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Branching in Relativistic Space-Times

We advertised Branching Space-Times theory as a theory of local indeterminism, playing out in our spatio-temporal world, where the space-time is to be at least rudimentarily relativistic. The theory thus promises to describe how to combine local indeterminism and relativistic space-times. In this chapter we will make good on this promise, as we will introduce here particular BST structures in which histories are similar to the space-times of relativistic physics. We can approach these tasks differently, with various degrees of modesty, and our first construction, the so-called Minkowskian Branching Structures (MBSs), is intended to be a modest one: we focus on how alternative histories, all happening on Minkowski space-time, can be seen as developing from some shared past. Of course, a history, realistically speaking, cannot be a bare Minkowski space-time, as it should involve some properties that are not spatio-temporal (e.g., matter fields). To have more full blooded objects, we first make a simplifying assumption regarding how physical properties are associated to space-time points, and then leave the matter to physics in order to check whether some of its theories support this construction. The simplifying assumption is that whatever content a history has, it comes from the ascription of some quantities (values, or strengths of a field) to *points* in the space-time. This is of course pointilisme pure and simple, and the resulting concept of history is a case of a Humean mosaic. The next idea is that whilst a physics theory is pointilistic as suggested earlier, perhaps it acknowledges alternative property ascriptions. How can that be the case? A natural response is what is known as the ill-posedness of the initial value problem, which means that the theory's equations of evolution allow that for some initial values, there are multiple global solutions to these equations, with the solutions representing evolutions occurring in Minkowski space-time. Whether there are such theories of physics, and whether their solutions satisfy some further constraints necessary for the construction to go through, we leave it for physics to judge. And if the judgement is in the negative, that would mean that, as far as physics goes,

Branching Space-Times: Theory and Applications. Nuel Belnap, Thomas Müller, and Tomasz Placek,
Oxford University Press. © Oxford University Press 2022. DOI: 10.1093/oso/9780190884314.003.0009

there is no local indeterminism of the sort described by BST, and occurring in Minkowski space-time. In the second part (Section 9.3) we turn toward linking BST to general relativity, which is both an ideologically different and more demanding project.

9.1 Minkowskian Branching Structures

We now turn to constructing a special class of BST structures, called Minkowskian Branching Structures (MBSs), in which each history is isomorphic to Minkowski space-time. Our aim here is to show that our MBSs satisfy the axioms of $\mathrm{BST_{NF}}$ structures (Def. 3.15). An alternative construction, in which an MBS comes out as a $\mathrm{BST_{92}}$ structure, was investigated in our earlier work.[1] In this book, we focus on MBSs that are $\mathrm{BST_{NF}}$ structures because we want the structures to admit locally Euclidean topologies; this provides a better integration with General Relativity (GR). Section 9.3 investigates the relations between BST and GR.

9.1.1 Basic notions

Our construction proceeds in terms of the assignments of physical properties (which we think of as values of physical fields) to space-time points. As we will see, MBSs seamlessly permit the introduction of space-time locations in the sense of Def. 2.9. For the construction to succeed, however, some additional physical conditions must be satisfied. As before, by Minkowski space-time we will understand the set $\mathbb{R}^n$ with the Minkowskian ordering, $<_M$, defined in the usual way (see Eq. (2.1)) as:[2]

$$x <_M y \quad \text{iff} \quad -(x^0 - y^0)^2 + \sum_{i=1}^{n-1} (x^i - y^i)^2 \leqslant 0 \text{ and } x^0 < y^0. \tag{9.1}$$

Usually we assume that $n = 4$. The ordering $<$ (or $\leqslant$) on the right-hand side refers to the natural strict (or non-strict) ordering of the reals. As usual, we

[1] See the papers by Müller (2002), Wroński and Placek (2009), and Placek and Belnap (2012).

[2] For the record, in physics it is common to take Minkowski space-time to be a set of points together with a metric, which allows one to derive the causal ordering. The structures we are working with thus contain less information, but are also simpler to handle, than those of the physicists.

define: $x \leqslant_M y$ iff ($x <_M y$ or $x = y$), and furthermore, $xSLR_M\,y$ iff neither $x \leqslant_M y$ nor $y \leqslant_M x$.

Some physical theories ascribe physical properties, typically the strengths of physical fields, to points of a space-time, or can be viewed as involving such an ascription. If the underlying space-time is a Minkowski space-time, the properties are ascribed to points of Minkowski space-time (i.e., elements of $\mathbb{R}^4$). A necessary condition for a theory to exhibit indeterminism is that it allows for many "scenarios" ascribing alternative possible properties to points of Minkowski space-time. In other words, one point of $\mathbb{R}^4$ may have alternative properties assigned, depending on the scenario.

Ultimately, we will define an MBS as a triple $\mathfrak{M} = \langle \Sigma, F, P \rangle$ (see Def. 9.5). We will begin by partly characterizing Σ, F, and P. To help capture the informal concept of possible "scenarios" abstractly, we assume a non-empty set Σ of labels, understood as labels for "scenarios." We let σ, η, γ range over Σ. We want to think of a scenario as Minkowski space-time filled with some "content," where the content of a scenario should be representable by an attribution of properties to each Minkowski space-time point. That is, the content of a single scenario, σ, may be represented by a function in the set $\mathbb{R}^4 \to \mathscr{P}(P)$, where P is a nonempty set of properties attributable to points of $\mathbb{R}^4$. Our purposes do not require putting any structure on P. A system of such contents can then be represented by a global attribution of properties $F : \Sigma \times \mathbb{R}^4 \to \mathscr{P}(P)$. We will call such an F a "property attribution on Σ and P," noting that it is in effect a modal notion because it refers to alternative possible properties for the same space-time point. Writing $\langle \sigma x \rangle$ for a pair from $\Sigma \times \mathbb{R}^4$, we may read "$F(\langle \sigma x \rangle)$" as "the set of properties instantiated at space-time point x in scenario σ". We informally think of the set in question as only containing compatible properties. The function F evidently dictates for each space-time point, x, whether two scenarios, σ, η, are qualitatively the same there ($F(\langle \sigma x \rangle) = F(\langle \eta x \rangle)$) or not ($F(\langle \sigma x \rangle) \neq F(\langle \eta x \rangle)$).—To avoid clutter, from now on we simply write "$F(\sigma x)$".

Clearly, many property attributions yield a pattern of scenarios without any similarity to what one might call indeterminism. Indeed, there is a consensus that indeterminism involves many scenarios that agree over some region (typically, an initial region) and then disagree over some (typically, later) region. In what follows, we will single out those special property attributions that we will call *proper* property attributions. We will find that proper property attributions lead to a pattern of indeterminism that is describable by Branching Space-Times. This means that we will derive from a triple $\mathfrak{M} = \langle \Sigma, F, P \rangle$ a BST-like pair $\langle B, <_R \rangle$, and BST-like notions of history and choice set, and ultimately show that the $\mathrm{BST_{NF}}$ axioms are satisfied in the defined model.

We turn to the task of defining a proper property attribution, F. A part of this task is to single out a set of particular points of $\mathbb{R}^4$, to be interpreted, loosely speaking, as locations of chanciness, as where the scenarios diverge. We will call such points "splitting points". We first assume that any two scenarios differ somewhere, i.e.,

$$\forall \sigma, \eta \in \Sigma \, (\sigma \neq \eta \rightarrow \exists x \in \mathbb{R}^4 \; F(\sigma x) \neq F(\eta x)), \qquad \text{(SDiff)}$$

where F is a property attribution on Σ and P.

Condition (SDiff) assures us that any two scenarios are qualitatively different. We further require that the pattern of differences for two scenarios be rather special: We postulate that for every two scenarios there is (at least) one point $s \in \mathbb{R}^4$ such that the scenarios disagree at s, but agree everywhere in the past of s. A point satisfying these two conditions will be defined as a splitting point for the two scenarios.

Definition 9.1 (Splitting points). Given $\mathfrak{M} = \langle \Sigma, F, P \rangle$, where F is a property attribution on Σ and P and $\sigma, \eta \in \Sigma$ and $s \in \mathbb{R}^4$, s is a *splitting point* between scenarios σ, η iff s satisfies the condition

$$F(\sigma s) \neq F(\eta s) \wedge \forall y \in \mathbb{R}^4 \; [y <_M s \rightarrow F(\sigma y) = F(\eta y)]. \qquad \text{(PastsAgree)}$$

$S_{\sigma\eta} \subseteq \mathbb{R}^4$ is defined as the set of all splitting points between scenarios $\sigma, \eta \in \Sigma$.

Splitting points for two scenarios allow us to define a region of $\mathbb{R}^4$ that we will soon prove to be the region in which the two scenarios are qualitatively the same (see Fact 9.1(4)). But note that the two scenarios are qualitatively different somewhere else as well. We allow for split scenarios to largely reconverge qualitatively. Regions of overlap were first introduced by Müller (2002).

Definition 9.2 (Region of overlap). Given $\mathfrak{M} = \langle \Sigma, F, P \rangle$, for $\sigma, \eta \in \Sigma$, $R_{\sigma\eta} := \{x \in \mathbb{R}^4 \mid \neg \exists s \, (s \leqslant_M x \wedge s \in S_{\sigma\eta})\}$.

Splitting is a qualitative notion that is derived from the differences of properties in scenarios, in contrast to the cause-like notion of choice sets of BST_{NF}. Note also that no $s \in S_{\sigma\gamma}$ can belong to $R_{\sigma\gamma}$. For a pattern of qualitative differences between scenarios to deliver a BST_{NF} structure, it must be somewhat restricted. The restrictions are incorporated in what we call "proper property attribution". Yet, before we state it we need to define the auxiliary notion of the set $\Sigma_\eta(x)$ of those labels for a given $x \in \mathbb{R}^4$ that specify one and the same event in an MBS.

Definition 9.3. Given $\mathfrak{M} = \langle \Sigma, F, P \rangle$, where F is a property attribution on Σ and P and $\eta \in \Sigma$:

$$\Sigma_\eta(x) =_{\text{df}} \{\sigma \in \Sigma \mid x \in R_{\sigma\eta}\}.$$

Note that $\Sigma_\eta(x)$ is not empty for any x, since $x \in R_{\eta\eta}$. This notion is needed in clause (2) of the definition below.

Definition 9.4 (Proper property attribution)**.** Given $\mathfrak{M} = \langle \Sigma, F, P \rangle$, F is a *proper property attribution* on a set of scenarios Σ and a set of properties P iff $F : \Sigma \times \mathbb{R}^4 \to \mathscr{P}(P)$, and F satisfies the condition SDiff that every two scenarios differ, and for all $\sigma, \eta \in \Sigma$,

1. $\forall x \in \mathbb{R}^4\ [F(\sigma x) \neq F(\eta x) \to \exists s \in S_{\sigma\eta}\ [s \leqslant_M x]]$;
2. for every chain l in $\langle \mathbb{R}^4, <_M \rangle$, if for each $x \in l$ there is a unique $\gamma_x \in \Sigma$ such that for any finite $Z \subseteq l$ we have $\bigcap_{x \in Z} \Sigma_{\gamma_x}(x) \neq \emptyset$, then $\bigcap_{x \in l} \Sigma_{\gamma_x}(x) \neq \emptyset$;
3. for every lower bounded chain l in $\langle \mathbb{R}^4, <_M \rangle$,

$$(\forall x \in l\ \exists s \in S_{\sigma\eta}\ [s \leqslant_M x]) \to (\exists s_0 \in S_{\sigma\eta}\ [s_0 \leqslant_M \inf l]).$$

The first clause links points of qualitative difference to splitting points: either a point of qualitative difference is a splitting point, or there is a splitting point below it. The second clause is to curb the wealth of splittings, and its significance comes out only later (in Fact 9.4), when we consider the form of histories in a BST_{NF} structure that implements an MBS.[3] The clause is automatically satisfied by any finite chain l. The last clause is to guarantee the satisfaction of the Prior Choice Principle, PCP_{NF}.

We can show that for any $\sigma, \eta \in \Sigma$, the set $S_{\sigma\eta}$ of splitting points induced by a proper property attribution on Σ and P, as well as the corresponding region of overlap $R_{\sigma\eta}$, have the following natural properties:

[3] The clause is modeled after the Chain Condition 7 of Wroński and Placek (2009).

Fact 9.1. *Assume that Σ and P are non-empty and that F is a proper property attribution on Σ and P. Then for any $\sigma, \eta, \gamma \in \Sigma$ and for any $x \in \mathbb{R}^4$:*

1. $\sigma = \eta$ *iff* $S_{\sigma\eta} = \emptyset$;
2. $S_{\sigma\eta} = S_{\eta\sigma}$;
3. $\forall s, s' \in S_{\sigma\eta}\ (s \neq s' \rightarrow s\, SLR_M\, s')$;
4. $x \in R_{\sigma\eta} \rightarrow F(\sigma x) = F(\eta x)$, *and*
5. $R_{\sigma\eta} \cap R_{\eta\gamma} \subseteq R_{\sigma\gamma}$.

Proof. (1) Immediate from Definition 9.4.

(2) Immediate from Definition 9.1.

(3) By (PastsAgree), neither $s <_M s'$ nor $s' <_M s$. So if $s \neq s'$, then $s\, SLR_M\, s'$;

(4) This is essentially the contrapositive of Definition 9.4 clause (1).

(5) For reductio, assume $x \in R_{\sigma\eta} \cap R_{\eta\gamma}$ but $x \notin R_{\sigma\gamma}$. The latter implies that there is an $s \in S_{\sigma\gamma}$ such that $s \leqslant_M x$ so (†) $F(\sigma s) \neq F(\gamma s)$. Since regions of overlap are evidently closed downward, we get $s \in R_{\sigma\eta} \cap R_{\eta\gamma}$. By item (4) of this Fact, $F(\sigma s) = F(\eta s)$ and $F(\eta s) = F(\gamma s)$, and hence $F(\sigma s) = F(\gamma s)$. Contradiction with (†). □

Observe that in the proof above we used clause (1), but neither clause (2) nor clause (3) of Def. 9.4.

9.1.2 Defining MBSs

After this preliminary work, we turn now toward defining MBSs and showing that they generate structures satisfying the postulates of $\mathrm{BST_{NF}}$. We thus first officially define Minkowskian Branching Structures, then prove desired facts about the form of B-histories, and finally show that the postulates of $\mathrm{BST_{NF}}$ are satisfied.

Definition 9.5 (MBS). A triple $\mathfrak{M} = \langle \Sigma, F, P \rangle$ is a *Minkowskian Branching Structure* (an *MBS*) iff Σ is a non-empty set of scenarios, P is a nonempty set of properties, and F is a proper property attribution on Σ and P.

In order to link MBSs to BST, our first task is to find a correlate for the BST notion of *Our World* (i.e., a base set), and for the BST ordering. Given $\mathfrak{M} = \langle \Sigma, F, P \rangle$, we take the elements of the base set to be equivalence classes of a certain relation $\equiv_R$ on $\Sigma \times \mathbb{R}^4$, where the “R” in the subscript indicates that the relation crucially depends on an assumed region of overlap R—the idea is to "identify" points in regions of overlap. The relation $\equiv_R$ is defined as below.

Definition 9.6 (MBS equivalence relation)**.** Given an MBS $\mathfrak{M} = \langle \Sigma, F, P \rangle$, the relation $\equiv_R$ on $\Sigma \times \mathbb{R}^4$ is defined as:

$$\langle \sigma, x \rangle \equiv_R \langle \eta, y \rangle \quad \text{iff} \quad x = y \text{ and } x \in R_{\sigma\eta}. \tag{9.2}$$

It is easy to check that $\equiv_R$ is an equivalence relation on $\Sigma \times \mathbb{R}^4$: Fact 9.1(1) implies $R_{\sigma\sigma} = \mathbb{R}^4$ from which reflexivity follows, symmetry follows by Fact 9.1(2), and transitivity follows by Fact 9.1(5). With this relation to hand, we introduce our candidate for a BST_{NF} structure via the equivalence classes under $\equiv_R$:

Definition 9.7 (MBS base set and MBS ordering)**.** Let $\mathfrak{M} = \langle \Sigma, F, P \rangle$ be an MBS. We define the *MBS base set for* $\mathfrak{M}$, B, to be

$$B =_{\text{df}} \{[\sigma x] \mid \sigma \in \Sigma, x \in \mathbb{R}^4\}, \quad \text{where} \quad [\sigma x] =_{\text{df}} \{\langle \eta, x \rangle \mid \langle \sigma, x \rangle \equiv_R \langle \eta, x \rangle\}.$$

The *MBS ordering* $<_R$ on B is defined by

$$[\sigma x] <_R [\eta y] \quad \Leftrightarrow_{\text{df}} \quad x <_M y \wedge \langle \sigma, x \rangle \equiv_R \langle \eta, x \rangle.$$

The pair $\langle B, <_R \rangle$ will be called *the structure generated by the MBS* $\mathfrak{M}$. As usual, we will write $\leqslant_R$ for the weak counterpart of $<_R$.

It is again easy to check that $<_R$ is a strict partial ordering on B. It is antireflexive because $<_M$ is antireflexive. Transitivity follows from Fact 9.1(5); see Exercise 9.1. Similarly, it is straightforward to prove density of $<_R$ (see Exercise 9.2). Note that if x is not in the region $R_{\sigma\eta}$ of overlap of σ and η, then $[\sigma x] \not<_R [\eta y]$ (for all y), but if x is in $R_{\sigma\eta}$, for $[\sigma x] <_R [\eta y]$ we only need to check the Minkowski ordering, $x <_M y$.

Given $\mathfrak{M} = \langle \Sigma, F, P \rangle$, a natural definition for the course of events corresponding to scenario σ is the set $\{[\sigma x] \mid x \in \mathbb{R}^4\}$ of equivalence classes. Knowing the set, that is, knowing each equivalence class from it, gives us all there is to be known about this course of events, that is, a property assignment for σ and every $x \in \mathbb{R}^4$. This motivates defining $\{[\sigma x] \mid x \in \mathbb{R}^4\}$ as a "B-history."

Definition 9.8 (B-histories)**.** Given $\mathfrak{M} = \langle \Sigma, F, P \rangle$, the *$B$-history corresponding to $\sigma \in \Sigma$* is defined to be $h_\sigma =_{\text{df}} \{[\sigma x] \mid x \in \mathbb{R}^4\}$. B-Hist is the set of all B-histories.

Given $\mathfrak{M} = \langle \Sigma, F, P \rangle$, our plan is to show that the pair $\langle B, <_R \rangle$ is a BST_{NF} structure. The next three facts about MBSs concern the form of histories:

Fact 9.2. *Suppose that $\langle B, <_R \rangle$ is a structure determined by MBS $\mathfrak{M} = \langle \Sigma, F, P \rangle$. Then every B-history is a maximal directed subset of B.*

Proof. Consider a B-history h_σ. Since for any $x_1, x_2 \in \mathbb{R}^4$ there is $x_3 \in \mathbb{R}^4$ such that $x_1 \leqslant_M x_3$ and $x_2 \leqslant_M x_3$, h_σ is directed. To argue that it is maximal directed, suppose for reductio that there is a directed set $g \subseteq B$ such that $h_\sigma \subsetneq g$. There is thus some $[\eta x] \in g \setminus h_\sigma$. Hence (†) $[\eta x] \neq [\sigma x]$. Since $[\sigma x] \in h_\sigma \subsetneq g$ and g is directed, there is $[\alpha y] \in g$ such that $[\sigma x] \leqslant_R [\alpha y]$ and $[\eta x] \leqslant_R [\alpha y]$. It follows that $[\sigma x] = [\alpha x] = [\eta x]$, which contradicts (†). □

Fact 9.3. *Let $\langle B, <_R \rangle$ be the structure determined by the MBS $\mathfrak{M} = \langle \Sigma, F, P \rangle$. Then*

(1) if $[\sigma_1 x_1], [\sigma_2 x_2] \in B$ and $[\sigma_1 x_1] \leqslant_R [\sigma_2 x_2]$, then $\Sigma_{\sigma_2}(x_2) \subseteq \Sigma_{\sigma_1}(x_1)$;

(2) for every directed set $h \subseteq B$ and every finite subset $X \subseteq h$, $\bigcap_x \{\Sigma_\gamma(x) \mid [\gamma x] \in X\} \neq \emptyset$.

Proof. (1) Let $[\sigma_1 x_1] \leqslant_R [\sigma_2 x_2]$, and $\eta \in \Sigma_{\sigma_2}(x_2)$, so $[\sigma_2 x_2] = [\eta x_2]$. Then $[\sigma_1 x_1] = [\eta x_1]$. Hence $\eta \in \Sigma_{\sigma_1}(x_1)$. Thus, $\Sigma_{\sigma_2}(x_2) \subseteq \Sigma_{\sigma_1}(x_1)$.

(2) By directedness of h and finiteness of X, there is an upper bound of X in h, say $[\sigma y]$. Thus, every element $[\eta x] \in X$ can be written as $[\sigma x]$, so $\sigma \in \Sigma_\eta(x)$ for every $[\eta x] \in X$. □

Fact 9.4. *Suppose that $\mathfrak{M} = \langle \Sigma, F, P \rangle$ is an MBS.*

1. *Every maximal directed subset of B is a B-history;*
2. *To every B-history there corresponds a unique $\sigma \in \Sigma$, i.e., for every pair h_σ, h_η of B-histories, $h_\sigma = h_\eta$ iff $\sigma = \eta$;*

Proof. (1) Let h be a maximal directed subset of B. We first show that $h = \{[\sigma x] \mid x \in Y\}$, for some $\sigma \in \Sigma$ and $Y \subseteq \mathbb{R}^4$. For reductio, let us suppose that this is not true, i.e., for any $\sigma \in \Sigma$ there is some $[\eta x] \in h$ such that $[\eta x] \neq [\sigma x]$. We will construct a chain in $\langle \mathbb{R}^4, <_M \rangle$ that contradicts clause 2 of Def. 9.4.

We pick some $[\eta y] \in h$, well-order $\Sigma_\eta(y)$ in some way, and define function $\Theta : \Sigma_\eta(y) \mapsto \mathscr{P}(\mathbb{R}^4)$ such that $\Theta(\sigma) = \{x \in \mathbb{R}^4 \mid [\sigma x] \notin h]\}$. $\Theta(\sigma)$ thus comprises all "bad for σ" points from $\mathbb{R}^4$; that is, those points that, taken with σ, produce no elements of h. By the reductio assumption, the value of Θ is never the empty set. To pick a representative from $\Theta(\sigma)$, we use a selection function $T : \mathscr{P}(\mathbb{R}^4) \mapsto \mathbb{R}^4$ such that $T(\Theta(\sigma)) \in \Theta(\sigma)$ for any $\sigma \in \Sigma_\eta(y)$. We label the values of the composition $T \circ \Theta$ so that we have $T \circ \Theta(\sigma_l) = x_l$ (note that l is a label, resulting from $\Sigma_\eta(y)$ being well-ordered). Accordingly, the ordering of $\Sigma_\eta(y)$ is carried over to the image $X =_{\text{df}} (T \circ \Theta)[\Sigma_\eta(y)]$. For each x_l, we pick $\sigma_{x_l} \in \Sigma_y(\eta)$ such that $[\sigma_{x_l} x_l] \in h$. Observe that by this construction, (†) for any $\sigma_l \in \Sigma_\eta(y)$: $\sigma_l \notin \Sigma_{\sigma_{x_l}}(x_l)$. Thus, $\bigcap_{x_l \in X} \Sigma_{\sigma_{x_l}}(x_l) = \emptyset$.

We now use the set $X \subseteq \mathbb{R}^4$ to produce a chain in $\langle \mathbb{R}^4, <_M \rangle$ above y. For simplicity's sake, in this construction we work directly in terms of coordinates of events like $y \in \mathbb{R}^4$. This amounts to choosing an arbitrary reference frame. The elements of this chain have different values of the first coordinate, whereas the remaining three coordinates are fixed. To construct the chain, we use a function $up : \mathbb{R}^4 \times \mathbb{R}^4 \mapsto \mathbb{R}^4$ that for any pair of points z_1, z_2 in $\mathbb{R}^4$ yields the minimal element of the set of their upper bounds that lie on a vertical line passing through z_1. Our chain is defined by $z_0 = y$, $z_1 = up(z_0, x_1), \ldots, z_{n+1} = up(z_n, x_{n+1})$. (For the details of the chain construction, in particular, for a limit step, see Exercise 9.3.) The result is a time-like chain $E \subseteq \mathbb{R}^4$ starting with y. Its cardinality is determined by the cardinality of $\Sigma_\eta(y)$ and the location of points of X.

In the next step, we observe that each $y, x_1, x_2, \ldots$, if associated with a proper label, is an element of h. Since h is directed, each $z_0, z_1, z_2, \ldots$, if associated with a proper label, is an element of h. For each z_l, we thus pick a label σ_{z_l} such that $[\sigma_{z_l} z_l] \in h$. Since h is directed, by Fact 9.3(2), for any finite subset $Z \subseteq E$, we have $\bigcap_{z_l \in Z} \Sigma_{\sigma_{z_l}}(z_l) \neq \emptyset$. Accordingly, the premise of clause (2) of Def. 9.4 is satisfied, so by this clause $\bigcap_{z_l \in E} \Sigma_{\sigma_{z_l}}(z_l) \neq \emptyset$. Since $[\sigma_{x_l} x_l] \leqslant_R [\sigma_{x_l} z_l]$, for every $x_l \in X$, $z_l \in E$, by Fact 9.3(1) $\Sigma_{\sigma_{z_l}}(z_l) \subseteq \Sigma_{\sigma_{x_l}}(x_l)$, and hence $\bigcap_{x_l \in X} \Sigma_{\sigma_{x_l}}(x_l) \neq \emptyset$. We thus have arrived at a contradiction with (†).

We thus showed that $h = \{[\sigma x] \mid x \in Y\}$ for some $\sigma \in \Sigma$ and $Y \subseteq \mathbb{R}^4$. By Fact 9.2, if $Y \subsetneq R^4$, then h is not maximal directed. Hence $Y = \mathbb{R}^4$.

(2) Clearly, if $\sigma = \eta$, then $\{[\sigma x] \mid x \in \mathbb{R}^4\} = \{[\eta x] \mid x \in \mathbb{R}^4\}$, so $h_\sigma = h_\eta$. In the other direction, if $\sigma \neq \eta$, then by Fact 9.1(1) $S_{\sigma\eta} \neq \emptyset$, so there are $x \in \mathbb{R}^4$ and $s \in S_{\sigma\eta}$ such that $s <_M x$. Hence $[\sigma x] \neq [\eta x]$. Hence $h_\sigma \neq h_\eta$. □

There is thus a perfect match between scenarios $\sigma \in \Sigma$, B-histories h_σ, and maximal directed subsets of B. As we now know what the histories in $\langle B, <_R \rangle$ look like, we next address the form of choice sets. Recall that a choice set is a particular subset of the base set B, whereas a splitting point $s \in S_{\sigma\eta}$ for two scenarios σ and η is just a point $s \in \mathbb{R}^4$. We will prove that the two notions correspond nicely. But since we need history-relative suprema in this proof, we first prove a fact concerning suprema and infima of bounded chains in $\langle B, <_R \rangle$, showing that the structure satisfies the two respective postulates of $\mathrm{BST_{NF}}$.

Fact 9.5. *Let $\langle B, <_R \rangle$ be a structure determined by the MBS $\langle \Sigma, F, P \rangle$. Then*

(1) $\langle B, <_R \rangle$ contains infima for all lower bounded chains,

(2) $\langle B, <_R \rangle$ contains history-relative suprema for all upper bounded chains.

Proof. Since every chain extends to a history, by the form of histories, every chain l in $\langle B, <_R \rangle$ can be written as $l = \{[\sigma x] \mid x \in l_x\}$ for some $\sigma \in \Sigma$ and chain l_x in $\langle \mathbb{R}^4, <_M \rangle$. Observe that if a chain $l \subseteq B$ is lower (upper) bounded, then the corresponding chain l_x is lower (upper) bounded. By the properties of $<_M$, l_x has then an infimum i_x (supremum s_x). Thus, for any history h_γ such that $l \subseteq h_\gamma$, $[\gamma i_x]$ is a history-relative (with respect to h_γ) infimum of l, and $[\gamma s_x]$ is a history-relative (with respect to h_γ) supremum of l. The second conjunct already proves part (2) of the fact. As for part (1), since histories are downward closed and $[\sigma i_x] \leqslant_R l$, $[\sigma i_x]$ belongs to every B-history that contains l. Accordingly, $[\sigma i_x]$ is an infimum, and not merely a history-relative infimum, of l. □

Having introduced history-relative suprema to MBSs, we are ready to return to the form of choice sets; we will prove the following correspondence:

Fact 9.6. *Let $\sigma, \eta \in \Sigma$ and h_σ, h_η be B-histories in structure $\langle B, <_R \rangle$ that is determined by MBS $\langle \Sigma, F, P \rangle$. Then*
$s \in S_{\sigma\eta}$ iff $[\sigma s], [\eta s]$ belong to a choice set $[\ddot{\sigma} s]$ at which h_σ and h_η branch.

Proof. $\Rightarrow$ By Defs. 9.1 and 9.6, $[\sigma s] \neq [\eta s]$. As there are no minimal elements in $\mathbb{R}^4$, $[\sigma s]$ and $[\eta s]$ are not minimal elements in B-histories, so the top clause of Def. 3.11 does not apply—we need to only check the bottom clause. Consider thus an arbitrary chain $l \in \mathscr{C}_{[\sigma s]}$. It means that $l \subseteq h_\sigma$, $\sup_{h_\sigma} l = [\sigma s]$, and $[\sigma s] \notin l$. It follows that $l_x = \{x \in \mathbb{R}^4 \mid [\sigma x] \in l\}$ has a supremum s in $\langle \mathbb{R}^4, <_M \rangle$. Further, since every element of l_x is below s

(with respect to $<_M$), it cannot be above some $s' \in S_{\sigma\eta}$, since then $s' <_M s$, which contradicts Fact 9.1 (3). Accordingly, $l_x \subseteq R_{\sigma\eta}$, so $l \subseteq h_\eta$. Thus, l has a supremum in h_η, $\sup_{h_\eta} l = [\eta s]$. Since l is arbitrary, $[\sigma s], [\eta s] \in \bigcap_{l \in \mathscr{C}_{[\sigma s]}} \mathscr{S}(l)$, and hence $[\ddot{\sigma s}]$ is a choice set, with $[\sigma s], [\eta s] \in [\ddot{\sigma s}]$. Further, by Def. 9.8 of B-histories, $h_\sigma \cap [\ddot{\sigma s}] = \{[\sigma s]\}$ and $h_\eta \cap [\ddot{\sigma s}] = \{[\eta s]\}$, and $[\sigma s] \neq [\eta s]$. It thus follows by Def. 3.13 that h_σ and h_η branch at $[\ddot{\sigma s}]$.
$\Leftarrow$ Let $[\sigma s], [\eta s], [\ddot{\sigma s}], h_\sigma$, and h_η be as the RHS of the Fact says. By Def. 3.13, $[\sigma s] \neq [\eta s]$, so there is $s' \in S_{\sigma\eta}$ such that $s' \leqslant_M s$. Let us next suppose that $s' <_M s$. Clearly, $[\sigma s'] \neq [\eta s']$, and for any $x \in \mathbb{R}^4$, if $[\sigma s'] \leqslant_R [\sigma x]$, then $[\sigma x] \notin h_\eta$. Pick then any chain $l \in \mathscr{C}_{[\sigma s]}$ that contains $[\sigma s']$. Since $[\sigma s'] <_R [\sigma s]$, by density there is non-empty upper segment l' of l such that $l' \cap h_\eta = \emptyset$. Thus, $[\eta s]$ is not a h_η-relative supremum of l, which entails $[\eta s] \notin [\ddot{\sigma s}]$, contradicting the Fact's premise. Thus, $s = s'$, with $s' \in S_{\sigma\eta}$. □

We next turn our attention to two more interesting postulates of BST_{NF}: PCP_{NF} and Weiner's postulate. We show that each is satisfied in structure $\langle B, <_R \rangle$ determined by an MBS.

Fact 9.7. *The structure $\langle B, <_R \rangle$ determined by an MBS satisfies PCP_{NF}, as defined by Def. 3.14.*

Proof. Let h_σ, h_η be B-histories in $\langle B, <_R \rangle$, and let $l = \{[\sigma x] \mid x \in l_x\} \subseteq h_\sigma$ be a lower bounded chain (so l_x is a lower bounded chain in $\langle \mathbb{R}^4, <_M \rangle$) such that $l \cap h_\eta = \emptyset$. Thus, for every $z \in l_x$: $[\sigma z] \neq [\eta z]$, and hence for every $z \in l_x$ there is $s \in S_{\sigma\eta}$ such that $s \leqslant_M z$. By the infima postulate, l has an infimum $[\sigma i_x]$, and hence l_x has an infimum i_x. Accordingly, l_x satisfies the premise of clause (3) of Def. 9.4, so by this very clause there is $s_0 \in S_{\sigma\eta}$ such that $s_0 \leqslant_M inf(l_x)$. It follows that $[\sigma s_0] \leqslant_R l$ and by Fact 9.6, $[\sigma s_0]$ gives rise to choice set $[\ddot{\sigma s_0}]$, at which h_σ and h_η branch, i.e., $h_\sigma \perp_{[\ddot{\sigma s_0}]} h_\eta$. □

Fact 9.8. *The structure $\langle B, <_R \rangle$ determined by an MBS satisfies Weiner's postulate 2.6.*

Proof. To check Weiner's postulate, let $l_1, l_2 \subseteq h_\sigma \cap h_\eta$ be upper bounded chains in B-histories h_σ and h_η. Let s_1, s_2 be relative to h_σ suprema of l_1 and l_2, respectively. If $s_1 = s_2$, then $s_1 = s_2 = [\sigma x]$ for some $x \in \mathbb{R}^4$, and hence there are $c_1 = c_2 = [\eta x] \in h_\eta$ that are h_η–relative suprema of l_1 and l_2, respectively. Analogously, if $s_1 <_R s_2$, then, as $s_1 = [\sigma x]$ and $s_2 = [\sigma y]$ for some $x, y \in \mathbb{R}^4$, it follows that $x <_M y$. Clearly, $c_1 = [\eta x]$ and $c_2 = [\eta y]$ are h_η–relative suprema of l_1, l_2, and $c_1 <_R c_2$. □

The last few facts testify that $\langle B <_R \rangle$ determined by an MBS satisfies the postulates of BST_{NF}. We are thus ready to state our main theorem about MBSs:

Theorem 9.1. *The structure $\langle B, <_R \rangle$ generated by an MBS (see Definition 9.7) is a BST_{NF} structure.*

Proof. We need to check if $\langle B, <_R \rangle$ satisfies all the BST_{NF} postulates, as stated in Def. 3.15. Given the definition of MBS, Σ is not empty, so B is not empty, either. As we noted (Def 9.7), $<_R$ is a dense strict partial order. Fact 9.5 states that the infima postulate and the suprema postulate are satisfied. Weiner's postulate is satisfied by Fact 9.8. PCP_{NF} holds by Fact 9.7 and it implies Historical Connection; see Exercise 3.5. Thus, all the postulates of BST_{NF} are satisfied by $\langle B, <_R \rangle$. □

Apart from being a BST_{NF} structure, an MBS has some additional welcome features. The fact below focuses on order-related similarities; that is, the existence of relevant order-isomorphisms.

Fact 9.9. *Let $\langle B, <_R \rangle$ be a structure determined by MBS $\langle \Sigma, F, P \rangle$. Then:*

1. *Every B-history in $\langle B, <_R \rangle$ is order-isomorphic to Minkowski space-time;*
2. *$\langle B, <_R \rangle$ permits the introduction of common space-time locations $\langle S, <_S \rangle$ in the sense of Def. 2.9;*
3. *$\langle S, <_S \rangle$ is order-isomorphic to Minkowski space-time.*

Proof. (1) A required isomorphism is $\varphi_j : \mathbb{R}^4 \mapsto h_\sigma$ such that $\varphi(x) = [\sigma x]$. It is clear to see that for $x, y \in \mathbb{R}^4$, $x \leqslant_M y$ iff $[\sigma x] \leqslant_R [\sigma y]$.

(2) By Def. 2.9, the set S of space-time locations should be a specific partition of B. For our $\langle B, <_R \rangle$ determined by MBS $\langle \Sigma, F, P \rangle$ we define: $S = \{\{[\sigma x] \mid \sigma \in \Sigma\} \mid x \in \mathbb{R}^4\}$. The ordering $<_S$ on S is given by: for $s, s' \in S$ such that $s = \{[\sigma x] \mid \sigma \in \Sigma\}$ and $s' = \{[\sigma y] \mid \sigma \in \Sigma\}$, $s <_S s'$ iff $x <_M y$. Its weak companion $\leqslant_S$ is defined via $s \leqslant_S s'$ iff $x \leqslant_M y$. Clearly, the intersection of any element $s = \{[\sigma x] \mid \sigma \in \Sigma\} \in S$ with any B-history h_η contains exactly one element, $[\eta x]$. And $\leqslant_S$ respects the ordering $\leqslant_R$, i.e., for $s, s' \in S$ and h_σ, h_η, if $s \cap h_\sigma = s' \cap h_\sigma$, then $s \cap h_\eta = s' \cap h_\eta$, and analogously for $<_S$ and $<_R$.(3)

The required isomorphism is $\varphi : S \mapsto \mathbb{R}^4$ with $\varphi(\{[\sigma x] \mid \sigma \in \Sigma\}) =_{\mathrm{df}} x$; by definition of $<_S$, φ is an order-isomorphism indeed. □

The similarity between a B-history and Minkowski space-time is even more intimate than an order-isomorphism. After all, as far as the structure goes, a B-history can be viewed as simply Minkowski space-time with the label $\sigma \in \Sigma$. The label σ is important, however, as it determines the physical content of the history, by the proper property attribution F and the set P of properties.

MBSs have interesting topological properties as well, to which we turn in Section 9.2.2, after introducing the required notions from the theory of differential manifolds. For the completeness of exposition, here we put down a few informal observations, while postponing rigorous arguments to the mentioned section. The natural topology on $\mathbb{R}^4$ is given by the base of open balls in $\mathbb{R}^4$. Since every B-history h_σ is, structurally speaking, just $\mathbb{R}^4 \times \{\sigma\}$, it admits a topology given by slightly fancier open balls, viz., standard balls with σ attached, which we call b-balls. We can show that, whenever o is an open ball on $\mathbb{R}^n$, $o_\sigma =_{\mathrm{df}} \{[\sigma x] \mid x \in o\}$ is indeed an *open* b-ball in the natural topology on h_σ. This means that the intersection of any two b-balls is the union of (possibly infinitely many) b-balls. Open b-balls generate a topology on h_σ. The open-ball topology on $\mathbb{R}^4$ and the open-b-ball topology on a B-history are thus topologically the same; they are homeomorphic. Further, the open ball topology on $\mathbb{R}^n$ and the open b-ball topology on h_σ share the same separation properties, including the Hausdorff property (see Def. 4.15). That is, if $[\sigma x]$ and $[\sigma y]$ are distinct elements of h_σ, they can be made centers of sufficiently small non-overlapping open b-balls in h_σ. One consequence of this is that the open b-ball topology on a B-history is both locally Euclidean (see Def. 4.16) and Hausdorff, so that a B-history is a topological manifold (see Def. 9.9 and footnote 7).

The first part of this observation carries over, perhaps somewhat surprisingly, to the base set B of a $\mathrm{BST_{NF}}$ structure $\langle B, <_R\rangle$ derived from an MBS $\langle \Sigma, F, P\rangle$, even if the structure comprises multiple B-histories. This topology is indeed Euclidean. For this construction to work, it is essential to have $\mathrm{PCP_{NF}}$ rather than $\mathrm{PCP_{92}}$, as the former guarantees that the intersection of any two b-balls is open. This construction would not work if $\langle B, <_R\rangle$ were a $\mathrm{BST_{92}}$ structure with multiple histories—see Exercise 9.4.

The second part of the observation, the one concerning Hausdorffness, does not carry over from a B-history to $\langle B, <_R\rangle$ if the structure comprises multiple histories. In this case, B contains at least one choice set with distinct elements, say, $[\sigma x]$ and $[\eta x]$. By Lemma 3.1, these two elements have exactly the same proper past. Accordingly, no matter how small the b-balls centered at these elements are that one chooses, they will overlap. Thus, the Hausdorff property fails in the b-ball topology on B.

To put these observations together, the b-ball topology on a B-history is both locally Euclidean and Hausdorff. In the terminology introduced in Chapter 9.2, a B-history is a topological manifold. In contrast, the b-ball topology on the base set B of a BST_{NF} structure derived from an MBS is locally Euclidean but typically non-Hausdorff. Such an object is called a generalized topological manifold. One may then ask if it is possible to extend a B-history by adding some elements from B, while preserving the local Euclidicity and Hausdorffness of the b-ball topology on the extended set. The answer turns out to be no (provided one assumes a further intuitive condition, connectedness). B-histories are the largest subsets of B on which b-balls deliver a locally Euclidean, Hausdorff, and connected topology. In short, B-histories are maximal connected topological sub-manifolds of the generalized topological manifold B.

Having seen these topological developments in Minkowskian Branching Structures and witnessing the debate in General Relativity about the status of the Hausdorff property in this theory's concept of space-time, it is tempting to try to provide topological foundations for BST (i.e., to define histories by a topological condition rather than by our order-theoretical one). More specifically, one might require that the base set W of *Our World* $\mathscr{W}$ admit a generalized (i.e., possibly non-Hausdorff) topological manifold structure, and then define a history in $\mathscr{W}$ to be a maximal subset of W that admits a connected Hausdorff topological manifold structure. In a strengthened version, one might further require that each history in $\mathscr{W}$ admit a differential manifold structure. We discuss this topic in Chapters 9.2 and 9.3, by first introducing the required topological notions, and then relating Branching Space-Times to General Relativity.

Before we turn to these tasks, let us take stock of MBSs.

9.1.3 Taking stock

Starting with three modest building blocks for Minkowskian Branching Structures, a set of labels for scenarios, a set of properties, and a function attributing properties to point-scenario pairs (where points are points of $\mathbb{R}^4$), we arrived at a BST_{NF} structure. The construction relied heavily on the constraints imposed on property attribution.[4]

An important result of this construction is that histories generated by an MBS are isomorphic to Minkowski space-time. Such histories are (and must be) different by having different physical contents (furnished by a property attribution), but they all share the same spatio-temporal structure. We do not claim that our construction is fully "physics-friendly". Even if the spatio-temporal structure of our universe were adequately represented by Minkowski space-time (which it is not), a few conditions must be satisfied on the part of physics for our construction of MBSs to be admitted. First, the physical description must come in the form of a property attribution to spatio-temporal points. Second, the property attribution must be quite specific: in Def. 9.4 we required it to be "proper". These notions are very much needed to guarantee that histories have the desired form, and that the prior choice principle, PCP_{NF}, is satisfied.[5]

Finally, each history in a BST_{NF} structure derived from an MBS can be viewed as a topological manifold (i.e., locally Euclidean and Hausdorff), whereas the structure itself can be seen as a generalized topological manifold (i.e., locally Euclidean but not necessarily Hausdorff). This establishes a

[4] Some remarks on the history of MBSs are in order. The notion of a Minkowskian Branching Structure was first introduced informally in Belnap (1992), denoting a BST structure in which every history is a Minkowski space-time. Placek (2000) first attempted to produce a BST model out of (copies of) Minkowski space-time, but failed. The first correct construction of MBSs (but with finitistic assumptions) is in Müller (2002). The construction presented here is a BST_{NF} version of the construction given by Placek and Belnap (2012), which in turn diverged from earlier constructions of Müller (2002) and Wroński and Placek (2009), as it aimed to be more physics-oriented. The latter authors begin their work with specifying a set Σ of labels for scenarios and a collection $\{S_{\sigma\eta}\}_{\sigma,\eta\in\Sigma}$ of sets of splitting points, where each $S_{\sigma\eta}$ possesses properties listed in Fact 9.1*(i)-(iv)*. Given the two primitive notions, that is, labels for scenarios and sets of splitting points, they define MBSs and show, on the assumption of certain additional conditions, that MBSs satisfy BST_{92} postulates. The authors diverge over these additional conditions: Müller assumes finitistic requirements whereas Wroński and Placek accepts a "topological" postulate that is equivalent to the chain condition. This difference notwithstanding, an MBS model is, in their sense, a pair $\langle\Sigma, \{S_{\sigma,\eta}\}_{\sigma,\eta\in\Sigma}\rangle$. In contrast, our point of departure is a property attribution to points in scenarios. Accordingly, an MBS model is, in our sense, a triple $\langle\Sigma, F, P\rangle$—cf. Def. 9.5. Splitting points are then a derived notion—see Def. 9.1 and, as Fact 9.1*(i)-(iv)* shows, they satisfy the conditions assumed by Müller and Wroński and Placek. Accordingly, given that $\mathfrak{M} = \langle\Sigma, F, P\rangle$ is an MBS in the sense of Definition 9.5, $\langle\Sigma, \{S_{\sigma,\eta}\}_{\sigma,\eta\in\Sigma}\rangle$ with $\{S_{\sigma,\eta}\}_{\sigma,\eta\in\Sigma}$ defined by Def. 9.1 is an MBS in the sense of Wroński and Placek (2009). If finitistic constraints concerning sets $S_{\sigma\eta}$ are assumed, $\langle\Sigma, \{S_{\sigma,\eta}\}_{\sigma,\eta\in\Sigma}\rangle$ is an MBS in the sense of Müller (2002). A discussion of the topological properties of the different ways of pasting together Minkowski space-times is given in Müller (2013).

[5] This is not to say that some other condition on property attributions and a matching definition of splitting points would not do the job. The point is that this notion must be quite regimented to be of use in producing BST models.

link between BST and General Relativity, and suggests adding topological notions to the foundations of BST theory. These are two topics to which we turn in the next two sections.

9.2 Differential manifolds and $\mathrm{BST_{NF}}$

In this section we first present some basic notions from the theory of differential manifolds. We will then use these notions to prove some facts about MBSs that we have already alluded to. We will also investigate $\mathrm{BST_{NF}}$ structures generally, asking if they admit differential manifold structure. We will need these notions later on, when we turn our attention to the space-times of General Relativity.

9.2.1 Differential manifolds

We introduce the already mentioned notions of topological manifold and generalized topological manifold. In General Relativity we need more complex structures, differential manifolds, which admit differential structure. We begin with the latter, defining the former as a special case.

Definition 9.9 (Chart, atlas, manifold: generalized, Hausdorff, non-Hausdorff)**.** Let M be a non-empty set and Γ an index set. A collection of pairs $\{\langle u_\gamma, \varphi_\gamma\rangle \mid \gamma \in \Gamma\}$, where each $u_\gamma \subseteq M$, is a C^r *n-atlas on M* iff $\bigcup_{\gamma\in\Gamma} u_\gamma = M$, each φ_γ is a bijection between u_γ and an open subset of $\mathbb{R}^n$, and for any two $\langle u_\gamma, \varphi_\gamma\rangle$ and $\langle u_\tau, \varphi_\tau\rangle$, if $u_\gamma \cap u_\tau \neq \emptyset$, then $\varphi_\gamma[u_\gamma \cap u_\tau]$ and $\varphi_\tau[u_\gamma \cap u_\tau]$ are open subsets of $\mathbb{R}^n$ and the composite functions $\varphi_\gamma \circ \varphi_\tau^{-1}$ and $\varphi_\tau \circ \varphi_\gamma^{-1}$ are C^r on their domains. Each $\langle u_\gamma, \varphi_\gamma\rangle$ is called a *chart* of the atlas.
A pair $\langle M, A\rangle$, where M is a non-empty set and A is a maximal C^r n-atlas on M, is a C^r *n-dimensional generalized differential manifold*.
If a C^r n-dimensional generalized differential manifold $\langle M, A\rangle$ satisfies the condition that for any distinct $p, q \in M$ there are $\langle u_\gamma, \varphi_\gamma\rangle, \langle u_\tau, \varphi_\tau\rangle \in A$ such that $p \in u_\gamma$, $q \in u_\tau$ and $u_\gamma \cap u_\tau = \emptyset$, then it is called a C^r n-dimensional Hausdorff differential manifold.[6]
If a C^r n-dimensional generalized differential manifold does not satisfy the

[6] An equivalent way of defining Hausdorff differential manifolds is to say that the induced topology is Hausdorff.

above condition, it is called a C^r *n-dimensional non-Hausdorff differential manifold*.

Next, in the degenerate case $r = 0$, we speak of a C^0 n-dimensional generalized (Hausdorff / non-Hausdorff) *topological* manifold.[7]

Finally, given a C^r n-dimensional generalized differential manifold $\langle M, A\rangle$, we say that the atlas A induces a topology $\mathscr{T}$ on the set M, which is given by the condition: $O \in \mathscr{T}$ iff for all $x \in O$ there is $\langle u_\gamma, \varphi_\gamma\rangle \in A$ such that $x \in u_\gamma$.

In what follows, if confusion is unlikely, we omit the qualifications "C^r, n-dimensional, differential", and just write "generalized" (or Hausdorff or non-Hausdorff) d-manifold, with "d" for "differential". By the above definitions, a generalized (or Hausdorff or non-Hausdorff) d-manifold counts as a generalized (or Hausdorff or non-Hausdorff) *topological* manifold as well.

9.2.2 Differential manifolds and MBSs

We are now going to make good on our informal observations about the topological features of MBSs, by proving some more general facts. The first fact says that any B-history admits a Hausdorff differential manifold structure, and hence, a Hausdorff topological manifold structure as well.

Fact 9.10. *Let h_σ be a B-history in the BST_{NF} structure $\langle B, <_R\rangle$ determined by an MBS. Then h_σ admits a C^∞ 4-dimensional Hausdorff d-manifold structure.*

Proof. Define b-balls as $o_\sigma =_{\text{df}} \{[\sigma x] \mid x \in o\}$, where o is an open ball in $\mathbb{R}^4$. For a b-ball o_σ, define $\varphi_\sigma : h_\sigma \mapsto \mathbb{R}^4$ such that $\varphi_\sigma([\sigma x]) =_{\text{df}} x$. Note next that the composition $\varphi_\sigma \circ \varphi_\sigma^{-1}$ is the identity function on $\mathbb{R}^4$, which is differentiable to an arbitrarily large degree. Thus, it follows immediately that $A =_{\text{df}} \{\langle o_\sigma, \varphi_\sigma\rangle \mid o \text{ an open ball in } \mathbb{R}^4\}$ is a C^∞ 4-dimensional atlas on h_σ. Thus, $\langle h_\sigma, A\rangle$ is a C^∞ 4-dimensional d-manifold, which is moreover Hausdorff: For distinct $[\sigma x], [\sigma y] \in h_\sigma$ there obviously are non-overlapping open balls $o^x, o^y \subseteq \mathbb{R}^4$ centered at x and y, respectively. These open balls determine non-overlapping b-balls centered at $[\sigma x]$ and $[\sigma y]$, o^x_σ and o^y_σ, respectively. □

[7] This is equivalent to the more typical definition of a topological manifold (Hausdorff or non-Hausdorff) as a locally Euclidean topological space (satisfying or not satisfying the Hausdorff condition).

Our next Fact concerns the base set of a BST_{NF} structure derived from an MBS. It says that the base set admits a 4-dimensional generalized differential manifold structure. Further, the manifold has to be non-Hausdorff if the structure comprises more than one B-history.

Fact 9.11. *Let $\langle B, <_R \rangle$ be the BST_{NF} structure determined by an MBS $\langle \Sigma, F, P \rangle$. Then B admits a C^∞ 4-dimensional generalized (possibly non-Hausdorff) d-manifold structure.*

Proof. Define a b-ball as before: $o_\sigma = \{[\sigma x] \mid x \in o\}$, where o is an open ball in $\mathbb{R}^4$ and $\sigma \in \Sigma$. For each open ball o and each $\sigma \in \Sigma$ define the function $\varphi_\sigma : o_\sigma \mapsto \mathbb{R}^4$ such that $\varphi_\sigma([\sigma x]) = x$. Next, consider intersections of the form $o_\sigma \cap o'_\alpha$, which are equal to $\{[\sigma x] \mid x \in o \cap o' \wedge \neg \exists s \in S_{\sigma\alpha}\ s \leqslant_M x\}$. Since the defining condition of this set picks an open subset of $\mathbb{R}^4$, clearly $\varphi_\sigma(o_\sigma \cap o'_\alpha)$ and $\varphi_\alpha(o_\sigma \cap o'_\alpha)$ are open (this holds even if the intersection is empty). Note, as above, that the composition of φ_σ and φ_α is the identity function on an appropriate domain in $\mathbb{R}^4$ (which may be empty), so it is differentiable to an arbitrarily large degree. Thus, the set $\{\langle o_\sigma, \varphi_\sigma \rangle \mid \sigma \in \Sigma, o \in \mathscr{B}^4\}$, with $\mathscr{B}^4$ the set of open balls in $\mathbb{R}^4$, induces a maximal C^∞ 4-dim atlas A on B. Thus, $\langle B, A \rangle$ is a C^∞ 4-dim generalized d-manifold.

If B comprises two B-histories, say h_σ and h_α ($\sigma, \alpha \in \Sigma$), then $\langle B, A \rangle$ is not Hausdorff, however. There is then a splitting point $s \in S_{\sigma\alpha}$, so that $[\sigma s] \neq [\alpha s]$. Then any b-ball centered at $[\sigma s]$ and any b-ball centered at $[\alpha s]$ overlap non-emptily, because $[\sigma s]$ and $[\alpha s]$ share the same proper past. □

We next show, for $\langle B, <_R \rangle$ derived from an MBS, that each B-history in this structure is a maximal subset of B that admits a connected Hausdorff d-submanifold structure; this manifold is a sub-manifold of the generalized d-manifold admitted by B.

Fact 9.12. *Let $\langle B, <_R \rangle$ be the BST_{NF} structure determined by an MBS $\langle \Sigma, F, P \rangle$. Then for every B-history h_σ in B, $\langle h_\sigma, A^\sigma \rangle$ is a maximal connected Hausdorff d-submanifold of the generalized d-manifold $\langle B, A \rangle$, where $\mathscr{B}^4$ is the set of open balls in $\mathbb{R}^4$, the maximal atlas A is induced by $\{\langle o_\gamma, \varphi_\gamma \rangle \mid \gamma \in \Sigma, o \in \mathscr{B}^4\}$, and the maximal atlas A^σ is induced by $\{\langle o_\sigma, \varphi_\sigma \rangle \mid o \in \mathscr{B}^4\}$.*

Proof. Observe first that $A^\sigma \subseteq A$, and that the inclusion is strict if Σ has more than one element; otherwise the two d-manifolds are identical. It is also immediate to see that h_σ is open and connected in the manifold topology induced on B by A (see Exercise 9.5). Suppose thus that $A^\sigma \subsetneq A$, and assume for reductio that $\langle h_\sigma, A^\sigma \rangle$ is not maximal; that is, there is an set $g \subseteq B$ endowed with topology induced by atlas A, so that g is an embedded sub-manifold in the manifold on A. Assume further that the topology on g is Hausdorff and connected, and that $h_\sigma \subsetneq g$. By definition (Lee, 2012, p. 99) g has no boundary in the sub-manifold topology. One immediately notes that in that topology h_σ is open. Thus, since the topology on g is connected, the boundary of $d =_{\mathrm{df}} g \setminus h_\sigma$, ∂d, is non-empty. Consider now some $e \in \partial d$. Then for any b-ball $o_\beta \subseteq g$, if $e \in o_\beta$, then $o_\beta \cap h_\sigma \neq \emptyset$. Since $e \notin h_\sigma$, it must be that $e = [\alpha x]$ for some $\alpha \in \Sigma$, $\alpha \neq \sigma$. Hence there must be some $s \in S_{\sigma\alpha}$ such that $s \leqslant x$. Moreover, it is impossible that $s < x$, because then the distance between s and x would be non-zero, so there would be a b-ball o_α such that $[\alpha x] \in o_\alpha$ and $o_\alpha \cap h_\sigma = \emptyset$. Thus, $x = s$. Observe next that $[\sigma s], [\alpha s] \in g$. We then argue, like at the end of the proof of Fact 9.11, that $[\sigma s]$ and $[\alpha s]$ witness a failure of the Hausdorff property in the topology on g that is induced by atlas A on B. Thus, g is not a connected and Hausdorff submanifold of $\langle B, A \rangle$, which contradicts our reductio hypothesis. Combined with Fact 9.10, this implies that any B-history is a maximal subset of the base set B, the topology on which is connected and Hausdorff; the history with the endowed differential structure is thus a maximal Hausdorff sub-manifold embedded in a generalized d-manifold on set B. □

Our next Fact concerns a relation between the diamond topology (see Def. 4.14) and the topology induced by a differential manifold. In Section 4.4.1 we advertised the diamond topology as a natural topology for BST, and wrote that one argument for naturalness is that this topology, if appropriately restricted, coincides with the standard open-ball topology on $\mathbb{R}^n$. The fact below says that this is indeed so: the manifold topology $\mathscr{T}^A$ induced by atlas A on history h_σ and the diamond topology $\mathscr{T}_{h_\sigma}$ on h_σ are identical. It follows that the diamond topology on a B-history is homeomorphic to the open ball topology on $\mathbb{R}^4$.

Fact 9.13. *Let h_σ be a B-history in $\langle B, <_R \rangle$ that is the BST_{NF} structure determined by an MBS $\langle \Sigma, F, P \rangle$ and $\langle h_\sigma, A \rangle$ be a Hausdorff d-manifold on h_σ. Then the manifold topology $\mathscr{T}^A$ induced by atlas A on h_σ and the diamond topology $\mathscr{T}_{h_\sigma}$ on h_σ are identical.*

Proof. Clearly, the two topologies $\mathscr{T}^A$ and $\mathscr{T}_{h_\sigma}$ have the same base set, h_σ. We need to see whether $Z \in \mathscr{T}^A$ iff $Z \in \mathscr{T}_{h_\sigma}$. To argue in the left to right direction, let $[\sigma x] \in Z \in \mathscr{T}^A$, so for some b-ball u_o: $[\sigma x] \in u_o \subseteq Z$, where o is an open ball in $\mathbb{R}^4$. Consider then the set $MC_{\langle \mathbb{R}^4, <_M \rangle}(x)$ of maximal chains in $\langle \mathbb{R}^4, <_M \rangle$ that contain x. By properties of real numbers, for every $t \in MC_{\langle \mathbb{R}^4, <_M \rangle}(x)$ there are $x_1, x_2 \in t \cap o$ such that $x_1 < x < x_2$, so the diamond $\tilde{D}_{x_1,x_2} \subseteq o$. As every diamond $\tilde{D}_{x_1,x_2}$ in $\langle \mathbb{R}^4, <_M \rangle$ determines a unique diamond $D_{[\sigma x_1],[\sigma x_2]}$ in $\langle h_\sigma, <_R \rangle$, we get that for every $t \in MC_{\langle h_\sigma, <_R \rangle}([\sigma x])$ there are $[\sigma x_1], [\sigma x_2] \in t$ such that $[\sigma x_1] < [\sigma x] < [\sigma x_2]$ and the diamond $D_{[\sigma x_1],[\sigma x_2]} \subseteq u_o \subseteq Z$, which proves $Z \in \mathscr{T}_{h_\sigma}$. In the opposite direction, $Z \in \mathscr{T}_{h_\sigma}$ means that for every $t \in MC_{\langle h_\sigma, <_R \rangle}([\sigma x])$ there are $[\sigma x_1], [\sigma x_2] \in t$ such that $[\sigma x_1] < [\sigma x] < [\sigma x_2]$ and the diamond $D_{[\sigma x_1],[\sigma x]} \subseteq Z$. This implies in particular that for the time-like chain $t^* \in MC_{\langle h_\sigma, <_R \rangle}$ there are $[\sigma x_1^*], [\sigma x_2^*] \in t^*$ such that $[\sigma x_1^*] < [\sigma x] < [\sigma x_2^*]$ and the diamond $D_{[\sigma x_1^*],[\sigma x^*]} \subseteq Z$. Diamond $D_{[\sigma x_1^*],[\sigma x_2^*]}$ determines diamond $\tilde{D}_{x_1^*,x_2^*}$ in $\langle \mathbb{R}^4, <_M \rangle$; thanks to time-likeness of t^*, there is an open ball $o \subseteq \tilde{D}_{x_1^*,x_2^*}$, with $x \in o$. Accordingly there is an associated b-ball u_o such that $u_o \subseteq D_{[\sigma x_1^*],[\sigma x_2^*]} \subseteq Z$, with $[\sigma x] \in u_o$. This proves $Z \in \mathscr{T}^A$. □

The very welcome message of this section is that the base set of a structure derived from an MBS admits a generalized (typically non-Hausdorff) d-manifold structure, whereas each B-history in this structure comes out as an embedded sub-manifold of the above manifold that is maximal with respect to having a connected and Hausdorff topology. This result ties in neatly with the situation in General Relativity, in which space-times are standardly identified with Hausdorff d-manifolds, but larger (generalized) d-manifolds are constructible as well. Before we explore this affinity, we need to take a look at BST_{NF} structures generally: Do they admit a d-manifold structure?

9.2.3 Differential manifolds and BST_{NF}, generally

Do the above results concerning MBSs carry over to BST_{NF} structures generally? Given the frugality of the BST_{NF} postulates, one should not expect this to be the case. Indeed, it is already problematic whether the base set of a BST_{NF} structure admits a d-manifold structure. One might try to find constraints on BST_{NF} structures that would ensure the admittance of a d-manifold structure on their base sets. The general case is, however, again unwieldy. A more promising line of enquiry is to see whether "nice" topological properties of histories in a BST_{NF} structure carry over to the structure itself. To some extent this is satisfied, as Theorem 4.1 testifies. It says that, for any BST_{NF} structure $\mathscr{W} = \langle W, < \rangle$, if there is an $n \in \mathbb{N}$ such that every history $h \in \text{Hist}(\mathscr{W})$ admits a generalized topological manifold structure $\langle h, A_h \rangle$ with dimension n, then W admits a generalized topological manifold structure $\langle W, A_W \rangle$ with dimension n as well. In short, we have "local Euclidicity in, local Euclidicity out" for the topological manifold structure admitted by BST_{NF} histories. But General Relativity needs differential manifolds to model space-times, rather than more frugal topological manifolds, which do not come with a differential structure. So a natural question is whether there is a generalization of Theorem 4.1 to C^r generalized d-manifolds with $r > 0$. Unfortunately, by simply putting together d-manifold structures admitted by individual BST_{NF} histories, we will not produce a (generalized) d-manifold that has as its base set the resulting BST_{NF} structure. This is a consequence of the fact that charts from atlases belonging to *different* d-manifolds need not properly combine in the way required by Def. 9.9. To see this, consider the BST_{NF} structure outlined at the bottom of Figure 3.1, with two histories $h_i = \{[\langle x, i \rangle] \mid x \in \mathbb{R}\}$, where $i = 1, 2$, with topologies given by the open intervals $u_{x_1,x_2} = \{[\langle x, 1 \rangle] \mid x_1 < x < x_2\}$ and $v_{x_1,x_2} = \{[\langle x, 2 \rangle] \mid x_1 < x < x_2\}$ $(x_1, x_2 \in \mathbb{R}, x_1 < x_2)$. Let the atlas of the manifold based on h_1 be induced by $\{\langle u_{x_1,x_2}, \varphi_{x_1,x_2} \rangle \mid x_1, x_2 \in \mathbb{R}, x_1 < x_2\}$, where $\varphi_{x_1,x_2} : u_{x_1,x_2} \mapsto \mathbb{R}$ such that $\varphi([\langle x, 1 \rangle]) = x$. Similarly, let the atlas of the manifold based on h_2 be induced by $\{\langle v_{x_1,x_2}, \psi_{x_1,x_2} \rangle \mid x_1, x_2 \in \mathbb{R}, x_1 < x_2\}$, where $\psi_{x_1,x_2} : v_{x_1,x_2} \mapsto \mathbb{R}$ such that $\varphi([\langle x, 2 \rangle]) = x^{1/3}$. Now, the charts from the atlas of the topology on h_1 properly combine, as do the charts from the atlas of the topology on h_1. In each case, the composite functions, $\varphi_{y_1,y_2} \circ \varphi^{-1}_{x_1,x_2}$ and $\psi_{y_1,y_2} \circ \psi^{-1}_{x_1,x_2}$ (if defined), are the identity functions on their domains, which are differentiable to an arbitrary high order. Thus, each d-manifold, on h_1 and on h_2, is C^∞. However, two charts, each from the atlas of a different topology, might fail to properly combine. Pick any two charts u_{x_1,x_2} and v_{x_1,x_2} such that $x_1 < 0 < x_2$. Then $\psi_{x_1,x_2} \circ \varphi^{-1}_{x_1,x_2} : (x_1, x_2) \mapsto \mathbb{R}$ is given by $(\psi_{x_1,x_2} \circ \varphi^{-1}_{x_1,x_2})(x) = x^{1/3}$, which is C^0 but no C^1 on (x_1, x_2). And such charts cannot be removed from the atlases, as they are needed for the distinct elements, $[\langle 0, 1 \rangle]$ and $[\langle 0, 2 \rangle]$, which form a choice set in the structure considered. Thus, although histories admit C^∞ d-manifold, the atlases of these manifolds do not produce a C^r atlas on the base set of the whole structure for any $r > 0$.

This illustration suggests a different approach, however, as one might choose "nicer" functions on the domains of charts in the topology on h_2. Generally, one might try to take advantage of Theorem 4.1. The idea is to begin with a collection of BST_{NF} histories, each of which admits a

C^r d-manifold of the same dimension. Since a C^r d-manifold counts as a topological manifold as well, by the theorem the base set of the whole BST_{NF} structure admits a *topological* manifold. To recall, the charts in the atlas of a topological manifold are only required to be C^0. One might thus hope that it is possible to smoothen these charts somewhat such as to make them C^r, for $r > 0$.[8] The question is thus whether a differential structure can be assigned to any topological manifold. The answer is that this is doable for many topological manifolds, but not for all. There are topological manifolds that do not admit a C^r atlas for any $r > 0$, as was proved by Kervaire (1960). Thus, to sum up, we do not know any (and there might be no) general method of obtaining a d-manifold admissible by a BST_{NF} structure from d-manifolds admissible by histories of that structure, but the smoothing approach can work in many cases.

The fact that the elegant results concerning MBSs and d-manifolds do not carry over to all BST_{NF} structures should not be seen as disconcerting, however. This is to be expected given the frugality of the BST_{NF} postulates. The important question is whether those d-manifolds and generalized (non-Hausdorff) manifolds that occur in the GR literature could be read as d-manifolds and generalized d-manifolds admissible by BST_{NF} histories and BST_{NF} structures, respectively. The results concerning MBSs and d-manifolds suggest a picture in which each space-time is a maximal connected Hausdorff d-submanifold within a generalized d-manifold, representing all space-times, such that any two of them share some initial segment.

Interestingly, this picture is similar to some situations in GR in which there is a failure of the initial value problem, to be discussed at length in Section 9.3.1. In these cases one has maximal Hausdorff manifolds, as well as larger non-Hausdorff manifolds. In the physics literature, the former are standardly interpreted as GR space-times, but the latter are problematic; i.e., their physical interpretation is unclear. Our suggestion is to read these non-Hausdorff d-manifolds very much like non-Hausdorff d-manifolds in MBS contexts, that is, as the representations of multiple alternative spatiotemporal histories, with each such history given by a maximal Hausdorff d-submanifold. Before we turn to this topic, however, we need to recall some notions of General Relativity.

[8] More precisely, it is the compositions of functions from different charts, like $\psi \circ \varphi^{-1}$, that are required to be C^r, if their domains are non-empty. Note also that the crucial step is from topological manifolds to C^1 d-manifolds, as a C^1 manifold can always be transformed into a C^r manifold, for any $r > 1$; see Hirsch (1976, Theorem 2.9).

9.2.4 Differential manifolds in GR

We now review some concepts and terminology needed for GR.[9] Standardly, an n-dimensional GR space-time is identified with a pair $\langle M, g_{ab}\rangle$, where M is a connected n-dimensional C^∞ Hausdorff d-manifold (without boundary) and g is a smooth, non-degenerate, pseudo-Riemannian metric of Lorentz signature $(-,+,\ldots,+)$ defined on M (aka a Lorentzian metric). To be explicit about the atlas, A_W, we sometimes write a GR space-time as $\langle W, A_W, g\rangle$, where $M = \langle W, A_W\rangle$ is a connected n-dimensional C^∞ Hausdorff d-manifold (without boundary). Two space-times $\langle M, g\rangle$ and $\langle M', g'\rangle$ are defined to be *isometric* if there is a diffeomorphism (smooth bijection) $\varphi : M \to M'$ such that the induced pull-back function φ^* satisfies $\varphi^*(g') = g$. A space-time $\langle M', g'\rangle$ is an *extension* of the space-time $\langle M, g\rangle$ if there exists an embedding $\Lambda : M \mapsto M'$ (i.e., Λ is a diffeomorphism onto its image) and $\Lambda^*(g' \mid_{\Lambda(M)}) = g$ and $\Lambda(M) \neq M'$. A space-time is *maximal* iff it has no extension.

With each point $p \in M$ there is associated a vector space, called the *tangent space*, M_p, on which g induces a cone structure, so that each vector $\xi^a \in M_p$ is either *timelike*, or *null*, or *spacelike*, depending on whether $\xi^a \xi^b g_{ab}$ is positive, zero, or negative, respectively. Here the superscripts a and b are abstract indexes, indicating that the object is a covariant vector. Time-orientable space-times permit a distinction between future and past lobes of light-cones; technically, a time-orientable space-time has a continuous timelike vector field on "its" manifold.

A continuous curve $\gamma : I \to M$ (where I is an interval of $\mathbb{R}$) is timelike (resp., spacelike, or null) iff its tangent vector ξ^a at each point in $\gamma[I]$ is timelike (resp., spacelike, or null). A curve is *causal* iff its tangent vector at each point is either null or timelike. A curve is *inextendible* iff it has no endpoints. A *geodesic* in a space-time $\langle M, g_{ab}\rangle$ is a curve $\gamma : I \to M$ that satisfies, for every vector $\xi^a \in M_p$ $(p \in \gamma[I])$ tangent to the curve, the geodesics equation: $\xi^a \nabla_a \xi^b = 0$, where ∇_a is the (unique) derivative operator compatible with g_{ab}. For any set $S \subseteq M$, the *domain of dependence* of S, written $D(S)$, is the set of points $p \in M$ such that every inextendible causal curve through p intersects S. S is an achronal subset of M iff no two points in S can be joined by a timelike curve. A Cauchy surface in $\langle M, g_{ab}\rangle$ is a smooth and achronal spacelike hypersurface such that $D(S) = M$.

[9] Explanations of the mathematics of GR can be found in mathematically oriented books on GR, such as Malament (2012, Chs. 1–2).

9.3 GR space-times

General Relativity is currently the best theory of space-time and matter. There are a variety of GR models, aka GR space-times, making the BST analysis of GR space-times rather complex. Yet, a fairly large class of GR-space-times, viz., time-orientable space-times that do not end abruptly and that do not contain closed causal curves, is easily amenable to a BST analysis (although, to be fair, some metric information provided by a GR space-time is not present in a bare BST structure). We put down this observation as a fact:

Fact 9.14. *Let $\langle M, g\rangle$ be a GR space-time that is (1) time-orientable, (2) without closed causal curves, and (3) subject to the condition that for any $x, y \in M$ there is $z \in M$ that is reachable from x and from y by future-directed causal curves. Then $\langle M, g\rangle$ induces a one-history BST_{NF} structure $\mathscr{W} = \langle M, <\rangle$, where for $e_1, e_2 \in M$, $e_1 \neq e_2$, we set $e_1 < e_2$ iff there is a continuous future-directed causal curve from e_1 to e_2. Being a one-history structure, that structure is also a BST_{92} structure.*

Proof. (Sketch) Asymmetry of $<$ comes from time-orientability (1) and the absence of closed causal curves (2). Transitivity of $\prec$ results from the composition of causal curves (a method of getting a continuous causal curve from piece-meal continuous causal curve is needed (see, e.g., Chruściel 2011, sec. 2). M is directed by (3), so it is a single BST history. The postulate of history-relative suprema thus simplifies to the condition of suprema simpliciter for upper bounded chains in $\langle M, <\rangle$. Both the infima postulate and the suprema postulate follow from the definition of causal curves. PCP_{92}, PCP_{NF}, and Weiner's postulate are vacuously satisfied since the structure has only one history. □

This result, of course, only constitutes a first step, as it does not touch upon the interesting question of whether BST can be used to model indeterminism occurring in a general relativistic world. Some GR models have indeterministic features. One indication of indeterminism, which is interesting from a BST perspective, arises in the context of the initial value problem (IVP). A failure of the IVP means, generally, that a theory's evolution equations allow for multiple global solutions that coincide over some region. If the region of coincidence is "nice", one may read a failure of IVP as the existence of multiple evolutions of a given system that develop from that common region. The picture of branching histories then naturally springs to mind. Accordingly, in our attempt to relate BST to GR, we begin with an overview of the initial value problem in GR: if there is a motivation for a non-trivial BST structure consisting of GR space-times, this motivation should come from a failure of the IVP in General Relativity.

In the next stage we will be concerned with two interconnected problems in our attempts to analyze GR space-times from a BST perspective. The first is that of GR space-times with closed causal curves. Given BST's reliance on an asymmetric ordering, such GR space-times cannot be modeled as BST structures, as long as the BST ordering is defined via causal curves. We will indicate how to resolve this problem, arguing that a BST-style theory with a *locally* asymmetric ordering (but not necessarily a globally asymmetric ordering on the whole base set) is available and is in line with the topological features of GR space-times—see Section 9.3.6. Conceptually more demanding is the second problem, which concerns the kind of indeterminism that might arise from failures of the IVP in GR. Recall that BST captures local indeterminism, the key idea being that a local entity, like a point-like event, has alternative possible futures. It might happen (the issue is not fully clear) that a failure of the IVP does not deliver local indeterminism: it could produce alternative developments of a region of GR space-time without there being an indeterministic trajectory of any entity (i.e., a trajectory that splits), with each continuation leading into a different possible development. In other words, it might be that each failure of the IVP produces a case of global indeterminism without there being local indeterminism. The big question is, of course, whether that odd combination reflects some incompleteness on the part GR, like its failure to accommodate quantum phenomena, or, alternatively, if it discloses a feature of our physical world.

The two problems, the existence of closed causal curves and the odd kind of indeterminism coming from the violation of the IVP, are related: the known cases of GR space-times that harbor indeterminism contain such curves or other causal anomalies. Furthermore, Theorem 2 of Clarke (1976) implies that a non-Hausdorff manifold with a Lorentzian metric has bifurcating curves (of the second kind – see below) or violates strong causality.[10] We would welcome bifurcating causal curves as they are very much needed

[10] The violation of strong causality means that, although a causal curve does not intersect itself, it comes arbitrarily close to intersecting itself.

in order to read a non-Hausdorff manifold as a BST structure with multiple histories (see Section 9.3.4). However, the data that we review below suggests that there might be no bifurcating causal curves in non-Hausdorff manifolds which are naturally constructible in GR (see Section 9.3.2). By Clarke's theorem then, such manifolds harbor violations of strong causality. Thus, in the context of the violation of IVP, causal anomalies might be inevitable. Our approach gives reasons for being optimistic with respect to accommodating closed causal curves in BST, and a violation of strong causality without closed causal curves is not problematic from a BST perspective. Therefore, in what follows we focus on what we believe to be more problematic for BST: the global variety of indeterminism in GR coming from a failure of the IVP. Accordingly, we now provide a short overview of the IVP in General Relativity.

9.3.1 The initial value problem in GR

The question of determinism presupposes the notion of a system evolving in time. This latter notion is not always well-defined in general relativity, as a GR space-time need not come with a distinguished time coordinate. Yet, a somewhat similar issue can be considered in GR: suppose we are given a 3-dimensional space Σ with possibly some data on it. (Technically, this should be a manifold with a metric "appropriate for space"; that is, i.e., a Riemannian metric.) The question now is: Can this space be uniquely extended to a 4-dimensional space-time–that is, to a manifold with a Lorentzian metric that satisfies the properties required from a manifold representing a GR space-time (listed in Chapter 9.2.4)–and in which the Einstein field equations (EFE) hold? The answer depends crucially on the kind of data assumed on the space and on the properties the sought-for space-time is supposed to have. As for the latter, one relevant factor is the existence of a matter field, and (if it is assumed to exist) the kind of model for the matter field; another issue is the value of the cosmological constant in the EFE. However, given the extension problem we will consider, a simple case is enough for our purposes. We focus on space-times with a vanishing Ricci tensor, the so-called vacuum space-times, and we consider the EFE without the cosmological constant. A satisfactory data set for this case consists of a Riemannian metric $\tilde{g}$ and a symmetric covariant 2-tensor $\tilde{k}$ that represents incremental changes of the metric in the direction normal to Σ. In this case the initial value problem amounts to constructing a 4-dimensional manifold M with a Lorentzian metric g and an embedding $i : \Sigma \to M$ such that if k is the second fundamental form on $i(\Sigma) \subseteq M$, then $i^*(g) = \tilde{g}$ and $i^*(k) = \tilde{k}$, where i^* is the pull-back function induced by the embedding i. Further, there is a set of equations relating $\tilde{g}$ and $\tilde{k}$, known as (vacuum) constraint equations, which guarantee the satisfaction of the EFE in the sought-for space-time. A space Σ with tensors $\tilde{g}$ and $\tilde{k}$ that satisfy the (vacuum) constraint equations is said to form a (vacuum) initial data set $\langle \Sigma, \tilde{g}, \tilde{k} \rangle$.

A result that is highly relevant to the initial value problem in the vacuum case was obtained by Choquet-Bruhat and Geroch (1969) in the context of globally hyperbolic space-times. Such space-times have particularly nice causal properties. To recall the definition, $\langle M, g_{ab} \rangle$ is said to be *globally hyperbolic* iff there is an achronal subset $S \subseteq M$ whose domain of dependence is the whole space-time (see Wald, 1984, Ch. 8). One consequence of this definition is that a globally hyperbolic space-time can be foliated by Cauchy surfaces (although the foliation is non-unique). Choquet-Bruhat and Geroch restrict their attention to globally hyperbolic space-times $\langle M, g \rangle$ that (1) are vacuum solutions to the EFE and which can be developed from a given vacuum initial data set such that (2) the image of the space Σ from that data set under the development embedding is a Cauchy surface in $\langle M, g \rangle$. A space-time satisfying these conditions is called a "vacuum Cauchy development" (VCD) of the initial data set. Note that condition (2) implies that a VCD is a globally hyperbolic space-time. The theorem proved by Choquet-Bruhat and Geroch says:

Theorem 9.2. *Let $\langle \Sigma, \tilde{g}, \tilde{k} \rangle$ be an initial vacuum data set. Then there is a unique, up to isometry, maximal VCD $\langle M, g \rangle$ of $\langle \Sigma, \tilde{g}, \tilde{k} \rangle$.*

The phrase "unique, up to isometry, maximal VCD" means that if there is another maximal VCD $\langle M', g' \rangle$ of the same initial data set, then there is a time-orientation preserving isometry $\varphi : M \to M'$. Thus, taking isometry to amount to the physical identity of vacuum space-times of GR (which is a typical move), the result ensures the uniqueness of maximal *globally hyperbolic* space-times compatible with vacuum initial data sets.

It is important to note that the theorem concerns globally hyperbolic developments only: it puts no restrictions on *other* developments of an initial vacuum data set. This raises the question of whether a maximal globally hyperbolic development of an initial data set can be further extended (where, of course, the resulting extension cannot be globally hyperbolic). Here, controversial questions of the physicality of such extensions become important. Some parties to the debate exclude non-globally hyperbolic space-times, on the basis that they involve causal anomalies, like closed causal curves, and these might be unphysical.[11] In recent research on the initial value problem in GR, some authors hold that non-globally hyperbolic developments of initial data sets are *rare*, in some measure-theoretical sense, with respect to a measure defined on the space of relevant solutions to the EFE. A view that is gaining ground is that "for generic initial data to Einstein's equations, the maximal globally hyperbolic development has no extension" (Ringström, 2009, p. 188). Without entering into the voluminous debate here, we nevertheless investigate here non-globally hyperbolic solutions, even if they are non-generic or rare.

To sum up, the Choquet-Bruhat and Geroch theorem has the consequence that evidence for the indeterminism of GR (if there is such evidence) in the vacuum case must consist of multiple non-isometric *extensions* of a maximal globally hyperbolic vacuum space-time. The question is, therefore, whether some maximal globally hyperbolic space-times (satisfying the EFE) have multiple non-isometric extensions (satisfying the EFE). The next section provides a positive answer to this question.

9.3.2 An example of the failure of the IVP: Non-isometric extensions of Taub space-time

We will describe below the construction of multiple non-isometric extensions of Taub space-time. Our discussion is based on a paper by Chruściel and Isenberg (1993), which also investigates a more realistic class of space-times, polarized Gowdy space-times, that also have multiple non-isometric extensions. But since these are mathematically more demanding, we describe here the simpler case of extensions of Taub space-time.

Taub space-time is a vacuum solution to the EFE. The manifold is $M = (t_-, t_+) \times S^3$, and the metric g is given by

$$ds^2 = -U^{-1}dt^2 + (2l)^2 U(d\psi + \cos\Theta d\varphi)^2 + (t^2 + l^2)(d\Theta^2 + \sin^2\Theta d\varphi^2),$$

[11] The considerable controversy over how to interpret Theorem 9.2 is related to the idea of cosmic censorship due to Penrose (1969), and especially to its later formulation in terms of the so-called Strong Cosmic Censorship Conjecture. That conjecture says (very roughly) that space-times that are not globally hyperbolic are unphysical.

where m and l are real positive constants, Θ, φ and ψ are Euler coordinates on the 3-sphere S^3, and

$$U(t) = \frac{(t_- - t)(t - t_+)}{l^2 + t^2}, \quad \text{where} \quad t_\pm = m \pm (m^2 + l^2)^{1/2}.$$

Note that $U(t_\pm) = 0$, and hence the metric is not defined at $t_\pm$. Taub space-time is globally hyperbolic, and maximally so, the Cauchy surfaces being identified by the condition $t = const$ for $t \in (t_-, t_+)$. As Newman, Tamburino and Unti (1963) showed, by using appropriate coordinate transformations $\langle M, g\rangle$ can be extended above t_+, the result being two non-hyperbolic space-times $\langle M^{\uparrow +}, g^{\uparrow +}\rangle$ and $\langle M^{\uparrow -}, g^{\uparrow -}\rangle$, known as Taub-NUT space-times. In a similar vein, Taub space-time can be extended below t_- into two non-hyperbolic space-times $\langle M^{\downarrow +}, g^{\downarrow +}\rangle$ and $\langle M^{\downarrow -}, g^{\downarrow -}\rangle$. Each of $\langle M^{\uparrow +}, g^{\uparrow +}\rangle$, $\langle M^{\uparrow -}, g^{\uparrow -}\rangle$, $\langle M^{\downarrow +}, g^{\downarrow +}\rangle$, and $\langle M^{\downarrow -}, g^{\downarrow -}\rangle$ satisfies the EFE and contains closed causal curves in the region new with respect to M.[12] As shown by Chruściel and Isenberg (1993), the pair $M^{\uparrow +}, M^{\uparrow -}$ and the pair $M^{\downarrow +}, M^{\downarrow -}$ are isometric.

To produce non-isometric extensions of Taub space-time, we need to glue together an upward extension together with a downward extension of Taub space-time. "Gluing" means, in mathematical parlance, finding an equivalence relation $\equiv$ on the union of two manifolds, say $M^{\downarrow +} \cup M^{\uparrow -}$, and then taking the set of equivalence classes with respect to this equivalence relation. The result is the quotient structure $(M^{\downarrow +} \cup M^{\uparrow -})/\equiv$.

Consider now four results of the gluing (for the equivalence relation used, consult Chruściel and Isenberg, 1993, p. 1619):

$$M^{xy} = (M^{\downarrow x} \cup M^{\uparrow y})/\equiv, \text{ where } x, y \in \{-, +\},$$

each result being associated to the metric g^{xy}, defined in terms of $g^{\uparrow x}$ and $g^{\downarrow y}$. Each $\langle M^{xy}, g^{xy}\rangle$ is a non-hyperbolic extension of Taub space-time $\langle M, g\rangle$ and satisfies the EFE. As for isometries, there are the following results (Chruściel and Isenberg, 1993, Theorem 3.1):

1. $\langle M^{+-}, g^{+-}\rangle$ is isometric to $\langle M^{-+}, g^{-+}\rangle$.
2. $\langle M^{++}, g^{++}\rangle$ is isometric to $\langle M^{--}, g^{--}\rangle$;

[12] See Misner and Taub (1969). The existence of closed causal curves raises the worry of the applicability of BST to such space-times; we address this concern in Section 9.3.6.

3. yet, $\langle M^{--}, g^{--}\rangle$ is not isometric to $\langle M^{-+}, g^{-+}\rangle$, and
4. $\langle M^{++}, g^{++}\rangle$ is not isometric to $\langle M^{+-}, g^{+-}\rangle$.

Each pair of the non-isometric extensions of Taub space-time above provide evidence for indeterminism in the sense of Butterfield's (1989) definition of determinism, which is tailored to applications to GR. It says:

Definition 9.10 (Butterfield's definition of determinism)**.** A theory with models $\langle M, O_i\rangle$ is **S**-deterministic, where **S** is a kind of region that occurs in manifolds of the kind occurring in the models, iff:
given any two models $\langle M, O_i\rangle$ and $\langle M', O'_i\rangle$ containing regions S, S' of kind **S** respectively, and any diffeomorphism α from S onto S':
if $\alpha^*(O_i) = O'_i$ on $\alpha(S) = S'$, then there is an isomorphism β from M onto M' that sends S to S', i.e., $\beta^*(O_i) = O'_i$ throughout M' and $\beta(S) = S'$.

Here the O_i stand for geometric object fields that are either definable in terms of a space-time's metric, or which characterize the matter field of the space-time. In our (vacuum) case the definition simplifies considerably, since in the absence of objects not definable in terms of the metric, the notion of isomorphy coincides with that of isometry, thus β can be an isometry and the condition on α^* above concerns only objects defined in terms of the metric.

To check that the definition yields the verdict of the indeterminism of GR, note that the space-time $\langle M^{++}, g^{++}\rangle$ contains the region $S^{++} = \Lambda^{++}[M]$, and the space-time $\langle M^{+-}, g^{+-}\rangle$ contains the region $S^{+-} = \Lambda^{+-}[M]$, where $\Lambda^{xy} : M \to M^{xy}$ is an embedding, which ensures that $\langle M^{xy}, g^{xy}\rangle$ is an extension of $\langle M, g\rangle$. For the diffeomorphism α we take $\alpha = \Lambda^{+-} \circ (\Lambda^{++})^{-1} : S^{++} \to S^{+-}$. Then the push-forward α^* induced by α satisfies $\alpha^*_{|S^{++}}(g^{++}) = g^{+-}$ (by the condition on embedding), and hence $\alpha^*(O_i) = O'_i$ for any object field defined in terms of the metric. On the other hand, however, $\langle M^{++}, g^{++}\rangle$ and $\langle M^{+-}, g^{+-}\rangle$ are not isometric, according to Chruściel and Isenberg's result quoted earlier. It follows that GR is indeterministic in the sense of Def. 9.10.

Physicists and philosophers alike have argued that non-isometric extensions of a globally hyperbolic space-time provide evidence for the indeterminism of GR.[13] Both from the perspective of Butterfield's definition

[13] To quote a philosopher's diagnosis, Belot (2011, p. 2876) says: "Instances in which globally hyperbolic solutions admit non-isometric extensions are instances of genuine indeterminism […]". For a similar assessment by a physicist, see Ringström (2009) or Costa et al. (2015).

of determinism and from the perspective of identifying determinism with unique solutions to a theory's evolution equations (EFE in GR), such cases *are* indeterministic. The crucial question for us is whether this kind of indeterminism can be captured by BST. Before we tackle this question head on, in the next section we discuss a "topological" characterization of indeterminism in GR. The gist of this characterization is to interpret a non-Hausdorff generalized d-manifold as a representation of a family of alternative space-times that extend a given spatio-temporal region.

9.3.3 Can non-Hausdorff manifolds in GR be interpreted modally?

Non-isometric extensions of a maximal globally hyperbolic space-time, like the extensions of Taub space-time just reviewed, are constructed to be Hausdorff d-manifolds, as they have to satisfy the conditions for a GR space-time. Can non-Hausdorff d-manifolds arise in the context of a failure of the IVP in General Relativity? The answer is yes.

Hawking and Ellis (1973, p. 173) exhibit extensions of so-called Misner space-time, which are Hausdorff d-manifolds that can be further extended to a d-manifold that is not Hausdorff. More precisely, there is a generalized (non-Hausdorff) d-manifold such that its maximal Hausdorff submanifolds are the mentioned extensions of Misner space-time. The same authors (p. 177) also discuss the above described non-isometric extensions of Taub space-time and show that they can be viewed as maximal Hausdorff d-submanifolds of some generalized (non-Hausdorff) d-manifold. More generally, Luc and Placek (2020) develop a pasting technique by means of which they prove some results concerning the relations between Hausdorff and non-Hausdorff d-manifolds (equipped with Lorentzian metrics). First, they show that non-isometric space-times (Hausdorff d-manifolds) that have some isometric regions can almost always be used to produce a generalized (non-Hausdorff) d-manifold with a Lorentzian metric. More precisely, in the collection of Hausdorff manifolds that is to be pasted any pair has to have isometric regions; and the "almost always" qualification comes from the fact that the technique yields topological non-Hausdorff manifolds rather than C^r non-Hausdorff manifolds. Thus, the final step in obtaining the sought-after d-manifold depends on whether the resulting topological manifolds admit a differentiable structure. As we already mentioned in

Chapter 9.2.3, there are (relatively rare) so-called Kervaire cases that do not admit such a structure. Second, Luc and Placek (2020) describe in detail the gluing together of non-isometric extensions of Taub space-time to form a generalized non-Hausdorff d-manifold. Finally, a result slightly generalizing Hájíček's (1971b) construction says that any non-Hausdorff d-manifold with a metric can be constructed by the gluing technique from a collection of Hausdorff d-manifolds with metrics (GR space-times). Thus, Kervaire cases aside, there is a general technique that permits one to take the non-isometric (Hausdorff) extensions of some globally hyperbolic space-time and paste them together into a generalized (non-Hausdorff) manifold with a metric. Maximal Hausdorff sub-manifolds of this generalized manifold will then be identifiable with the non-isometric extensions one started with.

This opens the door to a modal interpretation of non-Hausdorff d-manifolds, at least of the variety constructible in the context of a failure of the initial value problem by the mentioned technique. This interpretation sticks to the standard GR identification of space-times with Hausdorff d-manifolds (with some more constraints), and reads a non-Hausdorff manifold resulting from gluing together non-isometric extensions as representing alternative space-times, all of which develop from a certain common region, which is given by the maximal globally hyperbolic d-manifold. This interpretation gives physical meaning to a significant variety of non-Hausdorff d-manifolds occurring in GR. By the construction, a non-Hausdorff manifold of this variety witnesses indeterminism in Butterfield's sense, because it comprises non-isometric extensions of a space-time. Finally, the interpretation dissolves most objections leveled against non-Hausdorff manifolds in the GR literature; see Luc (2020). For more details on the modal interpretation of non-Hausdorff d-manifolds in GR, see Luc and Placek (2020).

The big question now is whether a generalized d-manifold constructed from non-isometric extensions of some space-time can be viewed as a BST structure. The next section suggests a negative answer to this question. This negative answer reflects a philosophically interesting point about indeterminism, which we explore in Section 9.3.5.

9.3.4 On bifurcating curves in GR

BST is a theory of local indeterminism. For such indeterminism, there is a small and well-defined locus at which one of a set of alternative possibilities is realized, whereas all the other alternatives stop being possible. The locus is idealized to a set of particular point events. Depending on whether we assume PCP_{92} or PCP_{NF}, there is either a maximal event in the intersection of any two histories, or there is a minimal event in the difference of any two histories. We can express the same point in terms of modal forks, by which we mean a pair of maximal chains that (1) share an initial segment, (2) the shared segment is fully within the overlap of some histories, $h_1 \cap h_2$, and (3) the separate segment of each chain is in a different difference of histories, $h_1 \setminus h_2$, or $h_2 \setminus h_1$. A modal fork is thus a pair of maximal chains that belong to two histories in a particular way. Since chains are obviously defined by the pre-causal ordering, read as "something can happen after something else", a maximal chain is readily interpreted as a potential trajectory of a point-like object. Thus, a modal fork can naturally be seen as representing initially coinciding alternative possible evolutions for some point-like object. Now, some modal forks have maximal elements in the shared segments—we call them *modal forks of the first kind*. Other modal forks have no maximal elements in the shared segments (they rather have minimal elements in each separate arm)—we call them *modal forks of the second kind*. Clearly, a BST_{92} structure with more than one history contains at least one modal fork of the first kind, whereas in BST_{NF} structures, all modal forks are of the second kind (cf. Fact 3.21).

To recall our topological discussion of Section 9.2.3, BST_{92} structures with multiples histories are non-starters for yielding generalized d-manifolds, whereas BST_{NF} structures stand a good chance. Indeed, we have already seen cases in which the base sets of BST_{NF} structures admit a generalized d-manifold structure. Thus, as BST_{92} is out of the game, and as BST_{NF} allows for modal forks of the second kind only, the question is whether generalized d-manifolds in GR admit bifurcating trajectories of point-like entities that are topologically like modal forks of the second kind. The required notion, introduced in the GR literature by Hájíček (1971a), is that of bifurcating curves of the second kind, defined as follows:

Definition 9.11. A *bifurcating curve of the second kind* on a C^r generalized d-manifold M is a pair $\langle C_1, C_2 \rangle$ of C^r-continuous curves $C_1 : [0,1] \to M$, $C_2 : [0,1] \to M$ such that for some $k \in (0,1]$: $\forall x \in I\ [x < k \Leftrightarrow C_1(x) = C_2(x)]$.

Thus, in a bifurcating curve of the second kind there is no maximal element in I at which C_1 and C_2 agree. A bifurcating curve of the first kind is defined analogously (by the condition $\forall x \in I\ [x \leqslant k \Leftrightarrow C_1(x) = C_2(x)]$), and topologically it is like a modal fork of the first kind. Note that, in contrast to modal forks, bifurcating curves of GR as defined by Hájíček (1971a) do not have any overt modal aspect.

We know that there are non-Hausdorff d-manifolds with bifurcating curves of the second kind; for a simple example, pick the bifurcating real line at the bottom of Figure 3.1 (p. 44). Relying on the visual intuition from that example, one might think that given a failure of Hausdorffness, such bifurcating curves should be easily available. After all, there must be a pair of points witnessing non-Hausdorffness, and one might think that one could simply pick two curves that overlap everywhere below the pair, but then pass through different elements of the pair. However, concrete cases from GR argue against this intuition: There are no bifurcating geodesics (a special class of causal curves) of the second kind in the generalized (non-Hausdorff) d-manifold representing non-isometric extensions of Misner space-time. Neither do the generalized (non-Hausdorff) d-manifolds resulting from pasting together non-isometric extensions of Taub space-time admit bifurcating geodesics of the second kind (Hawking and Ellis, 1973).[14]

Thus, the question of under what conditions non-Hausdorff d-manifolds admit bifurcating curves of the second kind is non-trivial. The question is answered, by providing necessary and sufficient conditions, by Hájíček's (1971a) theorem. To state it, we need one auxiliary notion, which is illustrated in Figure 9.1.

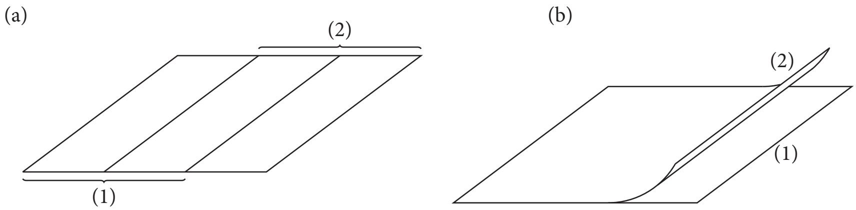

Figure 9.1 Gluing of two surfaces in two dimensions. (a) Not continuously extendible; (b) continuously extendible.

Definition 9.12 (Continuously extendible gluing). Let $\mathscr{W}_1 = \langle W_1, A_{W_1}, g_1 \rangle$ and $\mathscr{W}_2 = \langle W_2, A_{W_2}, g_2 \rangle$ be GR space-times. Then $\varphi : U_1 \mapsto U_2$, where $U_1 \subseteq W_1, U_2 \subseteq W_2$, is a *gluing function* if (1) U_1 is open and (2) φ is an isometry.

[14] To see why the visual intuition is wrong, note that these manifolds are constructed in a particular way, viz., by pasting via an equivalence relation. Margalef-Bentabol and Villaseñor (2014) show how non-Hausdorffness combines with the absence of bifurcating curves in Misner space-time.

Moreover, φ is said to be *continuously extendible* iff there exist U_1', U_2', φ' such that $U_1 \subsetneq U_1' \subseteq W_1, U_2 \subseteq U_2' \subseteq W_2$, $\varphi' : U_1' \mapsto U_2'$, φ' is continuous and $\varphi'|_{U_1} = \varphi$.

Clearly, the definition implies that U_2 is open as well, and that $U_2 \subsetneq U_2'$. Significantly, however, φ' above need not be an isometry between U_1' and U_2', thus it need not be a gluing function.

Hájíček's theorem says:

Theorem 9.3. *The necessary and sufficient condition for a d-manifold constructed by gluing together Hausdorff d-manifolds to admit bifurcating curves of the second kind is that the gluing be continuously extendible. (Hájíček, 1971a)*

According to the already mentioned results of Luc and Placek (2020), every non-Hausdorff d-manifold is constructible via the gluing of Hausdorff d-manifolds. Therefore, Hájíček's theorem produces a universal method to determine whether, for any non-Hausdorff d-manifold, it admits bifurcating curves of the second kind or not.

In plain English, Theorem 9.3 says that a non-Hausdorff d-manifold admits the sought-for bifurcating curves of the second kind exactly if among the component space-times that give rise to it, there are two particularly related space-times. These space-times are pasted together by a gluing function φ, with isometric domain and counter-domain U_1 and U_2, respectively. That gluing function φ can be extended to a function φ' on a larger domain, where φ' is to be continuous (but not necessarily an isometry). This does not mean that one can improve on the gluing φ; it merely says that φ, U_1, and U_2 are particularly related.

We have already encountered non-Hausdorff d-manifolds that result from gluing together Hausdorff d-manifolds. A case in point are non-Hausdorff d-manifolds on base sets of MBSs with multiple histories: see Fact 9.11. Another example is provided by the one-dimensional non-Hausdorff topological manifold M_b that results from gluing together two copies of the real line, sketched at the bottom of Figure 3.1 (p. 44).[15] Significantly, in each of the mentioned cases, it can be proved (see Exercise 9.7) that the gluing that is used is continuously extendible. Via Theorem 9.3, each of

[15] For an argument that this non-Hausdorff topological manifold gives rise to a C^∞ d-manifold, see Exercise 9.6. The Lorentzian metric g is trivially derived from the standard metric on the real line. The individual lines are Hausdorff d-manifolds, and in fact sub-manifolds of the bifurcating lines.

these non-Hausdorff d-manifold thus harbors bifurcating curves of the second kind. Do these MBS examples provide evidence that there are in GR non-Hausdorff d-manifolds, constructible from non-isometric GR space-times that arise from a failure of IVP for GR *and* that contain bifurcating curves? We do not think so. In the case of GR, one glues together non-isometric space-times. In our BST constructions, in contrast, the Hausdorff d-manifolds that form the basis of the resulting non-Hausdorff d-manifold are isometric. Even more importantly, in the case of a failure of the IVP in GR, the non-isometric solutions to the EFE are not globally hyperbolic—each extends a maximal globally hyperbolic space-time (guaranteed to exist by the Choquet-Bruhat and Geroch theorem). In contrast, the manifold on the real line, as well as Minkowski space-time which forms the base set of the mentioned MBS constructions, *are* globally hyperbolic. The fact that there are bifurcating curves of the second kind in MBS constructions does not imply that analogously bifurcating non-Hausdorff d-manifolds can be constructed from the multiples solutions to the EFE that are available from a failure of the IVP. And, to repeat, the non-Hausdorff d-manifolds resulting from pasting together extensions of Misner space-time[16] or non-isometric extensions of Taub space-time do not contain bifurcating geodesics of the second kind. It is a general and difficult open problem to prove whether there are non-Hausdorff d-manifolds with a Lorentzian metric that are constructible from non-isometric extensions of a maximal hyperbolic space-time by continuously extendible gluing.[17] Only such manifolds harbor bifurcating curves of the second kind. If they exist, one may hope to obtain a GR-based BST_{NF} structure with multiple histories that are individually identifiable with maximal Hausdorff d-submanifolds of the structure.

Yet, in the cases that we know, gluing together non-isometric extensions of a maximal hyperbolic space-time produces a non-Hausdorff d-manifold *without* bifurcating geodesics. Gluing the extensions of Taub space-time is a case in point. Interestingly, such non-isometric extensions witness indeterminism according to the definition that represents the received view in the philosophy of science (see Def. 9.10). We would like to capture this kind of indeterminism in BST as well. After all, BST theory promises to analyze *local* indeterminism, as occurring in spatio-temporal contexts. Yet BST cannot capture these GR cases, since it requires bifurcating curves.

[16] Extensions of Misner space-time are isometric, though, see Chruściel and Isenberg (1993).

[17] A qualification is needed to avoid, e.g., cunningly gluing two Taub space-times on regions that are a bit smaller than isometric regions and then presenting standard gluing as an extension.

It appears thus that the philosophy of science notion of determinism and indeterminism is different from that assumed by BST: the former is global, whereas the latter is local.[18] We turn to this issue in the next section.

9.3.5 Global and local determinism and indeterminism

There are two traditions of thinking about determinism, one centered on individual objects and the other centered on the entire universe. The former tradition focuses on relatively small objects or processes (that is, small if compared with the universe) and asks if these objects or processes could evolve differently than they actually did. The cloak story that Aristotle tells in *De Interpretatione* 19a clearly exemplifies this way of thinking: the issue is that the cloak might wear out, but that it could also be cut up first. If we deliberate over whether the cloak case argues in favor of indeterminism or not, the data we look at are limited in space and time. It is of course the cloak that matters, but some of its surroundings are relevant as well. However large these surroundings are compared to the cloak, we typically do not extend them to the entire universe. That is, we limit the data for the determinism question to a relatively small region of our spatio-temporal universe. This observation also applies to another great example we owe to Aristotle, that of tomorrow's sea battle. Although armadas of military vessels, together with sailors, their commanders, weather conditions, etc., occupy a relatively large area of the sea, this area is just a tiny spatio-temporal region of the entire universe. In some examples used in this tradition, namely those involving human agents and their decisions, the data is even further restricted—to a particular person, or more precisely, to some particular period in that person's life.

This local approach to determinism and future contingents is a characteristic feature of theories of agency. Putting philosophical disputes aside, it is this approach that is used in everyday contexts, including in science labs. A chemistry student investigating a catalytic reaction may wonder, seeing different outputs of seemingly identical processes in subsequent runs of her experiment: is the varying output due to the indeterministic nature of the process, or to some tiny differences in the reaction's initial conditions in

[18] A distinction between global and local notions of determinism has been argued for, e.g., by Belot (1995), Melia (1999), and Sattig (2015).

subsequent runs of the experiment? In an attempt to clarify the issue, she focuses on local matters of fact: is the catalyst, as well as other chemical substances used, sufficiently similar in subsequent runs of the reaction? Are the temperature, pressure, concentrations, and other relevant characteristics the same in all these runs? For these questions, the universe as a whole and its possible global evolutions play no role.

The second tradition centers on global notions like that of the universe, the world or its history, or a theory's models. That tradition is invariably linked to Laplace's vision of determinism. In Laplace's well known metaphor, a super intelligence is capable of "seeing" the entire past and future of the universe, thanks to its grasp of the instantaneous state of the universe and its knowledge of all the forces acting in the universe. After the removal of its epistemic overtones, signalled by words like "knowledge" or "seeing", the vision forms the backbone of the current received analysis of the determinism of *theories*. The basic intuition of this approach is that of "once similar, always similar", that is, a theory is deterministic iff whenever two models of the theory agree on initial segments, they agree as wholes. A theory's model, like a possible world, is a global notion. In GR, models are of course GR space-times, and similarity is identified with isometry. A general notion of determinism and indeterminism in the global style that is applicable to GR has been given via Def. 9.10.

The local and the global ways of thinking about determinism and indeterminism are in conflict. The combination of global determinism and local indeterminism is known from the literature (Belot, 1995; Sattig, 2015).[19] Non-isometric extensions of a maximal hyperbolic space-time point in the opposite direction, as these cases seem to combine global indeterminism with local determinism. We find this combination paradoxical, wondering how to conceive of a world that faces alternative possible evolutions, whereas each object in this world has a deterministically fixed evolution.

How then shall we define the *local* determinism and indeterminism of theories? Our point of departure is a given theory, together with some interpretation. An appeal to interpretation is needed, since in the next step we ask what individuals (that is, local objects that persist over time) are admissible,

[19] The context of their examples is Lewis's (1983) definition of determinism, applied to an ideally (axially) symmetric column with a critical weight on it, known as the 'buckling column'. Common sense and elasticity theory say that the direction of the column's buckling is not determined, whereas Lewis's analysis delivers the verdict "determinism". Lewis's analysis is global, whereas the common sense analysis is local.

and the response may vary from interpretation to interpretation of a given theory. We focus on systems admitted by a given theory, together with its interpretation. In GR, systems are space-times. We then ask if a given system of the theory is locally deterministic or locally indeterministic. To address this question, we need to consider smaller objects, the constituents of the system, that are admitted by the theory and its interpretation. There might be theories with systems without constituents—we leave such theories and systems aside. Considering a given system, we need to investigate whether it has constituents that evolve indeterministically. If a system contains at least one indeterministically evolving admissible object, we call the system *locally indeterministic*. If a theory has at least one locally indeterministic system, the theory is also called locally indeterministic accordingly.[20]

How can we learn that a small object admitted by a theory (plus its interpretation) evolves indeterministically? Here the theory's evolution equations come to the fore. Finding that the equations allow for multiple global solutions for a small object is a clear signal that the object might evolve indeterministically. Clearly, solutions are not to be outright identified with possible evolutions, as their difference might come from the mathematical features alone, that is, they might represent the same physical reality, with their difference residing in mathematical surplus structure only. In the parlance of physics, one says that such solutions exhibit gauge freedom. Further, since solutions to evolution equations describe states posited by a given theory, a further question is how these posited states are related to reality, as conceived by the theory (with its interpretation). Clearly, going local does not absolve one from the usual toil involved in attempts to infer determinism or indeterminism for particular interpreted theories.

Turning to GR, in order to get to grips with local determinism vs. indeterminism, we need the concept of the trajectory of a particle in that theory. In GR, the (potential) worldlines of free test point particles are standardly assumed to be geodesics (i.e., curves that satisfy the geodesic equation; see Section 9.2.4). Here "test" means that the particles do not alter the geometry of the space-time they move in. By the Einstein Field Equations, a particle's motion is governed by the space-time metric, but the metric is generally influenced by the particle's motion as well. In the case of *test* particles, however, their influence on the space-time metric is assumed to

[20] For a formal rendering of similar ideas, but assuming the simpler framework of Branching Time, see Müller and Placek (2018). For a different analysis of local indeterminism, see Sattig (2015).

be negligible; the test particle is thought of as moving before the background of an independent metric, following a geodesic.

Earlier, we distinguished bifurcating curves of the first vs. second kind; that distinction carries over to bifurcating geodesics. Now, quite generally, there are no bifurcating geodesics of the first kind in GR space-times, even if the usual requirement of Hausdorffness is dropped—this follows from the local uniqueness result for geodesics (see Chruściel, 2011, p. 6).[21] Thus, only bifurcating geodesics of the second kind are left on the stage. If the Hausdorff property is assumed, it is possible to glue together locally unique solutions to obtain a globally unique solution, and then there is no room for bifurcating geodesics of any kind. In summary, only bifurcating geodesics of the second kind stand a chance at all in GR, and they require non-Hausdorff d-manifolds.

The quest for bifurcating geodesics to support local indeterminism in GR might seem paradoxical, since they have been considered a bad thing by the physics community. To give some examples, Earman (2008, p. 200) asks: "how would such a particle [moving along a bifurcating geodesic] know which branch of a bifurcating geodesic to follow?" In a similar vein, Hawking and Ellis (1973, p. 174) opine that "a [bifurcating] behavior of an observer's world-line would be very uncomfortable", with "one branch going into one region and another branch going into another region". Hájíček (1971b, p. 79) observes that a system cannot have two solutions unless these solutions form a bifurcating curve, and concludes: "Therefore, in view of the classical causality conception coinciding with determinism it is sensible to rule out the bifurcate curves".

The underlying assumption of these objections is determinism. We agree that a bifurcating *actual* trajectory is barely understandable, echoing Hawking and Ellis's uneasiness of there being an observer present simultaneously in two regions. But this is not what a modal interpretation offers, as it takes separate branches of a bifurcating geodesic to be alternative *possible* trajectories of a test particle. Note also that given indeterminism, there are no answers to questions like "why did a particle go along a particular trajectory, which is but one of many alternative possible trajectories?" Taking indeterminism seriously means acknowledging that, sometimes, there are no such contrastive explanations for what happens.

[21] For the particular subtleties related to the uniqueness result, see Chruściel (1991, Appendix F).

We return to the case of non-isometric extensions of a maximal hyperbolic space-time, exemplified for instance by the extensions of maximal hyperbolic Taub space-time. As we saw above, the Taub example satisfies Def. 9.10 of global indeterminism (see Chapter 9.3.2). Is the example *locally* indeterministic as well? Suppose that there are some objects in Taub space-time. Could then at least one of these object face indeterministic evolutions, with each possible evolution going to a different non-isometric extension?[22]

To be more specific, what are the objects in Taub space-time and its non-isometric extensions? We focused on geodesics in the last section, which are standardly interpreted as trajectories of unaccelerated test particles. One might wonder what the trajectories of "real" particles in GR are. A dominant tradition, going back to Einstein and Grossmann (1913), assumes that particles of sufficiently small mass and size move along geodesics as well. That tradition is supported by topological theorems to the effect that, given certain idealizations are assumed, the particle moves along a geodesic. A theorem to this effect is proved by Ehlers and Geroch (2004).[23]

Consider, therefore, a Taub space-time inhabited only by photons which satisfy the required idealizations. In this case, all individual objects can be safely assumed to move along geodesics. Now consider a photon that moves along a lightlike geodesic in Taub space-time (such geodesics are called "null"). This space-time has two non-isometric extensions, $\langle M^{++}, g^{++} \rangle$ and $\langle M^{+-}, g^{+-} \rangle$. What happens to the photon as it leaves the initial region? That is, what does the photon's geodesic look like as the photon leaves Taub space-time and proceeds to a new region in one of the two extensions? By the discussions of Chapter 9.3.2, we know that there are no bifurcating geodesics. Thus, there are two classes of null geodesics in Taub space-time. Geodesics of the first class are completed in one extension, and geodesics of the other class are completed in the second extension (cf. Hawking and Ellis, 1973, pp. 170–178, and Chruściel and Isenberg, 1993, Lemma 3.2). The photon's evolution appears predestined: depending on which class the photon's geodesic belongs to, it will continue to one extension or to the other. Thus, no object living in Taub space-time faces an indeterministic evolution. The moral is that non-isometric extensions of Taub space-time inhabited by photons satisfying the mentioned idealizations are locally deterministic,

[22] More mathematical sophistication is needed to formulate this question precisely; see Chruściel and Isenberg (1993).

[23] Of course, the direct way to study a particle's trajectory is to find an exact solution to a (relevant) problem of motion of GR, yet there are only very few exact solutions of this kind.

yet they satisfy the definition of global indeterminism, Definition 9.10. We thus have a disturbing combination of global indeterminism and local determinism.

This result confronts us with a dilemma concerning how to further develop branching style-theories. The problem we face is metaphysical: How can we capture the notion of a possible history, which intuitively is a maximal possible course of events? Belnap (1992) opts for an order-theoretical criterion that is based on the "later witness" intuition (see Chapter 2.2): a history is a maximal directed subset of the base set W of a BST_{92} (or BST_{NF}) structure $\mathscr{W}$. That definition captures local indeterminism, but not the combination of global indeterminism plus local determinism that we discussed in this section. A remedy, suggested by the topological results concerning MBSs and discussed at the end of Chapter 9.1.2, might be to resort to topological foundations as an alternative to defining BST structures in terms of an ordering. The idea would be to identify a BST structure with a generalized manifold, and to define histories to be maximal Hausdorff submanifolds. This idea is further supported by the gluing technique applied to non-isometric extensions of maximal hyperbolic space-times of GR. With this remedy, a new BST-style theory could accommodate both varieties if indeterminism, local and global. Whether this remedy is attractive and worth pursuing depends on one's stance on the combination of global indeterminism and local determinism that we described: Does it reflect a feature of our world, or is it merely a mathematical gimmick coming from the theory of differential manifolds? We remain skeptical.

9.3.6 A note on closed causal curves and BST

As we have signaled, there is another relevant issue at the interface of GR and BST: some GR space-times admit closed causal curves, which means that the (strict) ordering determined by these curves is not asymmetric. This contradicts a basic a postulate of BST, viz., that $\mathscr{W} = \langle W, < \rangle$ is a (strict) partial order. Thus, in general, causal curves in GR allow one to define just an irreflexive and transitive relation $\prec$, called a strict pre-order, rather than the strict partial ordering that BST calls for.[24] In the following section we show that this ordering problem can be resolved by slightly generalizing

[24] A strict pre-order has a reflexive companion, $\preccurlyeq$, called a pre-order.

BST. For simplicity's sake, we limit our attention to structures without modal funny business as discussed in Chapter 5. We will construct modal structures (i.e., possibly with multiple histories) in which the ordering can be non-asymmetric. The construction of generalized BST, call it genBST, is motivated by the following theorem of GR.[25]

Theorem 9.4. *For every event p in an arbitrary GR space-time there exists an open set U with $p \in U$ such that for every $q, r \in U$ there is a unique geodesic connecting q and r, and staying entirely in U.*

Since geodesics fall into three classes, namely time-like, space-like, and null-like geodesics, the uniqueness of connectability means that in a time-orientable GR space-time the geodesics can be used to define a strict partial ordering $\prec$ on any U of the kind that the theorem above guarantees to exist: $q \prec r$ iff q and r are different events and q is connectible to r by a future directed time-like or null-like geodesic. On each U we can thus construct a BST structure (being topologically prudent, one might prefer BST_{NF}, preparing for cases with bifurcating geodesics). Since the U's (the "patches") cover the entire space-time, the genBST structure needs to somehow combine together all these little BST structures.

For our definitions, we recall the terminology introduced in Chapter 4.4.1: Let $\langle W, \prec \rangle$ be non-empty strict pre-order. Then

1. MC is the set of maximal chains in $\langle W, \prec \rangle$, and $MC(e) =_{\text{df}} \{t \in MC \mid e \in t\}$;
2. $t^{\prec x} = \{z \in t \mid z \prec x\}$, where $t \in MC(x)$ and $x \in W$ ($t^{\preccurlyeq x}$, $t^{\succ x}$, and $t^{\succcurlyeq x}$ are similarly defined).

Note that elements of MC, of $MC(e)$, as well as chains $t^{\prec x}$, $t^{\preccurlyeq x}$, $t^{\succ x}$, and $t^{\succcurlyeq x}$ can contain loops, i.e., there can be $y, z \in t$ for which both $y \prec z$ and $z \prec y$.

Definition 9.13 (genBST structure)**.** Let $W \neq \emptyset$, $\prec$ be a strict dense pre-order on W, and $\mathscr{O} \subseteq \mathscr{P}(W)$. A triple $\mathscr{W} = \langle W, \prec, \mathscr{O} \rangle$ is a *genBST structure*, iff for every $e \in W$ there is a patch $O_e \in \mathscr{O}$ around e such that:

1. $e \in O_e$;
2. $\langle O_e, \prec_{|O_e} \rangle$ is a nonempty dense strict partial order satisfying the following:

[25] See Wald (1984, Theorem 8.1.2).

(a) $\forall e' \in O_e\, \forall t \in MC(e')\, \exists x,y \in t \cap O_e\, [x \prec_{|O_e} e' \prec_{|O_e} y \wedge t^{\succ x} \cap t^{\prec y} \subseteq O_e]$;

(b) every chain in $\langle O_e, \prec_{|O_e} \rangle$ with a lower bound in O_e has an infimum in O_e;

(c) if a chain C in $\langle O_e, \prec_{|O_e} \rangle$ is upper bounded by $b \in O_e$, then $B_b =_{\mathrm{df}} \{x \in O_e \mid C \preccurlyeq_{|O_e} x \wedge x \preccurlyeq_{|O_e} b\}$ has a unique minimum,

(d) if $x, y \in O_e$ and $x \prec z \prec y$, then $z \in O_e$.

In a genBST structure $\langle W, \prec, \mathcal{O} \rangle$, W and $\prec$ form a non-empty dense strict pre-order, and $\mathcal{O}$ contains local patches, at least one for each $e \in W$. One may think of W and $\prec$ as a non-Hausdorff d-manifold with a Lorentzian metric and the ordering relation determined by geodesics on this manifold. A patch for e must satisfy some conditions: First, O_e contains e (it is a patch for e, after all) and the pre-order $\prec$ restricted to O_e is a dense strict *partial* order (asymmetric, thus containing no loops). Any maximal chain passing through e extends in O_e below and above e. The conditions (b) and (c) emulate the infima postulate and the suprema postulate of common BST structures. Note that (c) makes room for multiple history-relative suprema of a chain, provided that there are multiple upper bounds of the right kind. The next condition forbids O_e from having holes. A genBST structure may contain causal loops, but for any e there is a patch $O_e \in \mathcal{O}$, within which the order is partial, so the patch does not contain any causal loops.

In the usual way, for each O_e we define choice sets in O_e (see Def. 3.11). And we say that a subset $E \subseteq W$ is a choice set in $\mathscr{W}$ if it is a choice set in some $O_e \in \mathcal{O}$ for some $e \in W$. The existence of choice sets hinges on how the condition (2c) of Def. 9.13 is satisfied. If for every upper bounded chain in O_e there is just one minimum for all sets B_b defined in this condition, then there are no choice sets in O_e. This is exactly what happens in a generalized d-manifold (with a Lorentzian metric) with no bifurcating geodesics.

One may wonder how the global pre-order $\prec$ meshes with choice pairs. Somewhat worryingly, our postulates so far allow for distinct elements of a choice set to have an upper bound. Recall that we identified a choice set with something at which alternative possibilities become modally separated: while before the choice set all the relevant alternative possibilities are available, at each element of the choice set only one alternative possibility is available. Allowing for a common bound of distinct elements of a choice set thus sounds like permitting previously excluded alternative possibilities to be possible again. Before certain events, both alternative possibilities are open, then no matter how the world develops, only one of them is open and the other is excluded, but then again, we have both the alternative possibilities available. As this return of once excluded possibilities contradicts the basic intuition of no backward branching, we prohibit it by accepting the following postulate:

Postulate 9.1 (Separation)**.** *For every choice set* $\ddot{c} \subseteq W$*, and any* $x, x' \in \ddot{c}$*: if* $x \neq x'$*, then there is no* $z \in W$ *such that* $x \prec z \wedge x' \prec z$.

This postulate restricts Def. 9.13, as it implies that not every genBST structure is metaphysically sound. Significantly, it prohibits one kind of loop: those that pass through distinct elements of a choice set.[26] Note the interplay between local and global notions: if x and x' are separated by elements of a choice set $\ddot{c}$ in some patch O_e in the sense that $c \leqslant x$ and $c' \leqslant x'$ for $c, c' \in \ddot{c}$ with $c \neq c'$, then x and x' have no common upper bound, no matter how far we go along $\prec$, possibly outside O_e.

We next define consistency in order to anchor the notion of a history. Note that our definition excludes the possibility of modal funny business as discussed in Chapter 5.[27]

Definition 9.14 (Compatibility and consistency)**.** $e, e' \in W$ are compatible iff there is no choice set $E \subseteq W$ with distinct $x, x' \in E$ such that $x \preccurlyeq e$ and $x' \preccurlyeq e'$.
$A \subseteq W$ is consistent iff $\forall e, e' \in A:$ e and e' are compatible.

Provably, there are maximal consistent subsets of W in a genBST structure.[28] We identify them with histories:

Definition 9.15. Let $\mathscr{W} = \langle W, \prec, \mathscr{O} \rangle$ be a genBST structure. Histories in $\mathscr{W}$ are maximal consistent subsets of W.

It can be proved that histories in genBST are downward closed, and that genBST structures satisfy PCP_{NF}. However, histories in genBST are not necessarily directed, as they need not satisfy one direction of the later witness intuition: there might be e, e' in some history that have no upper bound in that history. However, genBST structures satisfy the following weaker condition:[29]

[26] Note that a bifurcating and reconvening geodesic that involves a choice set is different from a closed causal curve in a single GR space-time, as the latter does not involve a choice set.

[27] To accommodate MFB, one needs to keep track of which sets of elements of choice sets are consistent, and which are inconsistent. We do not discuss this topic, as the aim of this section is only to serve as an illustration of how to handle a non-asymmetric ordering in a branching approach.

[28] This is the proof of Lemma 9.1 of Placek (2014).

[29] See Fact 9.14 of Placek (2014). See that paper for a proof.

Fact 9.15. *If* $e, e', e^* \in W$ *and* $e \preccurlyeq e^*$ *and* $e' \preccurlyeq e^*$*, then there is a history* h *such that* $e, e', e^* \in h$.

In this sense, histories in genBST generalize the properties of histories in BST_{NF} structures.

Histories in genbBST structures and the structures themselves have welcome topological properties, as discussed in Placek (2014). We do not develop genBST here further, as it does not fully resolve the main obstacle to modeling the indeterminism of GR arising out of a failure of the IVP—the lack of bifurcating geodesics. The structures of genBST do, however, show that causal loops are not a fatal problem for BST.

9.3.7 Summary on General Relativity

In our discussion of indeterminism in GR and its modeling in BST, we focused our attention on indeterminism arising from a failure of the IVP for GR, restricted to vacuum solutions. The salient feature of the known cases of this sort is that a manifold representing the involved space-times is non-Hausdorff, but does not contain bifurcating geodesics of the second kind. To analyze such cases in BST, local Euclidicity (definitionally assumed in mainfolds) compels the use of BST_{NF}. Then, in order to represent indeterminism and satisfy PCP_{NF}, one needs modal forks of the second kind (see Sec. 9.3.4). The GR analogue of a modal fork of the second kind is a bifurcating causal curve of the second kind. Thus, a bare minimum for modelling indeterminism in GR in the BST_{NF} framework is the existence of bifurcating causal curves of the second kind in realistic GR manifolds.[30] However, the lack of bifurcating geodesics in the known GR constructions suggests that there might be no bifurcating causal curves of the second kind in GR.

Such GR manifolds, which are non-Hausdorff but do not contain bifurcating causal curves of the second kind, are curious. Our diagnosis is that such cases present a surprising combination of global indeterminism and local determinism. That is the reason why they cannot be modeled by BST, which focuses on local notions like choice sets or choice points. To contrast this with Minkowskian Branching Structures, which *are* BST structures: these are

[30] The word "realistic" is meant to put aside some artificial constructions like the one mentioned in footnote 17.

constructed by gluing together copies of Minkowski space-time (a GR space-time indeed), but *without* motivation provided by GR's dynamical laws. That is, the gluing does not come from the fact that these copies of Minkowski space-time, with different fields ascribed, are multiple solutions of GR's laws of evolution.

We do not know whether the absence of natural cases of bifurcating geodesics is just limited to known cases, or whether it is general (i.e., concerning the results of gluing together non-isometric extensions of *any* maximal hyperbolic space-time of GR). We thus do not know whether there are cases of local indeterminism resulting from a failure of IVP in GR. The known non-trivial cases of indeterminism in GR are global, not local.

Does this mean that GR has refuted the metaphysical version of local indeterminism underlying BST, showing that that it is not what the world is like? Undoubtedly, there is a conflict between BST and GR but, as to informing us about the world, GR is the theory of the large. It does not easily integrate with our best theory of the small, quantum mechanics. And, by the very nature of the BST project, it is the behavior of the small that is decisive for the success or failure of local indeterminism. Thus, the physics data for local indeterminism, if it is ever to emerge, will be from quantum gravity, a theory that would unify GR and QM, and not from GR alone.

9.4 Conclusions

In this chapter we made good on our promise of exhibiting BST_{NF} structures in which histories are isomorphic to Minkowski space-time, with their content being given by the attribution of physical properties to points. Our construction of Minkowskian Branching Structures relies on a number of conditions on property attributions. A significant result of this construction is that each history in a BST_{NF} structure derived from an MBS can be viewed as a Hausdorff d-manifold, whereas the whole structure itself can be seen as a generalized d-manifold, which is non-Hausdorff iff it contains multiples histories. These results suggest a more prominent role for topological notions in constructing BST structures.

We next discussed whether differential manifolds can generally be built on BST_{NF} structures. The BST postulates are too frugal to always allow for a topological, let alone a differential manifold structure. However, the slogan "nice input in, nice output out" is vindicated to a large extent. That is, if one begins with BST_{NF} histories that all admit Hausdorff C^r d-manifold for the same r, the entire structure will be a topological manifold; furthermore, if this manifold is not a Kervaire-like case, it admits a C^r generalized d-manifold structure.

Having reviewed some of the basic notions of GR, we attempted a BST-based analysis of indeterminism in GR. We described in detail one case of this sort, namely non-isometric extensions of Taub space-time. Although this case has the desired topological description, with the set of extensions interpretable as a generalized d-manifold, and each extension a maximal Hausdorff d-submanifold, it cannot be given a BST reading, because it does not contain bifurcating geodesics. Bifurcating causal curves are, however, needed to introduce alternative possibilities in the BST framework. Furthermore, bifurcating geodesics are not present in other non-Hausdorff manifolds naturally constructible from multiple solutions to the Einstein Field Equations, at least as far as we know. Whether this signals a general non-existence theorem on bifurcating geodesics in GR is an open problem.

Our diagnosis of this situation is that non-isometric extensions of Taub space-time (and similar systems) oddly combine global indeterminism with local determinism. Whether this combination is a universal feature of non-Hausdorff manifolds constructible from multiple space-times is not clear. Nevertheless, we take it that this opposition of local vs. global varieties of indeterminism is the major obstacle to modeling GR indeterminism in the order-theoretic framework of BST. The presence of closed causal curves in some space-times of GR, however, is not a devastating obstacle for a BST analysis, as a pertinent generalization of BST_{NF}, genBST, is available.

9.5 Exercises to Chapter 9

Exercise 9.1. Prove the transitivity of $<_R$ of Def. 9.7.

Exercise 9.2. Prove the density of $<_R$ of Def. 9.7.

Exercise 9.3. Furnish the detail of the chain construction in the proof of Fact 9.2(2).

Hint: This construction is given in the proof of Lemma 8 of Wroński and Placek (2009). We copy it in Appendix B.9.

Exercise 9.4. "Old" MBSs, as defined by Müller (2002), Wroński and Placek (2009), or Placek and Belnap (2012), yield BST_{92} (not BST_{NF}) structures of the form $\langle B, <_R \rangle$. Show that the open-ball topology on $\mathbb{R}^4$ does not yield the b-ball topology on B, if the structure comprises multiple B-histories.

Hint: By the premise, B has a choice point for some histories h_σ and h_η. This choice point is then in b-balls with label σ and in b-balls with label η. Observe that the intersection of such two b-balls with different labels is not open (i.e., cannot be constructed as an arbitrary union of b-balls). Thus, b-balls are not open and hence they fail to deliver a topology on B.

Exercise 9.5. Let $\langle B, <_R \rangle$ be the BST_{NF} structure determined by an MBS $\langle \Sigma, F, P \rangle$ and $\langle B, A \rangle$ be a generalized d-manifold on B. Show that for any B-history h_σ in B, h_σ is open and connected in the manifold topology induced by atlas A on B.

Exercise 9.6. Show that the non-Hausdorff topological manifold depicted by Figure 3.1(b) and described below it can be equipped with a C^∞ atlas A and a Lorentz metric g, the result being a C^∞ generalized non-Hausdorff d-manifold.

Hint: Consider an open ball $b = \{[\langle x, i \rangle] \mid x \in (x_1, x_2)\}$, where (x_1, x_2) is an open interval in the reals and $i = 1$ or 2, and the mapping is given by $\varphi([\langle x, i \rangle]) = x$, restricted to the ball. Take the atlas A generated by such maps. Assume that g has signature -1, so the metric is defined via $r_1 r_2 = -1 \mid r_1 \mid\mid r_2 \mid$, where r_i are co-vectors in this manifold. Argue finally that local Euclidicity is satisfied, but the Hausdorff condition is not.

Exercise 9.7. Show that our construction of MBSs involves continuously extendible gluing.

Hint: Write down the function that glues together two copies of Minkowski space-times into two MBS histories. Argue that their shared region is open.