees) to one of two possible outcomes: taking the letter, so that it will not arrive at Bob's place (E-takes), or letting it pass (E-passes). Given that BST represents concrete events, the initial E-sees can occur only after Alice sends the letter. The concrete event E-sees does not occur in any of the other, incompatible outcomes of A-undec. Now we assume again that the indeterminism of Eve's taking or not taking the letter is due to a chance set-up, this time the throw of an octahedron by Eve, which is represented by a set of eight basic outcomes $\boxed{i^E}$ of the choice point e^E at which the octahedron is thrown, corresponding to the eight basic transitions $\tau_i^E = e^E \rightarrowtail \boxed{i^E}$ $(i = 1, \ldots, 8)$. We assume that Eve lets the letter pass iff her octahedron shows 3, and that she takes the letter on all other outcomes. We also assume that the octahedron is fair, so that the causal probabilities are $\mu(e^E \rightarrowtail \boxed{i^E}) = 1/8$ $(i = 1, \ldots, 8)$. An adequate probability space for representing Eve's throw can be based on the sample space $S^E = \{\tau_1^E, \ldots, \tau_8^E\}$ and the symmetrical measure p^E that assigns the value $p^E(\tau_i^E) = 1/8$ for each of the τ_i^E $(i = 1, \ldots, 8)$.

As in Variant 1, the transition A-undec $\rightarrowtail$ A-sends, from Alice's being undecided to her having sent the letter, still has a single *causa causans*, namely, the transition $\tau_1^A = e^A \rightarrowtail \boxed{1^A}$. The transition B-noinf $\rightarrowtail$ B-receives from Bob's having no information about the letter to his personally receiving the letter again has two *causae causantes*, but this time, these are the basic transitions $\tau_1^A = e^A \rightarrowtail \boxed{1^A}$, which triggers Alice's sending the letter, and $\tau_3^E = e^E \rightarrowtail \boxed{3^A}$, which triggers Eve's letting the letter pass.

This causal story, together with our decision about probabilities having to be based exclusively on *causae causantes*, reaffirms that $\mu(\text{A-undec} \rightarrowtail \text{A-sends}) = p^A(\tau_1^A) = 1/6$. For Bob's transition from no information to having received the letter personally, we can start with the following partial answer:

$$\mu(\text{B-noinf} \rightarrowtail \text{B-receives}) = \mu(\{e^A \rightarrowtail \boxed{1^A}, e^E \rightarrowtail \boxed{3^E}\}) = ?$$

The probability of Bob's transition has to be the probability of its two-transition set of *causae causantes*. The question is whether we can use considerations about the causal set-up together with the known probabilities of the basic transitions to narrow down the possible answers, as above, or even to provide a unique answer.

It turns out that in the present case, the causal set-up provides the resources to secure the uniqueness of the causal probability. The adequate representation of this fact is, however, not completely straightforward from the perspective of standard probability theory.

Note, first, that since "B-receives" is a concrete outcome event, represented by an outcome chain or a scattered outcome, the set of *causae causantes* $CC(\text{B-noinf} \rightarrowtail \text{B-receives}) = \{e^A \rightarrowtail \boxed{1^A}, e^E \rightarrowtail \boxed{3^E}\} = \{\tau_1^A, \tau_3^E\}$ is consistent (see Def. 6.2). This does not imply, however, that any other set of basic transitions $\{\tau_i^A, \tau_j^E\}$ is also consistent. In the previous scenario in which we considered two *SLR* initials, the absence of MFB was sufficient to guarantee the consistency of all joint outcomes, and this made it possible to use the familiar probability-theoretic idea of building joint spaces via Cartesian products. In the present case, however, this construction makes no sense. Indeed, by Fact 4.7, only one outcome of the earlier choice point e^A is consistent with the occurrence of the later choice point e^E, namely, the outcome $\Pi_{e^A}\langle e^E\rangle = \boxed{1^A}$. For $i \neq 1$, any set of transitions $\{\tau_i^A, \tau_j^E\}$ $(j \in \{1, \ldots, 8\})$ is inconsistent. One might think that, as inconsistent sets of transitions cannot occur in any history, they could simply be assigned probability zero in a standard Cartesian product space, but this move has disastrous consequences. (As we will show in Chapter 7.3, this seemingly innocent move is one root of Humphreys's paradox.)

Before we discuss this problem, we can make a second observation: as above, we can establish a causally motivated marginal probability—only this time, limited to the *consistent* joint outcomes. Given that Alice's die showed 1, Eve's throw of the octahedron has to have one of its eight possible outcomes, and these together constitute the full set of alternatives; the transition from e^E to the disjunctive outcome consisting of all of e^E's basic outcomes, $e^E \rightarrowtail \{\boxed{1^E}, \ldots, \boxed{8^E}\} = e^E \rightarrowtail \check{\mathbb{1}}_{e^E}$, is deterministic. Accordingly, the disjunctive transition from the joint initial, $\{e^A, e^E\} \rightarrowtail \{\boxed{1^E}, \ldots, \boxed{8^E}\}$, only has one originating cause:[17]

$$CC\left(\{e^A, e^E\} \rightarrowtail \{\boxed{1^E}, \ldots, \boxed{8^E}\}\right) = \{e^A \rightarrowtail \Pi_{e^A}\langle e^E \rangle\} = \{e^A \rightarrowtail \boxed{1^A}\}.$$

Given that Alice threw her die and Eve's throw of her octahedron had any of its possible outcomes, it must be that Alice's die showed 1. So, given that causal probabilities depend only on the *causae causantes*, we have another marginal probability result, this time for the $<$-related initials e^A and e^E:

$$\mu\left(\{e^A, e^E\} \rightarrowtail \{\boxed{1^E}, \ldots, \boxed{8^E}\}\right) = \mu(e^A \rightarrowtail \boxed{1^A}) = 1/6.$$

The above result, we claim, is fundamental: given what causal probabilities are, it must be that full "upward" coarse-graining reduces the set of *causae causantes*, and thus makes a consideration of the upper chance event probabilistically superfluous.

We can now venture to propose a related principle of the upward multiplication of causal probabilities. That principle may, however, have some traces of empirical content—it is hard to be sure.[18] So we flag it as a causal-metaphysical Postulate rather than as a result of logico-metaphysical analysis:

Postulate 7.1. *(Finite-case Markov condition) Let $I \rightarrowtail O$ be an indeterministic BST transition to an outcome chain O such that $CC(I \rightarrowtail O) = \{e_0 \rightarrowtail H_0, e_1 \rightarrowtail H_1, \ldots, e_K \rightarrowtail H_K\}$, $e_0 < e_k$ for $k = 1, \ldots, K$, and $\mu(I \rightarrowtail O)$ is defined. Then*

$$\mu(\{e_0 \rightarrowtail H_0, e_1 \rightarrowtail H_1, \ldots, e_K \rightarrowtail H_K\}) =$$
$$\mu(\{e_0 \rightarrowtail H_0\}) \cdot \mu(\{e_1 \rightarrowtail H_1, \ldots, e_K \rightarrowtail H_K\}).$$

Two remarks are in order. Note first that in the precondition of Postulate 7.1, the set $\{e_0 \rightarrowtail H_0, e_1 \rightarrowtail H_1, \ldots, e_K \rightarrowtail H_K\}$ is required to be the full set of

[17] In the following, we make use of the interchangeability of proposition-like and event-like transitions, noting that the initial events are point-like.

[18] For some pertinent results, see Brierley et al. (2015) on quantum temporal correlations.

causae causantes for the transition $I \rightarrowtail O$ from initial I to the outcome chain O. That is, if $I \rightarrowtail O$ is to occur, exactly these basic transitions, no more and no less, are needed. Second, as O is an outcome chain, this set of basic transitions is consistent (see Def. 6.4), so in particular there is a history h_E such that $\{e_k \mid 0 < k \leqslant K\} \subseteq h_E$. This fact and the ordering relations $e_0 < e_k$ for $k = 1, \ldots, K$ imply that $H_0 = \Pi_{e_0}\langle h_E \rangle$. It is thus uncontroversial that the transition $I \rightarrowtail O$ is adequately represented as consisting of two consecutive steps. The first step is the basic transition $e_0 \rightarrowtail \Pi_{e_0}\langle h_E \rangle$ that enables the initials $e_1, \ldots, e_K$. The second step consists of K basic transitions from these initials, with the proviso that these transitions might work jointly. The fact that there are two consecutive, separate steps strongly supports the multiplication formula for causal probabilities. However, as the K basic transitions of the second step might work jointly, the second multiplicand need not to factor; that is, it can happen that $\mu(\{e_1 \rightarrowtail H_1, \ldots, e_K \rightarrowtail H_K\}) \neq \mu(e_1 \rightarrowtail H_1) \cdot \ldots \cdot \mu(e_K \rightarrowtail H_K)$.[19]

Given Postulate 7.1, the causal probability of two immediately consecutive basic transitions $\tau_0 = e_0 \rightarrowtail H_0$ and $\tau_1 = e_1 \rightarrowtail H_1$ is given by multiplying the respective individual causal probabilities. Whatever its ultimate metaphysical merit, this result is highly plausible. Given that in order for e_1 to occur, e_0 has to have had outcome $H_0 = \Pi_{e_0}\langle e_1 \rangle$, which is a possible event with causal probability $\mu(e_0 \rightarrowtail H_0)$, and that the causal probability for e_1 to result in outcome H_1 is $\mu(e_1 \rightarrowtail H_1)$, and that no other indeterminism occurs between e_0 and e_1, the causal probability for both transitions to occur consecutively is just given by multiplying the two individual values. Further support for our Postulate comes from considerations of causal probabilities attaching to histories. Earlier, we expressed reservations about that idea as a general approach (see note 3), but it provides an instructive illustration here. Assume that the number of histories under consideration is finite and that all histories are equiprobable. Then there has to be a natural number n of histories containing event e_0, and the size of the bundle of histories $H_0 = \Pi_{e_0}\langle e_1 \rangle$ is some natural number m. Given our assumptions, there are no further indeterministic happenings between e_0 and e_1, so that m is also the number of histories containing event e_1. Let the size of the history bundle H_1 be k. As we have $H_1 \subseteq H_{e_1} = H_0 \subseteq H_{e_0}$, we have $k \leqslant m \leqslant n$. By equiprobability of histories, one can read off the causal probabilities of the BST transitions under consideration as fractions:

[19] For a slightly more general discussion in terms of layered spaces, see Müller (2005).

$$\mu(e_0 \rightarrowtail H_0) = \frac{m}{n}; \quad \mu(e_1 \rightarrowtail H_1) = \frac{k}{m}; \quad \mu(e_0 \rightarrowtail H_1) = \frac{k}{n}.$$

It follows that, indeed, we can prove the required instance of Postulate 7.1:

$$\mu(e_0 \rightarrowtail H_1) = \frac{k}{n} = \frac{m}{n} \cdot \frac{k}{m} = \mu(e_0 \rightarrowtail H_0) \cdot \mu(e_1 \rightarrowtail H_1).$$

We can now use Postulate 7.1 to calculate the causal probability of Bob's personally receiving the letter in our "mail gets lost" scenario, as follows:

$$\begin{aligned}\mu(\text{B-noinf} \rightarrowtail \text{B-receives}) &= \mu\left(\{e^A \rightarrowtail \boxed{1^A}, e^E \rightarrowtail \boxed{3^E}\}\right) \\ &= \mu(e^A \rightarrowtail \boxed{1^A}) \cdot \mu(e^E \rightarrowtail \boxed{3^E}) \\ &= 1/6 \cdot 1/8 = 1/48.\end{aligned}$$

Having dealt with the numerical values of the causal probabilities, we can now look into their adequate representation via probability spaces. We said earlier that, as some (in fact, most) combinations of basic outcomes of e^A and of e^E are inconsistent and accordingly cannot have a causal probability, we cannot use the standard Cartesian product construction. The resulting sample space $S^A \times S^E$ contains many inconsistent joint outcomes, such as $\langle \tau_2^A, \tau_5^E \rangle$. We have already warned that it will not do to simply assign these inconsistent outcomes the probability zero and stick to the Cartesian product construction. Especially with a view to our discussion of Humphreys's paradox in Chapter 7.3, it is instructive to see what the consequences would be. Thus, assume that in fact all inconsistent sets of transitions—that is, all pairs $\langle \tau_i^A, \tau_j^E \rangle \in S^A \times S^E$ for which $i \neq 1$—are assigned probability zero. Recall that by the marginal property, we have

$$p^{A\times E}(\{\langle \tau_1^A, \tau_j^E \rangle \mid j = 1, \ldots, 8\}) = p^A(\tau_1^A) = 1/6.$$

Yet, on the other hand, all pairs of transitions in $S^A \times S^E$ whose first element is not equal to τ_1^A have probability zero, and the measure $p^{A\times E}$ has to be normalized. So it has to be that

$$\begin{aligned}1 = \sum_{i=1}^{6}\sum_{j=1}^{8} p^{A\times E}(\langle \tau_i^A, \tau_j^E \rangle) &= \sum_{j=1}^{8} p^{A\times E}(\langle \tau_1^A, \tau_j^E \rangle) \\ &= p^{A\times E}(\{\langle \tau_1^A, \tau_j^E \rangle \mid j = 1, \ldots, 8\}).\end{aligned}$$

This is a contradiction, so something has to give. We argue that, far from showing that the notion of a causal probability for a transition such as $\mu(\text{B-noinf} \rightarrowtail \text{B-receives})$ makes no sense, the contradiction in fact naturally vanishes once we provide a proper analysis of the causal background before which it arises.

The crucial question is: What is an adequate probability space in which the causal probability $\mu(\text{B-noinf} \rightarrowtail \text{B-receives})$ can be represented? Given the causal relations in our scenario, the causal alternatives to $CC(\text{B-noinf} \rightarrowtail \text{B-receives}) = \{e^A \rightarrowtail \boxed{1^A}, e^E \rightarrowtail \boxed{3^E}\}$ are of the following two kinds: (1) Alice's die in fact shows 1, but Eve's octahedron shows an outcome different from 3, or (2) Alice's die shows an outcome different from 1, and Eve never gets to throw her octahedron. That is, instead of the $6 \cdot 8 - 1 = 47$ alternatives according to the Cartesian product construction that was seen to make no sense, there are really only $7 + 5 = 12$ alternatives. Given Postulate 7.1, all of these have well-defined causal probabilities based on the causal probabilities characterizing Alice's and Eve's chance devices: (1) The causal probabilities $\mu(\{e^A \rightarrowtail \boxed{1^A}, e^E \rightarrowtail \boxed{j^E}\})$ can be computed by multiplying the causal probabilities for the respective outcomes of the two chance set-ups. (2) The causal probabilities $\mu(e^A \rightarrowtail \boxed{i^A})$ are all already given via the characterization of Alice's throw of her die. As one can easily check (see Eq. 7.5), on that representation, no problem with marginals ensues.

Building on our causal analysis, we posit the following probability space $\langle S^D, \mathscr{A}^D, p^D \rangle$ for an adequate representation of $\mu(\text{B-noinf} \rightarrowtail \text{B-receives})$: The sample space consists of 13 sets of transitions,

$$S^D =_{\text{df}} \{\{\tau_i^A\} \mid i = 2, \ldots, 6\} \cup \{\{\tau_1^A, \tau_j^E\} \mid j = 1, \ldots, 8\}.$$

The algebra is, as usual, the power-set algebra. For the probability measure, we assign:

$$p^D(\{\tau_i^A\}) = \mu(e^A \rightarrowtail \boxed{i^A}) = 1/6, \quad i = 2, \ldots, 6;$$
$$p^D(\{\tau_1^A, \tau_j^E\}) = \mu(e^A \rightarrowtail \boxed{1^A}) \cdot \mu(e^E \rightarrowtail \boxed{j^E}) = 1/6 \cdot 1/8 = 1/48,$$
$$j = 1, \ldots, 8.$$

We then recover the result of our causal analysis:

$$\mu(\text{B-noinf} \rightarrowtail \text{B-receives}) = p^D(\{\tau_1^A, \tau_3^E\}) = 1/48.$$

As a sanity check, note that our measure is indeed normalized:

$$p^D(\mathbf{1}^D) = \sum_{i=2}^{6} p^D(\{\tau_i^A\}) + \sum_{j=1}^{8} p^D(\{\tau_1^A, \tau_j^E\}) = 5 \cdot 1/6 + 8 \cdot 1/48 = 1. \quad (7.5)$$

Variant 3: Alice, Bob, and Eve. Let us quickly put the two previous scenarios together, such that Bob's transition B-noinf $\rightarrowtail$ B-receives now causally depends on Alice's die, on Bob's coin toss, and on Eve's octahedron, which is thrown iff Alice's die showed 3. That is, the *causae causantes* are now

$$CC(\text{B-noinf} \rightarrowtail \text{B-receives}) = \{e^A \rightarrowtail \boxed{1^A}, e^B \rightarrowtail T^B, e^E \rightarrowtail \boxed{3^E}\}.$$

We assume that the spatio-temporal relations are: $e^A\, SLR\, e^B$, $e^A < e^E$, $e^E\, SLR\, e^B$.

By logico-metaphysical analysis, as above, we have the following results concerning marginal probabilities:

$$\mu(\{e^A \rightarrowtail \boxed{1^A}, e^B \rightarrowtail T^B, e^E \rightarrowtail \mathbf{1}^E\}) = \mu(\{e^A \rightarrowtail \boxed{1^A}, e^B \rightarrowtail T^B\});$$
$$\mu(\{e^A \rightarrowtail \boxed{1^A}, e^B \rightarrowtail \mathbf{1}^B, e^E \rightarrowtail \boxed{3^E}\}) = \mu(\{e^A \rightarrowtail \boxed{1^A}, e^E \rightarrowtail \boxed{3^E}\});$$
$$\mu(\{e^A \rightarrowtail \boxed{1^A}, e^B \rightarrowtail \mathbf{1}^B, e^E \rightarrowtail \mathbf{1}^E\}) = \mu(\{e^A \rightarrowtail \boxed{1^A}\}).$$

Note, however, that the expression $\mu(\{e^A \rightarrowtail \mathbf{1}^A, e^B \rightarrowtail T^B, e^E \rightarrowtail \boxed{3^E}\})$, which also looks like a marginal probability, is undefined: as we said when discussing the previous scenario, it makes no sense to consider the causal probability of an inconsistent set of transitions, and that expression invokes, for example the inconsistent set $\{e^A \rightarrowtail \boxed{2^A}, e^B \rightarrowtail T^B, e^E \rightarrowtail \boxed{3^E}\}$.

As there are *SLR* transitions involved whose joint causal probability is only constrained, but not uniquely determined by the underlying individual causal probabilities, our analysis does not yield a unique verdict on the propensity of the transition B-noinf $\rightarrowtail$ B-receives. However, if there are no space-like correlations, we can simply multiply, which yields

$$\begin{aligned}\mu(\text{B-noinf} \rightarrowtail \text{B-receives}) &= \mu(e^A \rightarrowtail \boxed{1^A}) \cdot \mu(e^B \rightarrowtail T^B) \cdot \mu(e^E \rightarrowtail \boxed{3^E}) \\ &= 1/6 \cdot 1/2 \cdot 1/8 = 1/96.\end{aligned}$$

The important question is, again, which mathematical probability space is adequate for representing this causal probability. We take our lead from the previous case, in which the recipe was to *consider all consistent causal alternatives* that could be based on the local indeterministic happenings involved and on their local alternatives. That is, with a view toward a general recipe:

- We consider the set $\tilde{T}$ of all basic indeterministic transitions that are either a member of CC(B-noinf $\rightarrowtail$ B-receives), or a local alternative to such a member.
  Given that the *causae causantes* include transitions from the initials e^A, e^B, and e^E, we have the 16-element set

$$\tilde{T} = \{e^A \rightarrowtail \boxed{i^A} \mid i = 1, \ldots, 6\} \cup \{e^B \rightarrowtail T^B, e^B \rightarrowtail H^B\} \cup \{e^E \rightarrowtail \boxed{i^E} \mid i = 1, \ldots, 8\}.$$

- Many subsets of $\tilde{T}$ are inconsistent and thus cannot have a causal probability assigned.[20] We therefore take the sample space of our probability space to consist of consistent subsets of $\tilde{T}$ only. We already know that some consistent subsets, such as $\{e^A \rightarrowtail \boxed{1^A}\}$, come up via considerations of marginals. These sets should thus not be elements of the sample space, but elements of the algebra. As a general recipe, we take our sample space S to be the set of *maximal consistent subsets of* $\tilde{T}$.
- As this is a finite set, we let the algebra $\mathscr{A}$ be the power set algebra of S.
- The probability measure p on the space defined by S and $\mathscr{A}$ has to represent the corresponding causal probabilities μ. Given the causal structure of our scenario, the measure p is constrained (via marginals) by the measures p^A, p^B, and p^E of the probability spaces representing the individual chance devices at issue, which in turn represent the corresponding single-device causal probabilities. The measure p, however, is not uniquely determined by p^A, p^B, and p^E, due to the possibility of space-like correlations between outcomes of e^A and e^B, and between outcomes of e^E and e^B. Whether such correlations obtain is a fact about the corresponding propensities of sets of transitions.

[20] As stated in note 7 we do not categorically rule out extreme cases in which a real possibility may have probability zero. An inconsistent set of transitions, however, does not represent a real possibility at all, and consequently cannot have a probability assigned. Probabilities are, after all, graded possibilities.

- In the (here, highly plausible) case that such space-like correlations are absent, the measure is simply the product measure. But as we are here dealing with a construction that is not based on Cartesian products, we need to write the product measure in a somewhat non-standard way: For $T \in S$ a maximal consistent set of transitions, we have

$$p(T) = \prod_{\tau \in T} p^*(\tau),$$

where p^* is the appropriate single-device probability. Thus, for example, $p(\{\tau_4^A, \tau_2^B\}) = p^A(\tau_4^A) \cdot p^B(\tau_2^B) = 1/6 \cdot 1/2 = 1/12.$

Given this recipe and assuming no *SLR* correlations, we can thus compute

$$p(\{\tau_1^A, \tau_2^B, \tau_3^E\}) = p^A(\tau_1^A) \cdot p^B(\tau_2^B) \cdot p^E(\tau_3^E) = 1/6 \cdot 1/2 \cdot 1/8 = 1/96,$$

in full accordance with our causal analysis for the no correlation case.

7.2.4 General probability spaces in BST

Based on our discussion of a few simple but exemplary cases, we can now extract a general recipe for representing causal probabilities of BST transitions in mathematically well-defined probability spaces. We maintain our two simplifying assumptions: first, we assume that the structures we will be dealing with are all finite: we will always be dealing with transitions $I \rightarrowtail \mathscr{O}^*$ for which the set of *causae causantes* is finite, and for any choice point e that is an initial of one of the *causae causantes* (i.e., for any cause-like locus $e \in cll(I \rightarrowtail \mathscr{O}^*)$), the number of different local alternatives splitting off at e (the cardinality of Π_e) is finite.[21] Second, we assume that there is no modal funny business of the kind discussed in Chapter 5. Thus, for example, for any

[21] If a set of local alternatives from an initial e, Π_e, is infinite, one can use standard tools of measure theory (e.g., Borel sets) to generalize our approach. In case the set of initials itself is infinite, different approaches appear to be needed for the case in which there is an infinite set of *SLR* initials and for the case in which there is an infinite chain of initials. In the first case, the tools of standard probability theory for infinite product spaces (cylinder sets, zero-one laws) will apply. In the second case, the situation appears to be more challenging, as upward multiplication may trivialize the resulting probabilities. It is metaphysically interesting to investigate these cases, as, for example, in the modal theory of agency (Belnap et al., 2001), 'busy choice sequences' are analyzed whose probabilistic equivalent exactly requires a probability theory for infinite chains of transitions. We will leave this issue to one side and continue working with finite structures.

set of *SLR* initials, *all* combinations of local outcomes can be assumed to be consistent. That is, we assume that there are no *modal* correlations that would restrict the space of possibilities. This assumption, however, leaves open whether or not there are *probabilistic* correlations between distant outcomes, which will be the topic of Chapter 7.2.6.

In our preceding discussions it has been crucial to get the sample space right, and we motivated the choice of the sample space as a set of sets of transitions via a consideration of the causal alternatives of a given BST transition. Following exactly the recipe given above, we now work toward a general definition of a probability space based on a consistent set of basic transitions, such as provided via the *causae causantes* of an outcome chain or of a scattered outcome.

Definition 7.2 (Causal alternatives)**.** Let $\langle W, < \rangle$ be a BST_{92} structure, and let T be a consistent set of basic transitions, the initials of which form the set E. The *causal alternatives for* T are the sets of transitions in S, where

$$\begin{aligned} S &=_{\text{df}} \{T' \subseteq \tilde{T} \mid T' \text{ is maximally consistent}\}, \text{ and where} \\ \tilde{T} &=_{\text{df}} \{e \rightarrowtail H \mid e \in E \text{ and } H \in \Pi_e\}. \end{aligned} \tag{7.6}$$

Before we define probabilistic BST_{92} structures and causal probability spaces, we list again the constraints on the measure μ that have emerged in our examples above, and which we will postulate to hold in general.

Markov condition. First, for the sake of completeness, we repeat the statement of the finite-case Markov condition already introduced above:

Postulate 7.1. *(Finite-case Markov condition) Let $I \rightarrowtail O$ be an indeterministic BST transition to an outcome chain O such that $CC(I \rightarrowtail O) = \{e_0 \rightarrowtail H_0, e_1 \rightarrowtail H_1, \ldots, e_K \rightarrowtail H_K\}$, $e_0 < e_k$ for $k = 1, \ldots, K$, and $\mu(I \rightarrowtail O)$ is defined. Then*

$$\mu(\{e_0 \rightarrowtail H_0, e_1 \rightarrowtail H_1, \ldots, e_K \rightarrowtail H_K\}) = \mu(\{e_0 \rightarrowtail H_0\}) \cdot \mu(\{e_1 \rightarrowtail H_1, \ldots, e_K \rightarrowtail H_K\}).$$

Partial function. Fundamentally, we allow μ to be a partial function. We hold, however, that μ should not contain weird gaps in its domain: if μ is defined for some transitions, it should also be defined for parts of their *causae causantes* and for local causal alternatives to them. We can now spell out this constraint in a precise way, as the following causal-metaphysical Postulate:

Postulate 7.2 (Constraints on the domain of μ). *If μ is defined for the* causae causantes *of a BST transition $I \rightarrowtail \mathcal{O}^*$, then $\mu(I \rightarrowtail \mathcal{O}^*) = \mu(CC(I \rightarrowtail \mathcal{O}^*))$, and μ is also defined for the causal alternatives to $I \rightarrowtail \mathcal{O}^*$ and for all subsets of $CC(I \rightarrowtail \mathcal{O}^*)$.*

Unavoidable transitions have causal probability one. The causal probability of a non-trivial unavoidable transition, that is, of a transition to an exhaustive disjunction of alternatives, has to be one. Given that the initial occurs, the world has to continue in some way. All these ways together exhaust the possible alternatives, and therefore, their disjunction is certain to happen. We formulate this constraint as the general logico-metaphysical Postulate of the law of total causal probability:

Postulate 7.3 (Law of total causal probability). *Let $\{T_\gamma \mid \gamma \in \Gamma\}$ be a set of non-empty consistent sets of indeterministic transitions that are pairwise incompatible, i.e., for any $\gamma, \gamma' \in \Gamma$, if $\gamma \neq \gamma'$, then $H(T_\gamma) \cap H(T_{\gamma'}) = \emptyset$. Suppose further that there is an initial event (a consistent subset of W) I such that $H_{[I]} \subseteq \bigcup_{\gamma\in\Gamma} H(T_\gamma)$. Then, as these sets T_γ of transitions partition the possible ways in which I can occur, so that exactly one of them has to occur given that I occurs, if all the $\mu(T_\gamma)$ are defined, we have*

$$\sum_{\gamma\in\Gamma} \mu(T_\gamma) = 1.$$

This postulate applies naturally to transitions to disjunctive outcomes, as follows. Consider such a transition $I \rightarrowtail \check{\mathbf{O}}$, and let $T_\gamma = CC(I \rightarrowtail \hat{O}_\gamma)$, where $\check{\mathbf{O}} = \{\hat{O}_\gamma \mid \gamma \in \Gamma\}$. Then the T_γ fulfill the conditions of Postulate 7.3: We have $H(T_\gamma) \cap H(T_{\gamma'}) = \emptyset$ for $\gamma \neq \gamma'$ by the definition of a disjunctive outcome event (see Defs. 4.3(4) and 4.9), and $H_{[I]} = \bigcup_{\gamma\in\Gamma} H(T_\gamma)$ as $I \rightarrowtail \check{\mathbf{O}}$ is unavoidable. So $\mu(I \rightarrowtail \check{\mathbf{O}}) = \sum_{\gamma\in\Gamma} \mu(T_\gamma) = 1$.

Note that the premises of Postulate 7.3 are not trivial: there has to be a concrete initial I for which $H_{[I]} \subseteq \bigcup_{i\in\Gamma} H(T_i)$. Such an I may be hard to find. If one tries, for example, to take the set of minimal initials from the union of all the T_is, that set may well be inconsistent, which disqualifies it from being a BST initial. And if one tries to locate I in the common past of these initials (guided by the PCP), the premises may fail because the set $H_{[I]}$ stops being a subset of $\bigcup_{i\in\Gamma} H(T_i)$. Thus, the set $\{T_i \mid i \in \Gamma\}$ has to be of a special kind to admit an initial I satisfying the premises of the Postulate. Reading the premises informally, the T_is have to represent *all* alternative possible continuations that the world can take, provided that I occurs. In that case, the probability that any of these continuations will occur is one, and that is also the sum of the probabilities of each of the continuations taken separately, as these continuations are pairwise incompatible alternatives.[22]

Marginal probabilities. In a similar vein, we formulate a general Postulate for marginal probabilities, written in terms of sets of basic transitions:

Postulate 7.4 (General marginal probabilities). *Let $T = \{e_i \rightarrowtail H_i \mid i \in \Gamma\}$ be a set of basic transitions, and let $E =_{\text{df}} \{e_i \mid i \in \Gamma\}$ be the set of initials of transitions from T. Let $e \in W \setminus E$ be a new initial such that for any $e_i \in E$, either $e_i < e$ or $e_i \, SLR \, e$, so that e is maximal in the set $E \cup \{e\}$. Assume that $\mu(T)$ is defined and that also $\mu(T \cup \{e \rightarrowtail H_0\})$ is defined for some $H_0 \in \Pi_e$ (it is then defined for* all *such H, by Postulate 7.2). Then we have*

$$\sum_{H\in\Pi_e} \mu(T \cup \{e \rightarrowtail H\}) = \mu(T).$$

This postulate also results from logico-metaphysical analysis: the summation formula is based on the observation that the basic transitions from a maximal element of $E \cup \{e\}$ are truly alternative; that is, they have pairwise inconsistent outcomes, while not affecting the occurrence or non-occurrence of any of the other transitions (given no MFB). Note that, in contrast to the Cartesian product construction, and in line with our previous discussioin, the summation formula applies only to transitions issuing from a *maximal* element of $E \cup \{e\}$.[23]

[22] It is tempting to broaden the postulate to also cover trivial deterministic transitions, such as from a deterministic initial e to its only outcome $\Pi_e = \{H_e\}$, or to the reduced set of *causae causantes* for a deterministic transition to a disjunctive outcome. The problem in these cases is, however, that the set of *causae causantes* is then empty, so that the resulting algebra of subsets would have just one element, $\emptyset$—and there is no way to consistently define a probability measure on an algebra that has only one element, as one can see from Def. 7.1. Metaphysically, a trivial transition that happens anyway surely happens with certainty, but a probability space always has to include at least a probability-zero alternative, which is lacking if the set of *causae causantes* of a transition is empty.

[23] Note that the Cartesian product construction of standard probability theory is adequate precisely if all initials are maximal in the set of all initials, i.e., if they are all pairwise SLR (and there is no MFB). As standard probability theory formalizes neither space nor time (let alone space-time or modal correlations), this precondition of the Cartesian product construction is not—cannot be—made explicit. As we have shown, however, it is indeed crucial, and it fails to apply already in simple, everyday scenarios.

Based on the Postulates summarized above, we now define *probabilistic* BST_{92} *structures*:[24]

Definition 7.3 (Probabilistic BST structure). A *probabilistic* BST_{92} structure is a triple $\mathscr{W} = \langle W, <, \mu \rangle$, where $\langle W, < \rangle$ is a BST_{92} structure and the propensity function μ is a partial function defined on sets of indeterministic basic transitions in $\langle W, < \rangle$, $\mu : \mathscr{P}(\text{TR}(W)) \mapsto [0,1]$, that satisfies the Markov Postulate 7.1, Postulate 7.2 concerning the domain of μ, Postulate 7.3 of the law of total probability, and Postulate 7.4 of general marginal probabilities.

We can now give our general definition of causal probability spaces based on an arbitrary BST transition. We will proceed in two steps, first giving a definition for probability spaces based on transitions to outcome chains or scattered outcomes, and then for transitions to disjunctive outcomes. This division results from the fact that causal alternatives look different in these two cases, which implies that the base sets of the probability spaces are constructed differently. We begin with the first case. All the probability spaces discussed in our examples above fulfill this definition, as is easy to check.

Definition 7.4 (Causal probability spaces, $O/\hat{O}$ version). Let $\langle W, <, \mu \rangle$ be a probabilistic BST_{92} structure in which there is no MFB, and let $I \rightarrowtail \mathscr{O}^*$ be an indeterministic transition from an initial I to an outcome chain or a scattered outcome $\mathscr{O}^*$ for which $\mu(CC(I \rightarrowtail \mathscr{O}^*))$ is defined. The *causal probability space based on* $I \rightarrowtail \mathscr{O}^*$, $CPS(I \rightarrowtail \mathscr{O}^*)$, is the probability space $\langle S, \mathscr{A}, p \rangle$, where S is the set of causal alternatives for $CC(I \rightarrowtail \mathscr{O}^*)$ according to Def. 7.2, $\mathscr{A}$ is the power-set algebra over S, and the measure p on $\mathscr{A}$ is induced via the measure assigned to the elements $T \in S$ via $p(T) = \mu(T)$.

We have to establish that the object defined in this way, $\langle S, \mathscr{A}, p \rangle$, is in fact a probability space fulfilling Def. 7.1. This is the subject of the following lemma:

Lemma 7.1. *Let the conditions of Def. 7.4 hold for an indeterministic transition* $I \rightarrowtail \mathscr{O}^*$*, with* $\mathscr{O}^*$ *an outcome chain or a scattered outcome, and consider* $CPS(I \rightarrowtail \mathscr{O}^*) = \langle S, \mathscr{A}, p \rangle$*. That triple is in fact a probability space satisfying Def. 7.1. That is,* $CPS(I \rightarrowtail \mathscr{O}^*)$ *is well defined and* p *is a normalized measure on* $\mathscr{A}$*. Furthermore, we have that*

[24] Note the somewhat different status of the Postulates: While Postulates 7.3 and 7.4 result from purely metaphysical considerations, Postulates 7.1 and 7.2 also have causal underpinnings.

$$CC(I \rightarrowtail \mathscr{O}^*) \in S \quad \textit{and} \quad p(CC(I \rightarrowtail \mathscr{O}^*)) = \mu(I \rightarrowtail \mathscr{O}^*).$$

Proof. Let $E =_{\text{df}} cll(I \rightarrowtail \mathscr{O}^*)$, let $T_0 =_{\text{df}} CC(I \rightarrowtail \mathscr{O}^*)$, let $\tilde{T} =_{\text{df}} \{e \rightarrowtail H \mid e \in E, H \in \Pi_e\}$, and let S be the set of maximally consistent sets of transitions from $\tilde{T}$.

For well definedness, we have to prove that for any $T \in S$, $\mu(T)$ is defined. This follows directly from Postulate 7.2. As a consequence, $p(T)$ is defined via Def. 7.4, inducing the full measure p on $\mathscr{A}$.

To see that $CC(I \rightarrowtail \mathscr{O}^*) = T_0 \in S$, note first that according to Def. 7.2, E is the set of initials of basic transitions from T_0, so that $T_0 \subseteq \tilde{T}$. Furthermore, T_0 is consistent since any set of *causae causantes* of a transition to a scattered outcome or outcome chain is consistent. There is thus some $T_1 \in S$ for which $T_0 \subseteq T_1$. To see that $T_0 = T_1$, assume otherwise, so that there would have to be some $\tau_1 = e_1 \rightarrowtail H_1 \in T_1 \setminus T_0$. By the construction of S, it has to be that $e_1 \in cll(I \rightarrowtail \mathscr{O}^*)$. Note that there is some basic transition $e_1 \rightarrowtail H_0 \in T_0$, $H_0 \in \Pi_{e_1}$, as e_1 is a cause-like locus for $I \rightarrowtail \mathscr{O}^*$. As $\tau_1 \notin T_0$ by assumption, it has to be that $\tau_1 = e_1 \rightarrowtail H_1$ for some $H_1 \in \Pi_{e_1}$, $H_1 \neq H_0$. But then we have $H(T_1) \subseteq H_0 \cap H_1 = \emptyset$, showing that T_1 is inconsistent. So T_0 is in fact maximal consistent, and $T_0 \in S$. The claim that $p(T_0) = \mu(I \rightarrowtail \mathscr{O}^*)$ then follows by definition of p and by our general assumption that $\mu(I \rightarrowtail \mathscr{O}^*) = \mu(CC(I \rightarrowtail \mathscr{O}^*))$, which is part of Postulate 7.2.

It remains for us to show that p is indeed a normalized probability measure. By Def. 7.4, the measure on the algebra is induced in the standard way by the probabilities assigned to the elements of the sample space, so additivity holds by construction. Therefore it suffices to show that the probabilities assigned to the different elements of S sum to one, as then

$$p(\mathbf{1}_{\mathscr{A}}) = \sum_{T \in S} p(T) = 1.$$

Our proof uses the law of total probability in the form of Postulate 7.3. We thus have to show that the elements of S partition the set of histories in which the initial I occurs. First we have to show that for two different $T_1, T_2 \in S$, $T_1 \neq T_2$, we have $H(T_1) \cap H(T_2) = \emptyset$. This follows immediately from each T_1 and T_2 being a maximal consistent subset of $\tilde{T}$. We need next to show $H_{[I]} \subseteq \bigcup_{T \in S} H(T)$. Thus, for an arbitrary $h \in H_{[I]}$ we have to find some $T \in S$ for which $h \in H(T)$. There are two cases.

Case 1: $h \in H_{\langle \mathscr{O}^* \rangle}$ for the given $\mathscr{O}^*$ that defined $CPS(I \rightarrowtail \mathscr{O}^*)$. In this case, we have found the element $T_0 = CC(I \rightarrowtail \mathscr{O}^*) \in S$ such that $h \in H(T_0)$, and we are done.

Case 2: $h \notin H_{\langle \mathscr{O}^* \rangle}$. In this case, h is a witness for some $e_0 \in E = cll(I \rightarrowtail \mathscr{O}^*)$, such that $h \perp_{e_0} H_{\langle \mathscr{O}^* \rangle}$. We define the set

$$T' =_{\text{df}} \{e \rightarrowtail \Pi_e \langle h \rangle \mid e \in E \cap h\}.$$

By definition, $h \in H(T')$, so that T' is consistent. Furthermore, its initials belong to E, so we have $T' \subseteq \tilde{T}$. To show that T' is maximally consistent, take some $\tau' = e' \rightarrowtail H' \in \tilde{T} \setminus T'$, for which $e' \in E$ and $H' \in \Pi_{e'}$. We have to show that $T' \cup \{\tau'\}$ is inconsistent. There are again two cases. (1) Either $e' \in h$, whence $H' \neq \Pi_{e'}\langle h \rangle$ by $\tau' \notin T'$, showing that $T' \cup \{\tau'\}$ is (blatantly) inconsistent. (2) The other case is that $e' \notin h$. Let $h' \in H_{e'}$. We have $e' \in h' \setminus h$, and so, by PCP_{92}, there is some $c < e'$ for which $h \perp_c h'$. By Fact 3.8, $h \perp_c H_{e'}$. We claim that $c \in E$. Since $e' \in E = cll(I \rightarrowtail \mathscr{O}^*)$, there is some history $h^* \in H_{[I]}$ for which $h^* \perp_{e'} H_{\langle \mathscr{O}^* \rangle}$. So $H_{\langle \mathscr{O}^* \rangle} \subseteq H_{e'}$, which implies $h \perp_c H_{\langle \mathscr{O}^* \rangle}$. This, however, is the defining expression for $cll(I \rightarrowtail \mathscr{O}^*)$ (note that $h \in H_{[I]}$ by our initial assumption), i.e., we have $c \in E$, and as $c \in h$ as well, we have established that the transition $c \rightarrowtail \Pi_c\langle h \rangle \in T'$. On the other hand, we have $h \perp_c h'$, and as the new initial $c < e'$ and $e' \in h' \setminus h$, we have $H' \subseteq \Pi_c\langle e' \rangle$. But $\Pi_c\langle e' \rangle \neq \Pi_c\langle h \rangle$, so that $T' \cup \{\tau'\}$ is in fact inconsistent. So, in both cases, $T' \in S$ and $h \in H(T')$.

This establishes $H_{[I]} \subseteq \bigcup_{T \in S} H(T)$. With the premises of the law of total causal probability (Postulate 7.3) satisfied, we therefore have

$$\sum_{T \in S} p(T) = \sum_{T \in S} \mu(T) = 1,$$

showing that the measure p is indeed normalized. □

The Definition 7.4 of causal probability spaces requires a small modification to apply to a transition to a disjunctive outcome, $I \rightarrowtail \check{\mathbf{O}}$: For $\check{\mathbf{O}} = \{\hat{O}_\gamma \mid \gamma \in \Gamma\}$, we take S to be the set of maximal consistent subsets of $\tilde{T} =_{\text{df}} \bigcup_{\gamma \in \Gamma} \{e \rightarrowtail H \mid e \in cll(I \rightarrowtail \hat{O}_\gamma) \wedge H \in \Pi_e\}$. The transition to the disjunctive outcome is then represented not via an element of S, but via an element of the algebra $\mathscr{A}$.

Definition 7.5 (Causal probability spaces, $\check{\mathbf{O}}$ version). Let $\langle W, <, \mu \rangle$ be a probabilistic BST_{92} structure in which there is no MFB. Let $I \rightarrowtail \check{\mathbf{O}}$ be a transition to a disjunctive outcome $\check{\mathbf{O}} = \{\hat{O}_\gamma \mid \gamma \in \Gamma\}$, and let $\mu(CC(I \rightarrowtail \hat{O}_\gamma))$ be defined for every $\gamma \in \Gamma$. The *causal probability space based on* $I \rightarrowtail \check{\mathbf{O}}$, $CPS(I \rightarrowtail \check{\mathbf{O}})$, is the probability space $\langle S, \mathscr{A}, p \rangle$, where S is the set of maximal consistent subsets of $\tilde{T} =_{\text{df}} \bigcup_{\gamma \in \Gamma} \{e \rightarrowtail H \mid e \in cll(I \rightarrowtail \hat{O}_\gamma) \wedge H \in \Pi_e\}$, $\mathscr{A}$ is the power-set algebra over S, and the measure p on $\mathscr{A}$ is induced via the measure assigned to the elements $T \in S$ via $p(T) = \mu(T)$.

One might be worried by a difference, both set-theoretical and content-wise, between *causae causantes* for a transition to a disjunctive outcome, $I \rightarrowtail \check{\mathbf{O}}$, and the base set S of the causal probability space based on $I \rightarrowtail \check{\mathbf{O}}$. To recall, $CC(I \rightarrowtail \check{\mathbf{O}})$ is the *set* of $CCr(I \rightarrowtail \hat{O})$, with $\hat{O} \in \check{\mathbf{O}}$. This seems right, as each $CCr(I \rightarrowtail \hat{O})$ stands for a separate path leading to the production of disjunctive outcome $\check{\mathbf{O}}$. By taking the union of all $CCr(I \rightarrowtail \hat{O})$, we will typically lose the information about these alternative paths. Furthermore, the appeal to reduced sets, $CCr(I \rightarrowtail \hat{O})$, rather than *causae causantes* simpliciter, $CC(I \rightarrowtail \hat{O})$, stems from the fact that, given the disjunctive nature of an outcome in question, not every element of the latter set is needed for the production of that disjunctive outcome. Recall that at the extreme case of a deterministic transition to a disjunctive outcome, $CC(I \rightarrowtail \check{\mathbf{O}})$ is (the singleton of) the empty set.

As for set S, which is the set of transitions, it encodes information about alternative paths to a disjunctive outcome, as among members of S there are all sets $CC(I \rightarrowtail \hat{O})$, where $\hat{O} \in \check{\mathbf{O}}$. And in constructing S we appeal to $CC(I \rightarrowtail \hat{O})$ rather than to $CCr(I \rightarrowtail \hat{O})$ (literally speaking, to $cll(I \rightarrowtail \hat{O})$ rather than $cllr(I \rightarrowtail \hat{O})$), since even in a case of a deterministic disjunctive outcome, we want to accommodate all the underlying causal information. We take it that determinism should come up at a probabilistic level, by $I \rightarrowtail \check{\mathbf{O}}$ getting assigned probability $p(I \rightarrowtail \check{\mathbf{O}}) = 1$.

We need to check whether the object thus introduced is a probability space, that is, whether it satisfies Def. 7.1. The following Lemma states that this is indeed the case.

Lemma 7.2. *Let the conditions of Def. 7.5 hold for a transition* $I \rightarrowtail \check{\mathbf{O}}$ *to a disjunctive outcome* $\check{\mathbf{O}}$*, and consider* $CPS(I \rightarrowtail \check{\mathbf{O}}) = \langle S, \mathscr{A}, p \rangle$*. That triple is in fact a probability space satisfying Def. 7.1. That is,* $CPS(I \rightarrowtail \check{\mathbf{O}})$ *is well defined and* p *is a normalized measure on* $\mathscr{A}$*. Furthermore, we have that*

$$p(CC(I \rightarrowtail \hat{O}_\gamma)) = \mu(\{T \in S \mid CC(I \rightarrowtail \hat{O}_\gamma) \subseteq T\}) = \sum_{T \in S,\, CC(I \rightarrowtail \hat{O}_\gamma) \subseteq T} \mu(T);$$

$$p(CC(I \rightarrowtail \breve{\mathbf{O}})) = \sum_{\gamma \in \Gamma} p(CC(I \rightarrowtail \hat{O}_\gamma)).$$

Proof. For the proof, one needs to adapt the proof of Lemma 7.1. We leave this task as Exercise 7.1. □

Although officially, Defs. 7.4 and 7.5 require one to base a causal probability space on a concrete BST transition, such a space can be based on any set of consistent basic transitions, as any such set provides a well-defined notion of causal alternatives (see Def. 7.2).[25]

We are now in a position to provide a full answer to the question about the formal structures in which transitions are assigned probabilities.

Full answer to Question 7.2: When $\mu(I \rightarrowtail \mathscr{O}^*)$ is defined, that causal probability is represented in a causal probability space as defined via Defs. 7.4 and 7.5.

7.2.5 Representing transitions in different causal probability spaces

Before we end our introduction of causal probability spaces with a discussion of space-like correlations (which we will call "probabilistic funny business"), there is one topic left to discuss: how are concrete BST transitions represented in different causal probability spaces? This topic is important especially in light of the formal challenge of Humphreys's paradox, which will be treated in Chapter 7.3. In our general approach, BST transitions and their propensities are basic, while causal probability spaces are derivative. We have already stated that, in general, the propensity function μ is not a probability measure, but the connection of propensities to probabilities is, of course, intimate. We stress this connection by discussing the representation

[25] Indeed, as one can see from our definitions, it is enough to specify the set E of initials. This approach is investigated in Müller (2005). That paper also provides a discussion of a generalization of the Markov property and an extended discussion of the representation of transitions in different probability spaces. The latter discussion forms the background for the following remarks on representability.

of the propensity of a BST transition in different causal probability spaces.

The formal setting of this investigation is as follows. Let $\langle W, <, \mu \rangle$ be a probabilistic BST structure, and let $I \rightarrowtail \mathcal{O}^*$ be an indeterministic BST transition from initial I to outcome $\mathcal{O}^*$. We assume for simplicity's sake that $\mathcal{O}^*$ is an outcome chain or a scattered outcome. Assume that $\mu(I \rightarrowtail \mathcal{O}^*)$ is defined. Now let $CPS = \langle S, \mathscr{A}, p \rangle$ be some causal probability space definable on the basis of $\langle W, <, \mu \rangle$. We will discuss the following three questions:

- When is the transition $I \rightarrowtail \mathcal{O}^*$ representable in CPS?
- If that transition is representable, how is it represented?
- How can the numerical value of $\mu(I \rightarrowtail \mathcal{O}^*)$ be recovered from p?

The simplest case is, of course, $CPS = CPS(I \rightarrowtail \mathcal{O}^*)$, where the answers are immediate from Lemma 7.1. Almost equally simple is the case in which $T =_{\text{df}} CC(I \rightarrowtail \mathcal{O}^*) \in S$, that is, in which the transition in question is a causal alternative to the transition on which the space CPS is based; see again Lemma 7.1.

It is also clear that for $I \rightarrowtail \mathcal{O}^*$ to be representable in CPS, all the transitions in T have to occur in the transition sets that make up the sample space S. Let $\mathscr{S} =_{\text{df}} \bigcup S$ be the set of all transitions that belong to some element of the sample space. Then we can say that for representability of $I \rightarrowtail \mathcal{O}^*$ in CPS it is necessary that $T = CC(I \rightarrowtail \mathcal{O}^*) \subseteq \mathscr{S}$. If this condition is fulfilled, there are two cases, depending on how T is situated with respect to the other transitions present in $\mathscr{S}$. The crucial issue is whether or not $\mathscr{S}$ contains any transitions that precede (in the sense of the transition ordering $\prec$) any of the transitions from T.

Case 1: There is no $\tau' \in \mathscr{S}$ for which $\tau' \prec \tau$ for any $\tau \in T$. In this case, consider the element $a \in \mathscr{A}$ defined via

$$a =_{\text{df}} \{T' \in S \mid T \subseteq T'\}.$$

That element a is either equal to $\{T\}$ (leading back to the simplest cases above), or it is an extension (we might say, a fine-graining) of T via later or SLR transitions. By the marginal property (Postulate 7.4), perhaps applied a number of times, we therefore have

$$p(a) = \mu(a) = \mu(T) = \mu(CC(I \rightarrowtail \mathcal{O}^*)),$$

thus answering all our questions satisfactorily. For an illustration, consider our discussion of how to represent Alice's throw of her die in the "Variant 2" story above (p. 177).

Case 2: There is some $\tau' \in \mathscr{S}$ and some $\tau \in T$ for which $\tau' \prec \tau$; that is, the transition $I \rightarrowtail \mathscr{O}^*$ has causal preconditions (e.g., τ') that are made explicit in CPS. This case is somewhat more tricky—in fact, it is at the root of the problematic assumption (CI) in the statement of Humphreys's paradox (see Chapter 7.3.3). We can discuss this case in the context of the "Variant 2" story as well. To recall, a letter to Bob is transmitted iff first, Alice's throw of a die (initial e^A) has outcome $\boxed{1^A}$, and then, Eve's throw of an octahedron (initial e^E) has outcome $\boxed{3^E}$. As discussed earlier, the adequate causal probability space $CPS = CPS(\{e^A \rightarrowtail \boxed{1^A}, e^E \rightarrowtail \boxed{3^E}\})$ has a sample space consisting of the 13 maximally consistent combinations of basic transitions from e^A and basic transitions from e^E. Now consider the transition $\tau =_{\text{df}} e^E \rightarrowtail \boxed{5^E}$ of Eve's throwing her octahedron with result 5. Clearly, $\tau \in \mathscr{S}$, because the set of transitions $T =_{\text{df}} \{e^A \rightarrowtail \boxed{1^A}, e^E \rightarrowtail \boxed{5^E}\} \in S$. That set, however, also contains the transition $\tau' =_{\text{df}} e^A \rightarrowtail \boxed{1^A}$, for which $\tau' \prec \tau$.

In this case, the transition τ is sufficient to uniquely identify the set of transitions T: saying that Eve's octahedron has shown 5 amounts to saying that Alice's die has shown 1 and then Eve's octahedron has shown 5, because the former is a causal precondition of the latter. But the two sets of transitions, T and $\{\tau\}$, are different, and their propensities are different as well (unless τ' is inevitable). In fact, by the Markov property (Postulate 7.1), we have

$$\mu(T) = \mu(\tau') \cdot \mu(\tau).$$

It turns out that our questions have no simple answer in this case: there is no element a of the algebra $\mathscr{A}$ that represents τ such that the propensity of that transition ($1/8$, in our example) could be read off as $p(a)$. It is, however, possible to recover $\mu(\tau)$ as the conditional propensity of T given τ', as we will show in detail in Chapter 7.3.3.

7.2.6 Probabilistic funny business

For our discussion of quantum correlations in Chapter 8, there is one substantial task left: We need to provide a representation of non-local probabilistic correlations in BST. We provide details in the context of a discussion of random variables and their dependence or independence in the framework of causal probabilities.

What we call probabilistic funny business (PFB) consists of the probabilistic correlations between outcomes of space-like related events in a BST_{92} structure in which there is no MFB. Ultimately we will define PFB in terms of dependence of random variables. For ease of presentation, however, we begin with the rudimentary example of a non-local probabilistic correlation involving two basic outcomes $\tau_1 = e_1 \rightarrowtail H_1$ and $\tau_2 = e_2 \rightarrowtail H_2$, with $H_1 \in \Pi_{e_1}$, $H_2 \in \Pi_{e_2}$, and initials $e_1\, SLR\, e_2$. Given no MFB, the transitions τ_1 and τ_2 are compatible, i.e., $H_1 \cap H_2 \neq \emptyset$. For this simple example, the causal probability space $CPS = \langle S, \mathscr{A}, p\rangle$ is based on the set $\{\tau_1, \tau_2\}$, so $\{\tau_1, \tau_2\} \in S$. We then say that τ_1 and τ_2 exhibit non-local probabilistic correlations iff

$$p(\{\tau_1, \tau_2\}) \neq p(\tau_1) \cdot p(\tau_2),$$

where $p(\tau_1) = p(\bigcup_{T \in S, \tau_1 \in T} T) = \sum_{T \in S, \tau_1 \in T} p(T)$, and analogously for $p(\tau_2)$, are the marginal probabilities for τ_1 and for τ_2. Correspondingly, at the level of the causal probabilities μ represented via p, a non-local probabilistic correlation in this case is of the form

$$\mu(\{\tau_1, \tau_2\}) \neq \mu(\tau_1) \cdot \mu(\tau_2).$$

We can generalize this idea to two generic transitions with SLR initials I and I' and scattered outcomes $\hat{O}$ and $\hat{O}'$, respectively. Further generalizations to more than two transitions, or to transitions to disjunctive outcomes, are natural (see Placek, 2010).

Probabilistic correlations are often defined in terms of the dependence of random variables, and analyses of non-local quantum correlations are typically expressed in the framework of random variables. In Chapter 8 we embark on such an analysis: using probabilistic BST_{92} structures we ask whether, and under what conditions, non-local quantum correlations can be accounted for by hidden local factors. To this end, we need to define random variables and their (in)dependence in the framework of causal probabilities.

As in standard probability theory, in our theory a random variable X is a function defined on the base set S of a causal probability space $\langle S, \mathscr{A}, p\rangle$, where the values of X are real numbers. (One may consider other ranges of random variables, but we fix $\mathbb{R}$ as the range in order to be specific.) Just as in the standard theory, random variables allow for the definition of correlations and (probabilistic) dependence and independence:

Definition 7.6 (Independence and correlations). A family $X_1, \ldots, X_n$ of random variables defined on a causal probability space $CPS = \langle S, \mathscr{A}, p\rangle$ is called *independent* iff for any n-tuple $\langle x_1, ..., x_n\rangle$ of respective values of these variables, $p(X_1 = x_1 \wedge \ldots \wedge X_n = x_n) = p(X_1 = x_1) \cdot \ldots \cdot p(X_n = x_n)$. If the family $X_1, \ldots, X_n$ is not independent, it is called *dependent*, or *correlated*.

A causal probability space contains information about causal relations and the locations of events. It is therefore particularly well-suited to capturing the idea of concrete space-like related measurements, each capable of producing a number of alternative possible results. We now discuss how a setup of that sort is represented in a probabilistic BST_{92} structure with NO MFB, and, in particular, which random variables should be selected to describe non-local correlations. Let us thus consider finitely many pairwise-SLR initials $I_1, I_2, \ldots, I_K$. As the initials are thought of as representing a joint measurement, there must be a history in which they all occur. Thus, together the initials form a consistent set $E =_{\text{df}} \bigcup_{k=1}^{K} I_k$. Each initial I_k has a family of possible outcomes $\mathbf{1}_k =_{\text{df}} \{\hat{O}^k_\gamma \mid \gamma \in \Gamma(k)\}$, so we can represent an individual measurement via a deterministic transition to a disjunctive outcome, $I_k \rightarrowtail \mathbf{1}_k$, where $\bigcup_{\gamma\in\Gamma} H_{\langle\hat{O}^k_\gamma\rangle} = H_{[I_k]}$ and for $\gamma, \gamma' \in \Gamma(k)$, $H_{\hat{O}^k_\gamma} \cap H_{\hat{O}^k_{\gamma'}} = \emptyset$ if $\gamma \neq \gamma'$.[26]

Note now that by NO MFB, every element of $\mathbf{1}_1 \times \mathbf{1}_2 \times \ldots \times \mathbf{1}_K$ is consistent. Thus, by taking the set-theoretical union of an element of $\mathbf{1}_1 \times \mathbf{1}_2 \times \ldots \times \mathbf{1}_K$ (an n-tuple of sets) one obtains a scattered outcome. Since any two different elements of $\mathbf{1}_1 \times \mathbf{1}_2 \times \ldots \times \mathbf{1}_K$ are incompatible, the set of the unions below is thus a disjunctive outcome:

$$\mathbf{1}_E =_{\text{df}} \{\bigcup Z \mid Z \in \mathbf{1}_1 \times \mathbf{1}_2 \times \ldots \times \mathbf{1}_K\}. \tag{7.7}$$

One can immediately see that $\mathbf{1}_E$ is exhaustive and pairwise exclusive:

$$H_{\mathbf{1}_E} = H_{[E]}, \text{ and for } \hat{O}, \hat{O}' \in \mathbf{1}_E : H_{\langle\hat{O}\rangle} \cap H_{\langle\hat{O}'\rangle} = \emptyset \text{ if } \hat{O} \neq \hat{O}'.$$

By our assumption, E is consistent; furthermore, E is below $\mathbf{1}_E$ in the relevant sense of Def. 4.4. It follows that E together with $\mathbf{1}_E$ constitutes the deterministic transition $E \rightarrowtail \mathbf{1}_E$ to a disjunctive outcome. (The proof of the

[26] Compare the discussion following Definition 6.1 on p. 139.

above claims concerning exhaustiveness and the ordering between E and $\mathbf{1}_E$ is left as Exercise 7.2.)

The transition we just constructed, $E \rightarrowtail \mathbf{1}_E$, gives rise to a causal probability space $CPS(E \rightarrowtail \mathbf{1}_E) = \langle S, \mathscr{A}, p\rangle$ that is adequate to capture any PFB induced by the transitions $I_1 \rightarrowtail \mathbf{1}_1, \ldots, I_K \rightarrowtail \mathbf{1}_K$. That space is based on the set S determined by the causal alternatives to $E \rightarrowtail \hat{O}$, for $\hat{O} \in \mathbf{1}_E$ (see Def. 7.5). That is, S is the set of maximal consistent subsets of $\tilde{T} = \bigcup_{\hat{O}\in\mathbf{1}_E}\{e \rightarrowtail H \mid e \in cll(E \rightarrowtail \hat{O}) \wedge H \in \Pi_e\}$, $\mathscr{A}$ is the powerset algebra over S, and the measure p on $\mathscr{A}$ is induced by the measure assigned to the elements $T \in S$ via $p(T) = \mu(T)$. Now, in our construction of random variables that capture PFB, we need to represent sets like $CC(I_k \rightarrowtail \hat{O}_k)$ in $CPS(E \rightarrowtail \mathbf{1}_E)$; note the different initials, I_k vs. E. Representation is possible due to the following fact:

Fact 7.1. *Let $I_1, \ldots, I_K$ be pairwise SLR initial events in a BST_{92} structure with NO MFB, let $E =_{\text{df}} \bigcup_{k=1}^{K} I_k$, and for every $k \in \{1, \ldots, K\}$, let $I_k \rightarrowtail \mathbf{1}_k$ be a deterministic transition from I_k to the disjunctive outcome $\mathbf{1}_k$. Let S be the base set of $CPS(E \rightarrowtail \mathbf{1}_E)$, where $\mathbf{1}_E$ is defined by Eq. 7.7. Then for every $k = 1, \ldots, K$ and every $\hat{O}_k \in \mathbf{1}_k$, there is $T \in S$ such that $CC(I_k \rightarrowtail \hat{O}_k) \subseteq T$.*

Proof. Let $\tau = (e \rightarrowtail H) \in CC(I_k \rightarrowtail \hat{O}_k)$, so $e \in cll(I_k \rightarrowtail \hat{O}_k)$. There is thus $h \in H_{[I_k]}$ such that $h \perp_e H_{\langle \hat{O}_k\rangle}$. By no-MFB, $e < \hat{O}_k$ (Fact 6.1(4)), so $H = \Pi_e\langle \hat{O}_k\rangle$ and, again by no-MFB (as the I_k are pairwise SLR), there is $h' \in H_{[E]}$ such that $h' \perp_e H_{\langle \hat{O}_k\rangle}$. Hence $e \in cll(E \rightarrowtail \hat{O}_k)$. By no-MFB, and as each $\mathbf{1}_k$ is a disjunctive outcome, $(\hat{O}_1 \cup \ldots \cup \hat{O}_k \cup \ldots \cup \hat{O}_K)$ is consistent. It is also appropriately above E. Hence it is a scattered outcome of E. Also, as it is a scattered outcome, (†) $H_{\langle \hat{O}_1 \cup \ldots \cup \hat{O}_k \cup \ldots \cup \hat{O}_K\rangle} \subseteq H_{\langle \hat{O}_k\rangle}$. Thus, $e \in cll(E \rightarrowtail \hat{O}_1 \cup \ldots \cup \hat{O}_k \cup \ldots \cup \hat{O}_K)$, where each $\hat{O}_i \in \mathbf{1}_i$, so $(\hat{O}_1 \cup \ldots \cup \hat{O}_k \cup \ldots \cup \hat{O}_K) \in \mathbf{1}_E$. It follows that there is $T \in S$ such that

$$(e \rightarrowtail \Pi_e\langle \hat{O}_1 \cup \ldots \cup \hat{O}_k \cup \ldots \cup \hat{O}_K\rangle) \in T.$$

We can now show that the above transition is identical to our given τ: From (†) it follows that $\Pi_e\langle \hat{O}_1 \cup \ldots \cup \hat{O}_k \cup \ldots \cup \hat{O}_K\rangle = \Pi_e\langle \hat{O}_k\rangle = H$, and so we are done. □

The moral of this fact is that a set $CC(I_k \rightarrowtail \hat{O}_k)$, which belongs to the base set S_k of $CPS(I_k \rightarrowtail \hat{O}_k)$, is represented in $CPS(E \rightarrowtail \mathbf{1}_E) = \langle S, \mathscr{A}, p\rangle$ as the following element of the algebra $\mathscr{A}$:

$$\{T \in S \mid CC(I_k \rightarrowtail \hat{O}_k) \subseteq T\}.$$

Given the construction of our causal probability space and the above observation concerning the definition of our random variables, we are ready to state the general definition of probabilistic funny business, or PFB:

Definition 7.7 (PFB exhibited by a set of transitions and a set of random variables). Let $\mathscr{W} = \langle W, <, \mu \rangle$ be a probabilistic BST_{92} structure (Def. 7.3) with no MFB that contains the pairwise SLR initial events $I_1, \ldots, I_K$, and for every $k \in \{1, \ldots, K\}$, let $I_k \rightarrowtail \mathbf{1}_k$ be a deterministic transition to a disjunctive outcome $\mathbf{1}_k$. Consider the causal probability space $CPS(E \rightarrowtail \mathbf{1}_E) = \langle S, \mathscr{A}, p \rangle$ determined by the transition $E \rightarrowtail \mathbf{1}_E$ in the sense of Def. 7.5, where $E = \bigcup_{k=1}^{K} I_k$ and $\mathbf{1}_E = \{\bigcup Z \mid Z \in \mathbf{1}_1 \times \mathbf{1}_2 \times \ldots \times \mathbf{1}_K\}$. Let $\{X_1, \ldots, X_K\}$ be the set of random variables on $CPS(E \rightarrowtail \mathbf{1}_E)$ defined via

$$X_k : S \mapsto \Gamma(k) \text{ such that for every } T \in S : X_k(T) = \gamma \text{ iff } CC(I_k \rightarrowtail \hat{O}_\gamma) \subseteq T.$$

We say that the set of transitions $\{I_1 \rightarrowtail \mathbf{1}_1, \ldots, I_K \rightarrowtail \mathbf{1}_K\}$ *exhibits PFB* iff the random variables $\{X_1, \ldots, X_K\}$ are correlated in the sense of Def. 7.6.

Observe that each $T \in S$ corresponds to a unique element of $\mathbf{1}_1 \times \mathbf{1}_2 \times \ldots \times \mathbf{1}_K$. Hence, such a T corresponds to a unique sequence of indices $\langle \gamma_1, \gamma_2, \ldots, \gamma_K \rangle$, with $\gamma_k \in \{1, \ldots, \Gamma(k)\}$ $(k = 1, \ldots, K)$. We may thus view the random variable X_k as the projection function that projects the K-tuple $\langle \gamma_1, \gamma_2, \ldots, \gamma_K \rangle$ on the kth axis, yielding γ_k.

We can illustrate this definition by linking it to non-local quantum correlations which will be investigated in detail in Chapter 8. Assume that there are three space-like related measurement events, which we represent by pairwise SLR initials I_1, I_2, and I_3. Let these initial events have two, three, and four possible outcomes, respectively, i.e., $\Gamma(1) = 2, \Gamma(2) = 3$, and $\Gamma(3) = 4$. A joint outcome of these three measurements is thus represented by a scattered outcome—a union $\hat{O}^1_{\gamma_1} \cup \hat{O}^2_{\gamma_2} \cup \hat{O}^3_{\gamma_3}$, where $1 \leqslant \gamma_k \leqslant \Gamma(k)$ and $\hat{O}^k_{\gamma_k} \in \mathbf{1}_k$ $(k = 1, 2, 3)$. The base set S of the causal probability space $CPS = \langle S, \mathscr{A}, p \rangle$ thus comprises causal alternatives to each $CC(I_1 \cup I_2 \cup I_3 \rightarrowtail \hat{O}^1_{\gamma_1} \cup \hat{O}^2_{\gamma_2} \cup \hat{O}^3_{\gamma_3})$, for all alternative joint outcomes given by allowable values of γ_1, γ_2, and γ_3. Since we consider all possible outcomes of the initials in our setup, each element of S contains as a subset $CC(I_1 \rightarrowtail \hat{O}^1_{\gamma_1})$ for some γ_1. Analogously, it contains $CC(I_2 \rightarrowtail \hat{O}^2_{\gamma_2})$ and $CC(I_3 \rightarrowtail \hat{O}^3_{\gamma_3})$, for some γ_2 and γ_3. The three random variables, X_1, X_2, and X_3, are so defined that for any $T \in S$, $X_1(T) = \gamma_1$ iff $CC(I_1 \rightarrowtail \hat{O}^1_{\gamma_1}) \subseteq T$, $X_2(T) = \gamma_2$ iff $CC(I_2 \rightarrowtail \hat{O}^2_{\gamma_2}) \subseteq T$, and $X_3(T) = \gamma_3$ iff $CC(I_3 \rightarrowtail \hat{O}^3_{\gamma_3}) \subseteq T$. The setup then exhibits PFB if these random variables are correlated, i.e., if for some triple $\langle \gamma_1, \gamma_2, \gamma_3 \rangle$,

$$p(X_1 = \gamma_1 \wedge X_2 = \gamma_2 \wedge X_3 = \gamma_3) \neq p(X_1 = \gamma_1) \cdot p(X_2 = \gamma_2) \cdot p(X_3 = \gamma_3).$$

This means that for the $T \in S$ such that $T = CC(I_1 \rightarrowtail \hat{O}^1_{\gamma_1}) \cup CC(I_2 \rightarrowtail \hat{O}^2_{\gamma_2}) \cup CC(I_3 \rightarrowtail \hat{O}^3_{\gamma_3})$ we have, via the marginal probabilities:

$$\begin{aligned} p(T) \neq\ & p(\bigcup\{T' \in S \mid CC(I_1 \rightarrowtail \hat{O}^1_{\gamma_1}) \subseteq T'\}) \cdot \\ & p(\bigcup\{T' \in S \mid CC(I_2 \rightarrowtail \hat{O}^2_{\gamma_2}) \subseteq T'\}) \cdot \\ & p(\bigcup\{T' \in S \mid CC(I_3 \rightarrowtail \hat{O}^3_{\gamma_3}) \subseteq T'\}). \end{aligned}$$

This formula says that the probability of the joint (triple) outcome of the three SLR events I_1, I_2, and I_3 that corresponds to T does not factor into the probabilities of the three component single outcomes. A concrete system that shows such behavior will be discussed in Chapter 8.4.

We devote the rest of this chapter to a discussion of Humphreys's paradox. We will illustrate how our rigorous theory of causal probability can be employed to handle the problem, thereby also making good on our second condition of adequacy.

7.3 Fending off objections to propensities

As mentioned in Chapter 7.1.1, we consider two conditions of adequacy for our theory of causal probability: the formal condition of fulfilling Kolmogorov's axioms, and the condition of adequately responding to philosophical objections leveled against the notion of causal probabilities, or propensities. In the previous section we dealt with the first condition, showing how to define general causal probability spaces that fulfill the standard axioms of probability theory (starting with Def. 7.4 and Lemma 7.1). In this section we tackle the second task. The challenge we are facing is that our theory belongs to the category of theories of propensities, and propensities do not have a good reputation in philosophy. Propensity theories are often criticized on the grounds of certain "paradoxes" put forward by Humphreys (1985) and others. The aim of this section is to exhibit the reasons why these objections do not apply to our theory of propensities. In order to grasp propensities fully, we begin with a short survey of propensity-related concepts.

7.3.1 Some remarks on propensities

In the philosophical literature, propensities are typically assigned to singular entities. The English language suggests that these entities are objects, processes, or singular events.[27] Propensities can be graded in degrees of more or less, high or low, etc., which makes them at least similar to probabilities in this respect.

Propensities are valenced toward the future and they relate to a time-asymmetric situation. In 1973, Eddy Merckx has a propensity to win the *Tour de France* the next year; but after winning it in 1974, he does not have a propensity for having won the Tour in 1974. Pure probability theory is incapable of making such a distinction. The notion of a BST transition, however, is directly built on possibilities for the future and thereby provides the necessary resources to express the future-directedness of propensities.

As argued in Popper's (1959) influential essay on propensities, propensities are *unashamedly indeterminist*, since they "influence future situations without determining them". There being a propensity for such and such may make it likely, but is not itself a guarantee. Further, propensities are objective (Popper, 1959, p. 32). Both ideas suit causal probabilities of BST.

A further important issue that we want to stress is that *there is always some causal claim involved in an ascription of propensity.* Salmon (1989, p. 86) writes as follows: "Propensities, I would suggest, are best understood as some sort of probabilistic causes". BST theory makes good on this claim by building causal probability spaces directly out of the material provided by the BST analysis of indeterministic causation.

7.3.2 Humphreys's paradox

As propensities come in grades and are assignable to singular entities, a natural move, taken by Popper and others, is to identify propensities with a sub-species of probabilities. Propensities are put forward as providing an objective single-case interpretation of certain probabilities. Of course, everybody agrees that not *all* probabilities could be propensities, as there

[27] Phrases suggesting that propensities are ascribed to singular events are harder to find in English. Our usage of ascribing propensities to singular events, however, follows standard philosophical usage.

are, for example, also subjective probabilities, but the claim is that *some* probabilities are indeed propensities, and *all* propensities are probabilities.

The underlying idea that propensities are probabilities was fundamentally questioned by Humphreys's (1985) paper "Why propensities cannot be probabilities", which launched an intense discussion of the relation between these two concepts. Humphreys's claim can either be read as arguing that probability theory in its present form cannot serve as a true theory of propensities, or as arguing that the notion of a propensity as a specific kind of probability makes no sense. Many parties to the debate subscribe to the latter view.

Humphreys always has in mind that propensities are conditional, as in "there is a propensity Pr for an electron in the metal to be emitted, conditional upon the metal being exposed to light above the threshold frequency" (Humphreys, 1985, p. 558), and so his notation is generally of the form $Pr(A \mid B)$.[28] The principal lemma for Humphreys's claim that propensities cannot be probabilities is that *Bayes's theorem fails for propensities*. More generally, his strategy is to take issue with the idea that the inversion principles of probability theory apply to *conditional propensities*. Here "inversion" means that $p(A \mid B)$ and $p(B \mid A)$ are equally grammatical and inter-definable. The conclusion Humphreys draws is that "the theory of probability [is] an inappropriate constraint on any theory of single-case propensities" (Humphreys, 2004, p. 945).

To give a preview of our response to the objection, we agree with Humphreys that the five assumptions of his argument listed below lead to a contradiction. We take issue with them, however, via an analysis of what they could mean. In this analysis, our main goal is not to focus on an exegesis of Humphreys's semi-formal notation, but to attempt to construct a probability space in which all of the assumptions hold. As the assumptions are contradictory, we know that the attempted construction has to fail. By learning how it fails, we will learn which assumptions are untenable, and why. In this context, our causal probability spaces are highly relevant, as they can represent all the causal relations in Humphreys's photon story explicitly. We will argue that his problematic assumptions (iii) and (CI), see below, misrepresent the causal settings of his story. In fact, we already discussed a part of the problem in Chapter 7.2.3. We will thus reject the mentioned

[28] We uniformly write $Pr(A \mid B)$ for the propensity in order to heighten the contrast with the standard probability-theory notation for conditional probabilities, $p(A \mid B)$.

assumptions, and we will show how to represent Bayesian inversion in BST causal probability spaces. This means that the inversion principles which Humphreys blames for the contradiction are not problematic for propensities understood as BST causal probabilities. In Chapter 7.3.4 we analyze a more complex Humphreys-inspired case to further highlight the workings of causal probability spaces in the handling of inversion, and of conditional propensities.

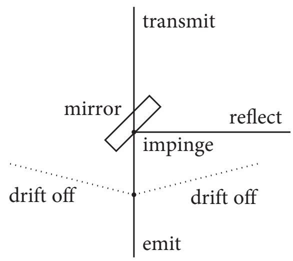

Figure 7.3 A nanosecond in the life of a photon. See the main text for details.

Humphreys's photon argument. Humphreys tells a story in which a Bayesian calculation gives the wrong answer. The story concerns a photon.[29] Figure 7.3 tells of a photon that has just been emitted in a laboratory. The photon either Impinges ($= I$) on a half-silvered mirror, with propensity q, or it drifts off somewhere. If it impinges on the mirror, it has a fixed propensity p to be Transmitted ($= T$) straight ahead through the mirror and onto a detector, and also of course a companion propensity of $r = (1 - p)$ to wind up Reflected ($= R$) off to the right. Figure 7.3 spells this out in a two-dimensional spatial diagram, looking at the apparatus from above.

The list below gives the assumptions of Humphreys's argument. The first four items on the list, namely, (i)–(iii) and (CI), are intended by Humphreys as local assumptions, governing just the emit-impinge-transmit set-up that is depicted in Figure 7.3. The rest, (TP) and (MP), express general principles of probability theory, including conditional-probability theory. The hypothesis under consideration is whether a theory of propensities can serve as an interpretation of conditional-probability theory. On the basis of that

[29] We take a few liberties with the story, such as not explicitly mentioning the background conditions holding at time t_1. The changes we make are irrelevant to the structure of the argument. Our identification of non-transmission ($\overline{T}_{t_3}$) with reflection (R_{t_3}) below is based on Humphreys's (1985) glosses on pp. 561 and 526.

hypothesis, according to Humphreys, those six assumptions should hold good for conditional propensities. Humphreys (1985), however, derives a contradiction and enters this as an argument that classical probability theory does not give a correct account of conditional propensities.

(i) $Pr_{t_1}(T_{t_3} \mid I_{t_2}) = p,\ p > 0$. [*The propensity at* t_1 *for* T *to occur at* t_3 *conditional upon* I *occurring at* t_2 *is* p.]

(ii) $Pr_{t_1}(I_{t_2}) = q, 0 < q < 1$. [*The propensity at* t_1 *for* I *to occur at* t_2 *is* q.]

(iii) $Pr_{t_1}(T_{t_3} \mid \bar{I}_{t_2}) = 0$. [*The propensity at* t_1 *for* T *to occur at* t_3 *conditional upon* $\bar{I}$ *occurring at* t_2 *is* 0.]

(CI) $Pr_{t_1}(I_{t_2} \mid T_{t_3}) = Pr_{t_1}(I_{t_2} \mid \overline{T}_{t_3}) = Pr_{t_1}(I_{t_2})$. [*The propensity for a particle to impinge upon the mirror is unaffected by whether the particle is transmitted or not.*]

(TP) $Pr_{t_i}(A_{t_j}) = Pr_{t_i}(A_{t_j} \mid B_{t_k}) \cdot Pr_{t_i}(B_{t_k}) + Pr_{t_i}(A_{t_j} \mid \overline{B}_{t_k}) \cdot Pr_{t_i}(\overline{B}_{t_k})$. [*A version of the principle of total probability is assumed for propensities.*]

(MP) $Pr_{t_i}(A_{t_j}B_{t_k}) = Pr_{t_i}(B_{t_k}A_{t_j}) = Pr_{t_i}(A_{t_j} \mid B_{t_k}) \cdot Pr_{t_i}(B_{t_k})$. [*The standard definition of conditional probabilities is assumed for propensities.*]

The notation $Pr_{t_i}(A_{t_j} \mid B_{t_k})$ is to be read as "the propensity at time t_i for A to occur at time t_j, conditional upon B occurring at time t_k" (cf. Humphreys, 1985, p. 561). Times are as follows: t_0 is the last moment before emission, t_1 is a time just after emission, t_2 is the time of impingement/no impingement, I_{t_2} is the (possible) event of the photon impinging upon the mirror at time t_2, and $\bar{I}_{t_2}$ is the (possible) event of the photon failing to impinge on the mirror at t_2, as it drifts off a moment before. t_3 is the time of transmission/no transmission. T_{t_3} is the (possible) event of the photon being transmitted through the mirror at time t_3. $\overline{T}_{t_3}$ is the (possible) event of the photon failing to be transmitted through the mirror at t_3 after impinging on the mirror, so $\overline{T}_{t_3} = R_{t_3}$, the (possible) event of the photon being reflected at t_3.

The contradiction that Humphreys (1985) derives from the above assumptions is this: We have

$$Pr_{t_1}(I_{t_2} \mid T_{t_3}) = Pr_{t_1}(I_{t_2}) = q < 1$$

by (CI) and (ii). But by (TP) and (MP) (or, equivalently, by Bayes's Theorem) and (i), (ii), and (iii) we get

$$\begin{aligned} Pr_{t_1}(I_{t_2} \mid T_{t_3}) &= \frac{Pr_{t_1}(T_{t_3} \mid I_{t_2}) \cdot Pr_{t_1}(I_{t_2})}{Pr_{t_1}(T_{t_3} \mid I_{t_2}) \cdot Pr_{t_1}(I_{t_2}) + Pr_{t_1}(T_{t_3} \mid \bar{I}_{t_2}) \cdot Pr_{t_1}(\bar{I}_{t_2})} \\ &= \frac{pq}{pq+0} = 1. \end{aligned}$$

7.3.3 Our diagnosis of Humphreys's paradox

We turn to our evaluation of the preceding assumptions. It has been noted that the notation used has some shortcomings.[30] We will not focus on this line of criticism here, although, for the record, the problematic assumption (iii) that we will discuss is also the target of the mentioned notation-driven criticism. One further notational worry can also be put to rest: the notation $\bar{I}_{t_2}$ seems to involve the negation, or the complement, of a singular event, which may not be well-defined. We can, however, identify $\bar{I}_{t_2}$ with the well-defined event of the photon drifting off just before hitting the mirror, which we take to be the single alternative to I_{t_2}.

The bottom line to our diagnosis is that Humphreys fails to motivate his assumptions by indicating what they could refer to in some probability space. In fact, there is no probability space which satisfies all the assumptions. This can either mean that there is no classical probability space for propensities (which is Humphreys's diagnosis),[31] or that the assumptions are causally flawed. The latter is our diagnosis. Our positive contribution will be to point out which two assumptions are flawed: Assumption (iii), which happens to be numerically salvageable on our analysis, suggests inappropriate causal assumptions, and assumption (CI) fails both causally and numerically. In addition, we can show which correct principles lie behind these problematic assumptions, thus (we hope) removing the whiff of a paradox from them. On top of this diagnosis, we will show how the BST theory of causal probabilities allows for the construction of an adequate probability space.

[30] Miller (1994, p. 113) suggests a modified reading of $Pr_{t_i}(A_{t_j} \mid B_{t_k})$, namely "The propensity of the world at time t_i to develop into a world in which A comes to pass at time t_j, given that it (the world at time t_i) develops into a world in which B comes to pass at the time t_k". He also shows that inversion holds after the modification. A detailed discussion of Humphreys's and Miller's notations is provided in Belnap (2007).

[31] Humphreys holds that standard probability theory is "an inappropriate constraint on any theory of single-case propensities" (Humphreys, 2004, p. 945).

We start with assumption (iii), which says: $Pr_{t_1}(T_{t_3} \mid \bar{I}_{t_2}) = 0$. Presumably, $Pr_{t_1}(T_{t_3} \mid \bar{I}_{t_2})$ must be zero because T_{t_3} cannot occur together with $\bar{I}_{t_2}$, because impinging (I_{t_2} rather than $\bar{I}_{t_2}$) is a causal precondition of transmission. Writing $Pr(AB)$ to signify the probability of the joint occurrence of A and B, this means $Pr_{t_1}(T_{t_3}\bar{I}_{t_2}) = 0$.

In the same vein, since reflection (R_{t_3}, assumed in the story to be identical to $\overline{T}_{t_3}$; see Humphreys, 1985, pp. 561 and 563) cannot occur together with $\bar{I}_{t_2}$, we should also have $Pr_{t_1}(R_{t_3} \mid \bar{I}_{t_2}) = 0$, and so $Pr_{t_1}(R_{t_3}\bar{I}_{t_2}) = 0$. But then, as reflection at t_3 means no transmission at t_3, and vice versa, we should have $Pr_{t_1}(\bar{I}_{t_2}) = Pr_{t_1}(T_{t_3}\bar{I}_{t_2}) + Pr_{t_1}(R_{t_3}\bar{I}_{t_2})$, and therefore it should be that $Pr_{t_1}(\bar{I}_{t_2}) = 0$, in contradiction to assumption (ii).

The background of this calculation is exactly the problematic construction that we discussed in our story "Variant 2: Mail gets lost" in Section 7.2.2 (p. 177): a probability space involving causally impossible elements in its sample space. A significant problem with Humphreys's analysis is therefore that the sample space that his notation suggests is inadequate—it fails to properly reflect the causal relations in the photon story. In the spirit of a standard Cartesian product construction, that notation requires there to be the 2×2 fine-grained combinations of impinge/not impinge at t_2 and transmit/reflect at t_3:

$$I_{t_2}T_{t_3}, \quad I_{t_2}R_{t_3}, \quad \bar{I}_{t_2}T_{t_3}, \quad \bar{I}_{t_2}R_{t_3}.$$

These should make up the sample space of the underlying probability space and thus should have probabilities assigned that add up to one. However, given Humphreys's photon story, the last two combinations are causally impossible. Humphreys has to assign them probability zero. This assignment then has numerical consequences that contradict the causal story: the occurrence of event I_{t_2} appears to be inevitable, as the corresponding marginal probability is one.

The problematic consequence of (iii), inevitability of the occurrence of event I_{t_2}, follows from (CI) as well; the problem here is the mirror image of the problem we have just discussed. Given that I_{t_2} is compatible with both T_{t_3} and with R_{t_3}, it is correct to calculate the marginal probability according to principle (TP) as follows:

$$Pr_{t_1}(I_{t_2}) = Pr_{t_1}(I_{t_2} \mid T_{t_3}) \cdot Pr_{t_1}(T_{t_3}) + Pr_{t_1}(I_{t_2} \mid R_{t_3}) \cdot Pr_{t_1}(R_{t_3}),$$

as we will confirm in our analysis below. On that basis, however, assumption (CI) implies

$$Pr_{t_1}(I_{t_2}) = Pr_{t_1}(I_{t_2}) \cdot Pr_{t_1}(T_{t_3}) + Pr_{t_1}(I_{t_2}) \cdot Pr_{t_1}(R_{t_3}),$$

from which it follows that

$$Pr_{t_1}(T_{t_3}) + Pr_{t_1}(R_{t_3}) = 1.$$

As the occurrence of I_{t_2} is a causal precondition of the occurrence of either T_{t_3} or R_{t_3}, it again follows, absurdly, that the occurrence of I_{t_2} is inevitable at t_1.

Given a 2×2 Cartesian product space, both assumption (iii) and assumption (CI) thus allow one to argue that $Pr_{t_1}(I_{t_2}) = 1$, in contradiction to assumption (ii). The Cartesian product space is, however, clearly causally inadequate. Given the causal details of the story there are just three, not four, really possible combinations of the four events in question:

$$I_{t_2}T_{t_3}, \quad I_{t_2}R_{t_3}, \quad \bar{I}_{t_2}.$$

These objects, or rather the corresponding sets of basic transitions, form the base set S of an adequate causal probability space according to Def. 7.4. Notably, the causally impossible combination $\bar{I}_{t_2}T_{t_3}$ is missing from S, so Humphreys's premise (iii) is not even statable without further analysis.

We can construct an adequate causal probability space to substantiate our point. To begin with the impingement I_{t_2}, which is supposed to have a single alternative, $\bar{I}_{t_2}$, we assume that there is a choice event e_1 with exactly two possible outcomes $H_I, H_{\bar{I}} \in \Pi_{e_1}$, to be read as "impingement" and "drifting-off", respectively. These outcomes give rise to two basic transitions, $\tau_1^I = e_1 \rightarrowtail H_I$ and $\tau_1^{\bar{I}} = e_1 \rightarrowtail H_{\bar{I}}$.

In a similar vein, as T_{t_3} and R_{t_3} are the only alternatives at t_3 and each can only occur in the outcome I_{t_2}, we posit a second choice event e_2 that is above e_1 in the I_{t_2}-outcome and that has exactly the two basic outcomes $H_T, H_R \in \Pi_{e_2}$. The resulting basic transitions are $\tau_2^T = e_2 \rightarrowtail H_T$ and $\tau_2^R = e_2 \rightarrowtail H_R$. We now exhibit a causal probability space $CPS = \langle S, \mathscr{A}, p\rangle$ that is adequate to represent the photon story. According to Def. 7.4, given the set of initials $E = \{e_1, e_2\}$, the base set S is

$$S = \{\{\tau_1^{\bar{I}}\}, \{\tau_1^I, \tau_2^T\}, \{\tau_1^I, \tau_2^R\}\}.$$

The algebra $\mathscr{A}$ is the power-set of S, $\mathscr{A} = \mathscr{P}(S)$, and the full measure p on $\mathscr{A}$ is generated from the measure on the elements of S in the standard way, assuming that the three causal alternatives that make up the set S are assigned propensities by μ, so that

$$p(\tau_1^{\bar{I}}) = \mu(\tau_1^{\bar{I}}); \quad p(\{\tau_1^I, \tau_2^T\}) = \mu(\{\tau_1^I, \tau_2^T\}); \quad p(\{\tau_1^I, \tau_2^R\}) = \mu(\{\tau_1^I, \tau_2^R\}).$$

Note that Postulate 7.1 implies

$$\mu(\{\tau_1^I, \tau_2^T\}) = \mu(\tau_1^I) \cdot \mu(\tau_2^T) \quad \text{and} \quad \mu(\{\tau_1^I, \tau_2^R\}) = \mu(\tau_1^I) \cdot \mu(\tau_2^R),$$

and Postulate 7.4 implies

$$\mu(\tau_1^I) = \mu(\{\tau_1^I, \tau_2^T\}) + \mu(\{\tau_1^I, \tau_2^R\}).$$

These relations carry over to the measure p, noting (in accordance with our discussion in Section 7.2.5) that the BST transition τ_1^I is represented in *CPS* as a fine-grained element of $\mathscr{A}$, viz., as $\{\{\tau_1^I, \tau_2^T\}, \{\tau_1^I, \tau_2^R\}\}$.

As Humphreys alleges that propensities do not satisfy the standard inversion principles of probability theory, it will be useful to show that in our causal probability space, inversion is not problematic at all. To illustrate, here is how our theory relates the conditional probabilities of impingement and transmission in both ways.

As we just said, the concrete event τ_1^I of the photon impinging on the mirror at time t_2 is represented in *CPS* not by an element of the sample space, but by the following element of the event algebra $\mathscr{A}$:

$$I_{CPS} = \{\{\tau_1^I, \tau_2^T\}, \{\tau_1^I, \tau_2^R\}\},$$

which consists of the two elements of the sample space that include the "impinge" transition τ_1^I. The concrete event of the photon being transmitted through the mirror at time t_3 is represented by the following element of the event algebra $\mathscr{A}$:

$$T_{CPS} = \{\{\tau_1^I, \tau_2^T\}\},$$

which consists of the single element of the sample space that includes the "transmit" transition τ_2^T. Note that this element of the sample space also includes the "impinge" transition, as this transition is a causal precondition of the "transmit" transition that is represented in our probability space. The "transmit" transition, of course, has lots of other preconditions, such as the installation of the apparatus, but these are not represented in our locally based probability space CPS. In precisely the same way, the "reflect" transition is represented as $R_{CPS} = \{\{\tau_1^I, \tau_2^R\}\}$.

With a view to conditional probabilities, we next need to find out which element of the event algebra corresponds to the conjunctive event "impinge and transmit". As our algebra $\mathscr{A}$ is just the power set algebra of S, this is simple: we have

$$I_{CPS}T_{CPS} = \{\{\tau_1^I, \tau_2^T\}, \{\tau_1^I, \tau_2^R\}\} \cap \{\{\tau_1^I, \tau_2^T\}\} = \{\{\tau_1^I, \tau_2^T\}\} = T_{CPS}.$$

Note that I_{CPS} is a proper superset of T_{CPS}, so assuming that there is a non-zero probability for reflection, probability theory alone suffices to guarantee that $p(T_{CPS}) < p(I_{CPS})$.

We can now calculate conditional probabilities in our space in the standard way. The probability of impingement conditional on transmission turns out to be one, as it should, because impingement is a causal precondition of transmission that is represented in CPS:

$$p(I_{CPS} \mid T_{CPS}) = \frac{p(I_{CPS}T_{CPS})}{p(T_{CPS})} = \frac{p(T_{CPS})}{p(T_{CPS})} = 1.$$

On the other hand, given impingement, transmission is contingent:

$$p(T_{CPS} \mid I_{CPS}) = \frac{p(T_{CPS}I_{CPS})}{p(I_{CPS})} = \frac{p(I_{CPS}T_{CPS})}{p(I_{CPS})} = \frac{p(T_{CPS})}{p(I_{CPS})} < 1.$$

These conditional probabilities clearly fulfill Bayes's theorem. As a sanity check, here is how to calculate $p(I_{CPS} \mid T_{CPS})$ (see below for the expansion of the denominator via the law of total probability):

$$p(I_{CPS} \mid T_{CPS}) = \frac{p(T_{CPS} \mid I_{CPS}) \cdot p(I_{CPS})}{p(T_{CPS})} = \frac{\frac{p(T_{CPS})}{p(I_{CPS})} \cdot p(I_{CPS})}{p(T_{CPS})} = 1.$$

It will be illuminating now to look at Humphreys's assumption (iii), which involves a causally impossible combination of events. In our framework, we have

$$T_{CPS}\bar{I}_{CPS} = \{\{\tau_1^I, \tau_2^T\}\} \cap \{\{\tau_1^{\bar{I}}\}\} = \emptyset.$$

As $\emptyset$ is a valid element of the event algebra $\mathscr{A}$, with probability $p(\emptyset) = 0$ by normalization, this implies $p(T_{CPS}\bar{I}_{CPS}) = 0$, and accordingly,

$$p(T_{CPS} \mid \bar{I}_{CPS}) = \frac{p(T_{CPS}\bar{I}_{CPS})}{p(\bar{I}_{CPS})} = 0,$$

in accord with the numerical claim of Humphreys's assumption (iii). We even recapture the relevant instance of the law of total probability,

$$p(T_{CPS}) = p(T_{CPS} \mid I_{CPS}) \cdot p(I_{CPS}) + p(T_{CPS} \mid \bar{I}_{CPS}) \cdot p(\bar{I}_{CPS}),$$

because the first term evaluates as

$$p(T_{CPS} \mid I_{CPS}) \cdot p(I_{CPS}) = \frac{p(T_{CPS}I_{CPS})}{p(I_{CPS})} \cdot p(I_{CPS}) = p(T_{CPS}I_{CPS}) = p(T_{CPS}),$$

and the second term is zero, as we have just shown.

As we pointed out earlier, however, the causal structure of the set-up is misrepresented by the suggestion that one should work with the Cartesian product of impinge/not impinge and transmit/reflect. Numerically, this misrepresentation shows up as a problem with assumption (CI). To repeat, (CI) says

$$Pr_{t_1}(I_{t_2} \mid T_{t_3}) = Pr_{t_1}(I_{t_2} \mid R_{t_3}) = Pr_{t_1}(I_{t_2}).$$

In our preceding discussion, we have already evaluated the first term involved, and the calculation for the second term is exactly analogous. So, on our analysis,

$$p(I_{CPS} \mid T_{CPS}) = 1; \quad p(I_{CPS} \mid R_{CPS}) = 1; \quad \text{but} \quad p(I_{CPS}) = q < 1.$$

Assumption (CI) is therefore not just causally, but also numerically incorrect on our analysis, and this is how Humphreys's contradiction is avoided. In order to dispel the air of paradox, we should also be able to point out which correct principles lie behind assumption (CI). For starters, it is correct that the propensity at t_1 (in our analysis, at initial event e_1) for I to occur is independent of what happens afterward. As we indicated above, there is a number $q = \mu(\tau_1^I)$ that represents the causal probability of the local indeterministic transition from initial e_1 to the outcome "impinge", and we follow Humphreys's assumption that $0 < q < 1$ (the photon is neither certain to impinge nor certain not to impinge). That number q is the number it is, and it describes a local propensity of a concrete event. In this sense, the gloss of Humphreys's assumption (CI) holds on our analysis as well: the propensity to impinge in unaffected by whether the particle is transmitted or not. But, as we have just shown, the equation (CI) does not hold. One might think that the following (fallacious) reasoning provides good intuitive support for (CI):

> There is a number $q = \mu(\tau_1^I)$, the propensity for the photon to impinge, and given that the photon has impinged, there is a number $p = \mu(\tau_2^T)$ for the photon to be transmitted. The propensity for the photon to impinge and then to be transmitted is $\mu(\tau_1^I, \tau_2^T) = \mu(\tau_1^I) \cdot \mu(\tau_2^T)$, by the Markov condition. So we can calculate the propensity for the photon to impinge, conditional on its being transmitted, as
>
> $$\mu(\tau_1^I \mid \tau_2^T) = \frac{\mu(\tau_1^I, \tau_2^T)}{\mu(\tau_2^T)} = \frac{\mu(\tau_1^I) \cdot \mu(\tau_2^T)}{\mu(\tau_2^T)} = \mu(\tau_1^I),$$
>
> in support of (CI); and the calculation involving R is analogous.

The problem with this argument is that it fails to take into account how the transitions in question are represented, skipping a crucial step in deriving a mathematically well-defined probability measure p on the algebra $\mathscr{A}$ of a probability space from the propensity function μ. It is correct that μ assigns a value between 0 and 1 to basic transitions and to sets of basic transitions (with the proviso that μ need not be a total function). But it is not correct to assume that μ itself is a probability measure. We already pointed out in Section 7.1.3 that this assumption is untenable. There is some work involved in using the Nature-given causal probabilities (propensities) μ to construct causal probability spaces, and that work depends crucially on how the relevant transitions are represented. Once an adequate probability space has been constructed, all assumptions of probability theory, including conditional probability theory and its inversion principles, hold without any reservations.

At this point we can take up the discussion of case 2 of representing BST transitions in causal probability spaces from Section 7.2.5. The issue with the fallacious reasoning above is exactly that the "transmit" transition τ_2^T is *not* represented in isolation in a causal probability space that includes the "impinge" transition τ_1^I, which is a causal precondition of transmission. There is no element of the event algebra $\mathscr{A}$ of *CPS* that represents the "transmit" transition alone, and therefore, the above calculation involving a conditional propensity is undefined in our *CPS*. It is correct, due to the Markov condition, that

$$\mu(\{\tau_1^I, \tau_2^T\}) = \mu(\tau_1^I) \cdot \mu(\tau_2^T). \tag{7.8}$$

The propensity for the two-step transition from e_1 (before impinging) to transmission, which has the two *causae causantes* τ_1^I and τ_2^T, factors into the propensity for the photon to impinge and the propensity for the photon then to be transmitted. It is, therefore, also correct to calculate

$$\mu(\tau_2^T) = \frac{\mu(\{\tau_1^I, \tau_2^T\})}{\mu(\tau_1^I)} = \frac{p(T_{CPS})}{p(I_{CPS})} = \frac{p(T_{CPS})}{p(T_{CPS}) + p(R_{CPS})}.$$

In this calculation we have used the probability measure p defined on *CPS* wherever possible. But the expression $\mu(\tau_2^T)$, which is well-defined in the probabilistic BST structure in which we are working, has no counterpart in our causal probability space, which represents the two consecutive events of impingement and transmission. In any such space, the transmission event is not represented in isolation, but only together with its causal precondition, impingement. Eq. 7.8 thus connects *two different* causal probability spaces. Conditional probabilities, however, are only defined within *one single* probability space.

Viewed in this light, we can repeat our initial assessment that Humphreys's claim, read charitably, is indeed correct: raw propensities are not probabilities because there is no probability space whose measure they could be. In order to show that, no Bayesian inversion is needed, however: it is enough to note that two incompatible deterministic transitions, each of which has propensity one, would have to give rise to a disjunctive event with propensity 2 (see p. 165). Thus, Humphreys's argument is a red herring. Once propensities have been used to construct causal probability spaces, standard probability theory holds, and there is no problem. Humphreys's paradox is no threat to propensity theory as implemented via BST causal probability theory.

7.3.4 Salmon's corkscrew story: More on conditional propensities and inversion

In this final section we will show how causal probability theory can also handle Bayesian inversion in a more complex case. We will proceed in terms of an analysis of Salmon's (1989) corkscrew story, which provides a standard case illustrating Bayesian inversion and which Salmon had invoked in defending his reluctance to give a probabilistic reading to propensities.[32] Here is how Salmon (1989, p. 88) put the matter.

> Imagine a factory that produces corkscrews. It has two machines, one old and one new, each of which makes a certain number per day. The output of each machine contains a certain percentage of defective corkscrews. Without undue strain, we can speak of the relative propensities of the two machines to produce corkscrews (each produces a certain proportion of the entire output of the factory), and of their propensities to produce defective corkscrews. If numerical values are given, we can calculate the propensity of this factory to produce defective corkscrews. So far, so good. Now, suppose an inspector picks one corkscrew from the day's output and finds it to be defective. Using Bayes's theorem we can calculate the *probability* that the defective corkscrew was produced by the new machine, but it would hardly be reasonable to speak of the *propensity* of that corkscrew to have been produced by the new machine.[33]

In retelling the story, we shall rely on Figure 7.4. Each h_j $(j = 1, \ldots, 5)$ in that figure is a history. e_0 is a reference point event before all the action. The ellipses are machines M_i $(i = 1, 2)$, with e_i the (idealized) point event of production by M_i. e_3 is the choice for the Inspector to pick one out of the two defective corkscrews. d_i $[g_i]$ is a defective [good] corkscrew produced

[32] Salmon told (at least) two "inversion stories", the can opener story, in Salmon (1984), and the corkscrew story, in Salmon (1989). With these stories he subscribed to the position that propensities "make sense as direct probabilities [. . .], but not as inverse probabilities (because the causal direction is wrong)" (Salmon, 1989, p. 88).

[33] Salmon's language attaches propensities to things; without further comment, we translate his story into language that attaches propensities to transitions.

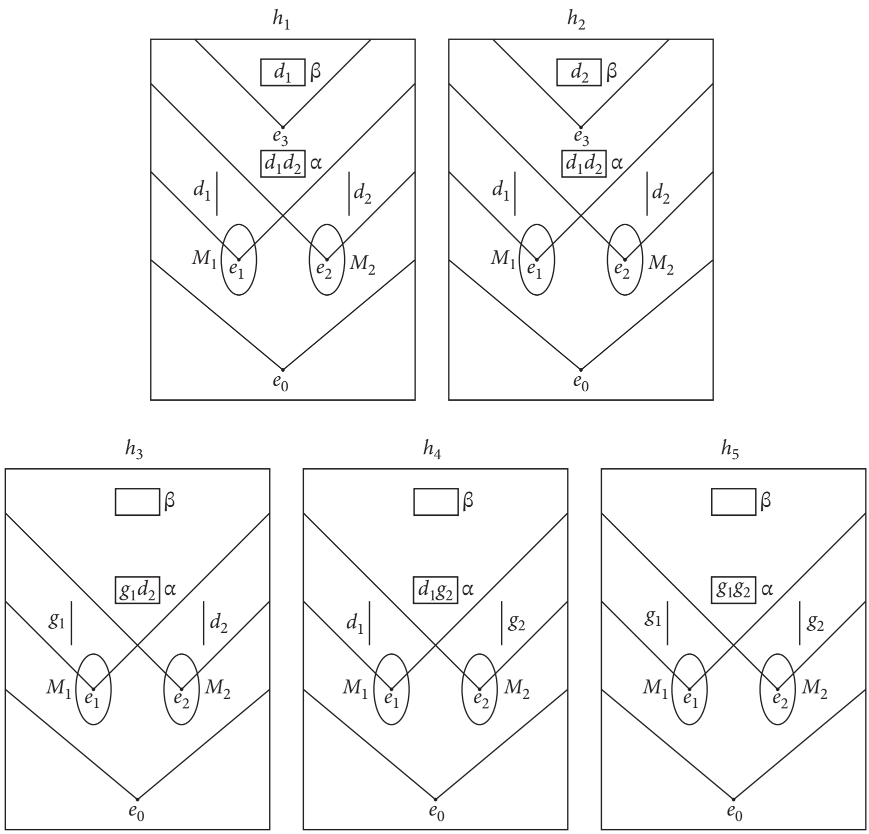

Figure 7.4 The Corkscrew Story. See main text for details.

by machine M_i. In turn, $d_i^*(i = 1, 2)$ stand for the Inspector's picking of a defective corkscrew produced by M_i from Box-α at e_3. To simplify Salmon's story, we assume that each machine makes just one corkscrew per day. The causal analysis of the story as pictured leads us to six basic transitions. First there are four transitions related to the output (good or defective) of the machines:

$$\tau_1 = e_1 \rightarrowtail g_1, \;\; \tau_1' = e_1 \rightarrowtail d_1, \;\; \tau_2 = e_2 \rightarrowtail g_2, \;\; \tau_2' = e_2 \rightarrowtail d_2. \qquad (7.9)$$

The remaining two basic transitions result from possible actions of the Inspector, who ignores any good output in Box α, paying attention only to defective corkscrews. Accordingly, she only has a choice if two defective corkscrews are in Box α, which occurs in histories h_1 and h_2. In these histories the Inspector indeterministically chooses which defective corkscrew to transfer to Box β, and her two possible picks are represented by two alternative basic transitions,

$$\tau_3 = e_3 \rightarrowtail d_1^* \text{ and } \tau_3' = e_3 \rightarrowtail d_2^*. \tag{7.10}$$

The following table gives exemplary values for the propensities of all the basic transitions involved which we will use in our calculations below:

$$\begin{aligned} \mu(\tau_1) = 0.4 = p_1, \;\; \mu(\tau_2) = 0.9 = p_2, \;\; \mu(\tau_3) = 0.7 = p_3, \\ \mu(\tau_1') = 0.6 = p_1', \;\; \mu(\tau_2') = 0.1 = p_2', \;\; \mu(\tau_3') = 0.3 = p_3'. \end{aligned} \tag{7.11}$$

From this information we can calculate the propensities of relevant non-basic transitions, such as the transition $e_0 \rightarrowtail d_1^*$ from the initial state to the inspector picking the first corkscrew.[34] To this end, one must first locate the sets of *causae causantes* of these transitions:

$$CC(e_0 \rightarrowtail d_1^*) = \{\tau_1', \tau_2', \tau_3\}, \quad CC(e_0 \rightarrowtail d_2^*) = \{\tau_1', \tau_2', \tau_3'\}, \tag{7.12}$$

$$CC(e_0 \rightarrowtail (g_1 \cup d_2)) = \{\tau_1, \tau_2'\}, \quad CC(e_0 \rightarrowtail (g_2 \cup d_1)) = \{\tau_1', \tau_2\}, \tag{7.13}$$

$$CC(e_0 \rightarrowtail (g_1 \cup g_2)) = \{\tau_1, \tau_2\}. \tag{7.14}$$

As an illustration, we will calculate the propensity of $e_0 \rightarrowtail d_1^*$ as represented in an adequate causal probability space $CPS = \langle S, \mathscr{A}, p\rangle$. First, we list the set of alternatives $\tilde{T}$ to elements of $CC(e_0 \rightarrowtail d_1^*)$:

$$\tilde{T} = \{\tau_1, \tau_1', \tau_2, \tau_2', \tau_3, \tau_3'\}. \tag{7.15}$$

Then, given the causal set-up of our story, the base set S, i.e., the set of maximal consistent subsets of $\tilde{T}$, is

$$S = \{\{\tau_1, \tau_2\}, \{\tau_1, \tau_2'\}, \{\tau_1', \tau_2\}, \{\tau_1', \tau_2', \tau_3\}, \{\tau_1', \tau_2', \tau_3'\}\}. \tag{7.16}$$

$\mathscr{A}$ is then the set-theoretic Boolean algebra over S, as it should be. The propensity of $e_0 \rightarrowtail d_1^*$ is thus to be analyzed in the probability space $\langle S, \mathscr{A}, p\rangle$. Given the assignment of propensities μ stated in Eq. 7.11, we need

[34] Recall that the g_i, d_i, and d_i^* are scattered outcome events, so if any two of them are consistent, their set-theoretical union is a scattered outcome as well. But if such a pair is inconsistent (like g_1, d_1^*), they do *not* yield a scattered outcome.

to find the numerical value of $p(e_0 \rightarrowtail d_1^*) = \mu(e_0 \rightarrowtail d_1^*) = \mu(CC(e_0 \rightarrowtail d_1^*)) = \mu(\{\tau_1', \tau_2', \tau_3\}) = p(\{\tau_1', \tau_2', \tau_3\})$. To calculate this number, some extra information is needed, over and above the propensities of the basic transitions involved. If each individual *causa causans* works separately and independently of each other, we may multiply probabilities of all ingredient transitions. On the other hand, the SLR *causae causantes* could work jointly. In that case, we have probabilistic funny business, and we cannot just multiply probabilities.

Here we give the numerical values that result from the basic propensities of Eq. 7.11, the Markov condition (Postulate 7.1), and the reasonable assumption of no probabilistic funny business:

$$\begin{aligned} p(e_0 \rightarrowtail d_1^*) &= p_1' \cdot p_2' \cdot p_3 = .6 \cdot .1 \cdot .7 = .042 \\ p(e_0 \rightarrowtail d_2^*) &= p_1' \cdot p_2' \cdot p_3' = .6 \cdot .1 \cdot .3 = .018 \\ p(e_0 \rightarrowtail (g_1 \cup d_2)) &= p_1 \cdot p_2' = .4 \cdot .1 = .04 \\ p(e_0 \rightarrowtail (g_2 \cup d_1)) &= p_1' \cdot p_2 = .6 \cdot .9 = .54 \\ p(e_0 \rightarrowtail (g_1 \cup g_2)) &= p_1 \cdot p_2 = .4 \cdot .9 = .36 \end{aligned}$$

Since our causal analysis requires that, given the occurrence of e_0, exactly one of the five scattered outcomes must occur, it is hardly a surprise that the sum of these five propensities is 1 (see Postulate 7.3).

For future reference, let us calculate the propensity of the transition from e_0 to "the Inspector picks a defective corkscrew". In our BST structure, this is the transition from e_0 to the disjunctive outcome $\check{\mathbf{O}} = \{\{d_1^*\}, \{d_2^*\}\}$, which in *CPS* is represented as the element $\{\{\tau_1', \tau_2', \tau_3\}, \{\tau_1', \tau_2', \tau_3'\}\} \in \mathscr{A}$. The calculation of the probability is not difficult; just add, using the numerical results above:

$$p(e_0 \rightarrowtail \check{\mathbf{O}}) = .042 + .018 = 0.06. \tag{7.17}$$

Returning to Salmon's main problem, the crucial question of the story is the following:

Question 7.3. What shall we say about the chosen corkscrew's propensity to have been made by M_1?

Salmon's answer is: "it would hardly be reasonable to speak of the *propensity* of that corkscrew to have been produced by" M_1 (Salmon, 1989, p. 88). Indeed: would anyone want to say that the corkscrew lying quietly in their hand has a certain propensity to—what? A propensity to *have been made* by machine M_1 rather than M_2? The reason why this answer sounds nonsensical has precious little to do with propensities, probabilities, or Bayes: It is a purely causal issue. The corkscrew has either been made by M_1 or by M_2, and even though the Inspector may not know which, that is a settled matter.

Although the appeal to a past-directed propensity does not make sense, we would still like to reformulate Question 7.3 in some propensity-friendly way. We offer the following as an adequate replacement that makes sense.

Question 7.4. What is the propensity for the event e_0 (in Figure 7.4) to give rise to the Inspector's taking a corkscrew made by machine M_1, given that e_0 gives rise to his picking a defective corkscrew?

The two transitions involved are $e_0 \rightarrowtail d_1^*$ and $e_0 \rightarrowtail \breve{\mathbf{O}}$. In order to compute the probability of the former conditional on the latter, we also need the element of $\mathscr{A}$ that represents their intersection, which comes out as

$$\{CC(e_0 \rightarrowtail d_1^*)\} \cap CC(e_0 \rightarrowtail \breve{\mathbf{O}}) = \{\{\tau_1', \tau_2', \tau_3\}\} = \{CC(e_0 \rightarrowtail d_1^*)\}.$$

Given all this, we can compute the probability of the Inspector taking a corkscrew made by M_1 conditional on the Inspector picking a defective corkscrew:

$$p(\{\{\tau_1', \tau_2', \tau_3\}\} \mid \{\{\tau_1', \tau_2', \tau_3\}, \{\tau_1', \tau_2', \tau_3'\}\}) = \frac{p(e_0 \rightarrowtail d_1^*)}{p(e_0 \rightarrowtail \breve{\mathbf{O}})} = \frac{0.042}{0.06} = 0.7.$$

The inversion, although of little interest, makes perfectly good technical sense as well:

$$p(\{\{\tau_1', \tau_2', \tau_3\}, \{\tau_1', \tau_2', \tau_3'\}\} \mid \{\{\tau_1', \tau_2', \tau_3\}\}) = \frac{p(e_0 \rightarrowtail d_1^*)}{p(e_0 \rightarrowtail d_1^*)} = 1.$$

The propensity for e_0 to give rise to "The Inspector taking a defective corkscrew" given that e_0 gives rise to "The Inspector taking a corkscrew made by machine M_1", is a boring 1, since in our story, among the corkscrews made by M_1, the Inspector takes only defective corkscrews. We leave a sanity check of the corresponding instance of Bayesian inversion as Exercise 7.4.

7.4 Conclusions

In this chapter, we have used the analysis of causation in BST in terms of *causae causantes* as a background for a formal theory of objective single-case probabilities, or propensities, defined on BST transitions $I \rightarrowtail \mathscr{O}^*$. Working from a set of simple examples, we established a number of constraints on the (partial) propensity function μ that constitutes an additional ingredient in a probabilistic BST structure (Def. 7.3). We distinguished the propensity function μ from the probability measure p in causal probability spaces according to Defs. 7.4 and 7.5, and we discussed in which way a BST transition $I \rightarrowtail \mathscr{O}^*$ can be represented in different causal probability spaces (Section 7.2.5). With a view especially to later applications (see Chapter 8), we analyzed the notion of space-like probabilistic correlations, or probabilistic funny business (Def. 7.7), in terms of the dependency of random variables.

Our analysis fulfills two crucial conditions of adequacy, which in our view singles it out from among other accounts of propensities. First, the formal structures that we have defined, causal probability spaces, are standard Kolmogorovian probability spaces, showing how our causal probability theory fits in with the mainstream accounts of probabilities. The main distinction to standard approaches is the way in which probability spaces combine. As we have shown, the standard Cartesian product construction is only adequate if causally separated probability spaces with space-like related initials are combined. In the general case, the adequate way to construct the sample space of a causal probability space is to consider a set of causal alternatives (Def. 7.2). Second, we showed that our account of propensities via causal probability spaces is immune to the criticism of Humphreys's paradox. Conditional probabilities and Bayesian inversion constitute no problems for our approach, and we were able to pinpoint exactly which assumption for Humphreys's impossibility result is fallacious. We ended by showing in which way conditional propensities also make sense in more complex scenarios.

To sum up, BST-based probability theory is not an alternative to standard probability theory, but a fine-grained application of it that makes that theory able to treat objective single-case probabilities in a formally perspicuous way.

7.5 Exercises to Chapter 7

Exercise 7.1. Prove Lemma 7.2.

Hint: Consider the finite number of disjuncts separately and note that these disjuncts are mutually exclusive, allowing for the summation of their probabilities. A full proof is given in Appendix B.7.

Exercise 7.2. Let $I_1, I_2, \ldots, I_K$ be a set of pairwise SLR initial events that give rise to deterministic transitions to disjunctive outcomes $I_1 \rightarrowtail \mathbf{1}_1, I_2 \rightarrowtail \mathbf{1}_2, \ldots, I_K \rightarrowtail \mathbf{1}_K$. Let

$$\mathbf{1}_E = \{\bigcup Z \mid Z \in \mathbf{1}_1 \times \mathbf{1}_2 \times \ldots \times \mathbf{1}_K\}.$$

Prove that $\mathbf{1}_E$ is a disjunctive outcome and that $E =_{\text{df}} \bigcup_k^K I_k$ and $\mathbf{1}_E$ form a transition (i.e., that E is below $\mathbf{1}_E$ in the relevant sense of Def. 4.4), and that that transition is deterministic.

Exercise 7.3. Exhibit the causal probability space induced by set $CC(e_0 \rightarrowtail g_1)$ in the corkscrew story of Chapter 7.3.4 and then show how this set, $CC(e_0 \rightarrowtail g_1)$, is represented in the probability space CPS defined on the basis of Eq. 7.16.

Exercise 7.4. Perform a sanity check of the principle of Bayesian inversion for the corkscrew story of Chapter 7.3.4; that is, verify that, with the appropriate A and B,

$$p(A \mid B) = \frac{p(B \mid A)}{p(B \mid A) \cdot p(A) + p(B \mid \overline{A}) \cdot p(\overline{A})}.$$