3

The Refutation of Singularism?

In mathematics, linguistics, and science more generally, pluralities are often eliminated in favor of sets or mereological sums. This affords unification and theoretical economy. In the philosophical literature, by contrast, it is easy to find arguments with sweeping conclusions to the effect that we need primitive plurals, and that this need cannot be filled by any of the singularist alternatives, such as adding *sets* of objects already recognized, adding *mereological sums* of such objects, or using *second-order logic* to quantify over all the ways for the objects already recognized to be. For example, Alex Oliver and Timothy Smiley write that "changing the subject"—which is their name for singularist attempts to eliminate plurals—"is simply not on" (2001, 306). Similar views have been defended by Byeong-uk Yi (1999, 2005, 2006), Tom McKay (2006), and others. In this chapter, we clarify and evaluate these arguments.

If successful, these arguments establish something important. Not only are primitive plurals available in English and many other natural languages, they are also scientifically legitimate and indeed indispensable. Since these are strong claims, however, we will play devil's advocate and examine whether primitive plurals might, after all, be dispensed with for scientific purposes.

3.1 Regimentation and singularism

It is useful to begin by asking: for what purposes are the singularist alternatives "not on"? Sweeping conclusions like the one just mentioned are usually made in the context of discussions about regimentation. Let us elaborate on our understanding of this notion, which we briefly discussed in Section 2.7. The process of regimentation takes as input sentences of a meaningful object language ($\mathcal{L}_O$) and yields a translation into a regimenting language ($\mathcal{L}_R$). This may be a natural language or a formal one. Even when $\mathcal{L}_R$ is a formal language, we may follow Quine in treating it as a "special

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part [. . .] of ordinary or semi-ordinary language" (Quine 1960, 145). From this perspective, both $\mathcal{L}_O$ and $\mathcal{L}_R$ are interpreted languages. We take $\mathcal{L}_O$ to be a fragment of natural language containing plurals.

To say that we need primitive plurals for regimentation does not immediately answer our initial question. *The adequacy of a regimentation is always relative to some purpose.* Whether a particular regimentation succeeds will thus depend on the purpose of the regimentation. Let us recall some of the main theoretical purposes that regimentation has served.

One of the most widespread uses of regimentation concerns "the application of logical theory" (Quine 1960, 145) and is illustrated by the process of translation into logical notation familiar from any logic course. Here the translation provides a perspicuous model of the object language that enables us to formulate a precise account of deductive reasoning and logical consequence. To provide such a model, it is not necessary to capture faithfully the meanings of the sentences translated. As emphasized by Quine (1960), a translation might convey more or less information than the sentence it translates. What matters is that, in virtue of the vocabulary being analyzed, the translation mostly reflects what follows from what in the object language.[1]

Regimentation can also serve the purpose of representing ontological commitments. The ontological commitments of statements of the object language are not always fully transparent. The translation might help clarify them. Following Donald Davidson, one might for instance regard certain kinds of predication as implicitly committed to events. As a result, one might be interested in a regimentation that, by quantifying explicitly over events, brings these commitments to light.

Our focus in this chapter is largely on "the application of logical theory". We will discuss some arguments purporting to show that singularist regimentations mischaracterize logical relations in the object language or mischaracterize the truth values of some sentences. There are various requirements one could put forward in this context. A minimal requirement is that the regimentation be *logically faithful* in the sense that, if an argument in $\mathcal{L}_O$ is invalid, then so should be its regimentation. Let τ be a translation. Then logical faithfulness requires that

[1] Note that this use of regimentation is consistent with different attitudes towards logical consequence. In particular, it is consistent with logical monism as well as logical pluralism. Of course, one's view about logical consequence will be reflected in one's approach to regimentation. So, unlike the pluralist, the monist will see regimentation as a tool to capture the "correct" notion of logical consequence for the object language.

$$\text{if } \tau(\varphi_1), \ldots, \tau(\varphi_n) \vDash \tau(\psi), \text{ then } \varphi_1, \ldots, \varphi_n \vDash \psi.$$

The converse requirement, which we call *logical adequacy*, is that if an argument in $\mathcal{L}_O$ is valid, then its regimentation be valid as well:

$$\text{if } \varphi_1, \ldots, \varphi_n \vDash \psi, \text{ then } \tau(\varphi_1), \ldots, \tau(\varphi_n) \vDash \tau(\psi).$$

The requirement of logical adequacy is less compelling than that of faithfulness, as can be seen by considering the case of sentential logic. There are arguments in natural language whose validity depends on their quantificational structure and thus cannot be captured by the usual regimentation in sentential logic. While this regimentation is logically faithful, it is not logically adequate. Logical adequacy is sometimes more plausible, however, when relativized to a theory T. That is, one may require that if an argument in $\mathcal{L}_O$ is valid, its regimentation be valid *when supplemented with the axioms of* T. Thus, we have:

$$\text{if } \varphi_1, \ldots, \varphi_n \vDash \psi, \text{ then } \tau(\varphi_1), \ldots, \tau(\varphi_n), T \vDash \tau(\psi).$$

Some of the analyses of plurals we will examine satisfy only this weaker adequacy condition.

There are parallel requirements concerning truth. Here it makes sense to require both faithfulness and adequacy, that is, to demand that a regimenting sentence be true if and only if the regimented one is true.

We turn now to some arguments to the effect that primitive plurals are needed for regimentation. The bone of contention is whether, for logical purposes, we can dispense with plurals in the regimentation of $\mathcal{L}_O$. This presupposes that we can determine whether the regimentation contains plurals. Since $\mathcal{L}_R$ is an interpreted language, however, we can presumably establish whether it contains plurals by relying on an antecedent understanding of the distinction between singular and plural expressions.

A singularist regimentation attempts to paraphrase away plural expressions. The alternative approach advocated by Boolos resists this elimination by taking plurals at face value. On this alternative, which we call *regimentation pluralism*, $\mathcal{L}_R$ does contain plural expressions. The languages $\mathcal{L}_{\text{PFO}}$ and $\mathcal{L}_{\text{PFO+}}$ are the main examples of regimentation pluralism.

Which approach is correct? As observed, the recent philosophical literature abounds with arguments against regimentation singularism. The principal aim of this chapter is to assess some of these arguments and gain a better understanding of the limits of regimentation singularism. While we think that regimentation pluralism has important applications, we also think that regimentation singularism is a more serious rival to regimentation pluralism than the mentioned literature suggests. So we want to give it a fair hearing. In fact, we identify some conditions under which regimentation singularism is perfectly benign. Since these conditions tend to be satisfied in the cases that interest linguists, their singularist proclivities are less problematic than many philosophers claim.

3.2 Substitution argument

One argument against regimentation singularism, put forward by Yi (2005, 471–2), turns on a substitution of plural and singular terms. We therefore dub it the *substitution argument*. This argument is meant to apply to any singularist regimentation, no matter how it paraphrases plural expressions.[2] For concreteness, we focus on a regimentation that uses sets.

Consider the plural term 'Russell and Whitehead' and its set-theoretic regimentation, the set term '{Russell, Whitehead}'. Letting 'Genie' abbreviate this set term, we now formulate the following sentences:

(3.1) Genie is one of Genie.

(3.2) Genie is one of Russell and Whitehead.

While (3.1) is arguably true (and logically so), (3.2) is false. But given the way in which 'Genie' was introduced, aren't 'Genie' and 'Russell and Whitehead' intersubstitutable *salva veritate*? If so, it follows that the two sentences have the same truth value. But this appears not to be the case.

Let us examine the argument more closely. Does it concern sentences of $\mathcal{L}_O$ or $\mathcal{L}_R$? Since $\mathcal{L}_R$ is supposed to be free of plurals, the argument must be concerned with sentences of $\mathcal{L}_O$.

Thus understood, the argument assumes that $\mathcal{L}_O$ contains plural resources and is able to express claims about sets (or whatever other objects are used

[2] Note that a semantic version of this argument is also present in Yi's discussion. The semantic version targets the view that a plural term *denotes* a set or a set-like entity. In this chapter, our focus is on regimentation and thus on the view that plural expressions can be *paraphrased* by means of singular constructions. We think that it is important to keep the two views separate. See Chapter 7 for a discussion of the semantics of plurals.

in $\mathcal{L}_R$ to paraphrase plurals). The problem—if there is one—stems from an *unintended interaction* between these plural resources and talk about the objects that are used to represent pluralities. Suppose $\mathcal{L}_O$ could not talk about the objects used to represent pluralities. Then 'Genie' would not be part of $\mathcal{L}_O$, and the argument would not get off the ground. This suggests that the argument can be blocked by denying the assumption that $\mathcal{L}_O$ can talk about the objects used to represent pluralities. We explore this option in the next section. In the remainder of this section, we will show that the argument can be resisted even when this assumption is granted.

The argument relies on the reasonable requirement that a proper regimentation of (3.1) and (3.2) do justice to the fact that the two sentences differ in truth value. But it is not hard to think of a simple translation that meets this demand. For example, we may translate (3.1) and (3.2) as respectively:

(3.3) Genie = Genie.

(3.4) Genie ∈ {Russell, Whitehead}.

This regimentation captures the truth values of (3.1) and (3.2). It maps a (logically) true sentence to a (logically) true sentence, and it maps a false sentence to a false sentence.

It might be objected that we didn't translate 'is one of' uniformly. However, this non-uniformity seems justified by the peculiar grammatical status of (3.1). One might even complain that (3.1) is ungrammatical, since 'is one of' requires a plural term in its second argument place.

Yi proposes a variant of the argument intended to avoid this complication. Consider the following two sentences:

(3.5) Genie is one of Genie and Frege.

(3.6) Genie is one of Russell and Whitehead and Frege.

Again, the two sentences appear to differ in truth value: while (3.5) is (logically) true, (3.6) seems false. (Presumably, something is one of Russell and Whitehead and Frege just in case it is identical to one of the three named logicians.)

Even in this case it is not hard to think of a translation that captures the difference in truth value. For instance, we may translate (3.5) and (3.6) as respectively:

(3.7) Genie $\in$ {Genie, Frege}.

(3.8) Genie $\in$ {Russell, Whitehead, Frege}.

The translation of 'is one of' is now uniform. In addition, the translation preserves the truth value of the two sentences. The moral is that, even though regimentation singularism paraphrases a plural term like 'Russell and Whitehead' by means of a singular expression such as '{Russell, Whitehead}', it need not license in $\mathcal{L}_O$ the intersubstitution *salva veritate* of the two terms.

We conclude that the substitution argument does not undermine regimentation singularism. First, the argument relies on an assumption that may be resisted, namely that $\mathcal{L}_O$ can talk about the objects used in $\mathcal{L}_R$ to paraphrase plurals. Second, it overlooks the potential of some singularist regimentations to capture the intuitive truth values of the relevant sentences of $\mathcal{L}_O$.

We now turn to two further objections to regimentation singularism that share with the substitution argument the assumption that $\mathcal{L}_O$ can talk about the entities used to paraphrase plurals in $\mathcal{L}_R$.

3.3 Incorrect existential consequences

A colorful formulation of the next objection is contained in Boolos's famous passage quoted in Chapter 2:

> There are, of course, quite a lot of Cheerios in that bowl, well over two hundred of them. But is there, in addition to the Cheerios, also a set of them all? [...]
>
> It is haywire to think that when you have some Cheerios, you are eating a set [...]. [I]t doesn't follow just from the fact that there are some Cheerios in the bowl that, as some who theorize about the semantics of plurals would have it, there is also a set of them all. (Boolos 1984: 448–9)

In one reading of the passage, the objection is that a singularist regimentation validates incorrect inferences in $\mathcal{L}_O$ and, in particular, incorrect existential generalizations.

If the purpose of regimentation is the application of logical theory, we want the regimentation to be logically faithful. Consider the relevant inference (with '*cc*' naming the Cheerios in the bowl):

(3.9) $\dfrac{\text{George ate } cc.}{\text{George ate a set.}}$

As emphasized by Boolos, this inference is invalid. Compare now a set-theoretic translation of the argument:

(3.10) $\dfrac{\text{George ate* } \{cc\}.}{\text{George ate* a set.}}$

where 'ate*' is the translation of 'ate' and '{*cc*}' is the set-theoretic rendering of '*cc*' (say, '{*x*: *x* is a Cheerio in the bowl}').[3]

Unlike (3.9), (3.10) is valid. So we have a violation of logical faithfulness: an invalid inference has a valid translation.

The argument can be extended to other types of singularist regimentation. The general point is that regimentation singularism seems to permit illicit existential generalizations, allowing us to transition *as a matter of logic* from some objects to a single object that comprises or somehow represents these objects.

As in the case of the substitution argument, it is assumed that $\mathcal{L}_O$ can talk about the objects used in $\mathcal{L}_R$ to paraphrase plurals. Let us call these objects *proxies*. The alleged problem stems from an unintended interaction between the plurals and the talk of proxies. Thus, if $\mathcal{L}_O$ was precluded from talking about the proxies, the argument could not get off the ground. How might this be achieved? Since the object language is just given to us, it is not an option to ban certain expressions from $\mathcal{L}_O$ if they already occur in it. By contrast, the regimenting language $\mathcal{L}_R$ is not given but can freely be chosen to serve our needs. So we might well be able to choose our proxies so as to avoid problematic interactions with any resources found in $\mathcal{L}_O$. In many of the cases studied by linguists, such a choice is indeed possible.

Philosophical analysis, on the other hand, often aims for greater generality. Suppose that $\mathcal{L}_O$ already talks about sets. Then we might be able to avoid the problematic interaction by finding some other proxies—let us call them "supersets"—with which to regiment the plurals of $\mathcal{L}_O$. But even if an appropriate notion of superset can be found, we are not done. We may want to include talk of supersets in $\mathcal{L}_O$. Thus, for *any* kind of object that extensions

[3] Since 'ate' is used here as plural predicate, it is regimented by means of a singular counterpart 'ate*'. By contrast, the predicate 'set' is singular and thus remains unchanged in the regimentation.

of $\mathcal{L}_O$ might talk about, we must be able to regiment plural talk about such objects using proxies of a *new* kind.

Is this possible? The answer will depend on the generality to which our analysis aspires. Suppose we want a fully specified regimentation strategy that works for any given object language whatsoever. We thus specify a certain kind of proxy that will always be used to paraphrase plurals. When this general strategy is applied to an object language that talks about proxies of this kind, the problem under discussion arises. We conclude that the singularist's only hope is that her regimentation strategy need *not* be fully specified, thus allowing her to wait and see what expressive resources a given object language contains and *only then* choose her proxies—in a way that avoids the problematic interactions. This might well be doable. Thus, even if the most ambitious form of regimentation singularism succumbs to the objection from incorrect existential consequences, there are less extreme forms that avoid it. These forms will likely suffice for linguists' purposes. To achieve the kind of generality that philosophers seek, however, any viable form of regimentation singularism must refrain from a fixed choice of proxies.

3.4 The paradox of plurality

What is often regarded as the most serious objection to regimentation singularism is *the paradox of plurality*, first foreshadowed in Section 2.1. Suppose that we use sets to eliminate plurals and that $\mathcal{L}_O$ has the resources to talk about sets. Then the following sentence of $\mathcal{L}_O$ appears to be a truism:

(3.11) There are some objects such that any object is one of them if and only if that object is not an element of itself.

The contention is that set-theoretic singularism is bound to regiment (3.11) as follows:

(3.12) There is a set of which any object is an element if and only if that object is not an element of itself.

In symbols:

(3.13) $\exists x\big(\mathrm{set}(x) \land \forall y(y \in x \leftrightarrow y \notin y)\big)$

But this is an instance of the familiar, inconsistent Russell sentence. Thus, the true sentence (3.11) is regimented by means of the false sentence (3.12), which is unacceptable.[4]

Let us make explicit the generality of the argument.[5] As before, let proxies be the entities used in $\mathcal{L}_R$ to paraphrase plurals. Thus, a proxy need not be a set, but could equally well be a class, a mereological sum, or a group. Plural quantification is regimented as singular quantification over proxies. Let η be the translation into $\mathcal{L}_R$ of the one-many relation 'to be one of'. So '$x\eta y$' regiments the statement that x is one of the objects represented by the proxy y. Since η is a meaningful predicate, nothing precludes its introduction into $\mathcal{L}_O$—or so the argument goes. Thus we may suppose that the predicate is available in $\mathcal{L}_O$ as well.

We now face an awkward dilemma. Is η reflexive? Suppose not. Then there is an object that satisfies the open formula '$\neg(x\eta x)$'. Applying plural comprehension to this formula, we obtain:

(3.14) There are some objects such that any object is one of them if and only if that object does not bear η to itself.

The singularist regimentation of this sentence is:

(3.15) There is a proxy to which any object bears η if and only if that object does not bear η to itself.

And this, in turn, is formalized as:

(3.16) $\exists x[\text{proxy}(x) \wedge \forall y(y\eta x \leftrightarrow \neg(y\eta y))]$

But (3.16) is inconsistent! So again, a true sentence is regimented by means of a false one, which is unacceptable.

Alternatively, suppose that η is reflexive. Then there is no object that satisfies the open formula '$\neg(x\eta x)$'. This blocks the previous argument. Instead, another problem arises: the reflexivity of η entails that different pluralities must be represented by one and the same proxy and hence cannot

[4] The proposed regimentation also involves a violation of logical faithfulness. For (3.11) is not only true but intuitively valid, whereas (3.12) is not only false but logically so.

[5] The argument, inspired by Boolos (1984b, 440–4), is also discussed in Lewis 1991, 68; Schein 1993, Chapter 2, Section 3.3; Higginbotham 1998, 14–17; Oliver and Smiley 2001, 303–5; Rayo 2002, 439–40; and McKay 2006, 31–2.

be distinguished in the regimentation. To prove this entailment, suppose there are at least two objects, a and b. When η is reflexive, the singleton plurality of a must have itself as its proxy, and likewise for b. Consider now the plurality of a and b, and let c be its proxy. By the reflexivity of η, we have $c\eta c$. By the definition of a proxy, we also know that only a and b bear η to c. Hence c is identical with either a or b. But we observed that each of a and b is already used as a proxy for a distinct plurality, namely the singleton pluralities of a and b. Thus, as promised, different pluralities are represented by one and the same proxy.

Just like the previous two arguments, the paradox of plurality relies on the assumption that talk of proxies is available in $\mathcal{L}_O$. The lesson is that, if $\mathcal{L}_O$ can talk not only about pluralities but also about their proxies, then the regimentation validates unintended interactions of the sort just seen. To block the paradox, we would therefore have to prevent such problematic interactions.

One possibility, suggested by our discussion in the previous section, is to refrain from making a fixed choice of proxies to be used in the analysis of all object languages. Instead, the singularist can let her choice of proxies depend on the particular object language she is asked to regiment. All she needs to do is to choose new proxies, not talked about by the given object language. In this way, the problematic interactions are avoided.

In fact, there are other responses to the paradox of plurality that are compatible even with a fixed choice of proxies. One such response is that there is variation in the range of the quantifiers involved in the paradoxical reasoning.[6] In particular, one can avoid the paradox by assuming that in (3.16) the quantifier '$\exists x$' has a wider range than '$\forall y$'. To see why this assumption blocks the paradox, consider the reasoning leading from (3.16) to contradiction:

$$\exists x\big[\text{proxy}(x) \wedge \forall y(y\eta x \leftrightarrow \neg(y\eta y))\big] \tag{3.16}$$

$$\forall y(y\eta r \leftrightarrow \neg(y\eta y)) \tag{3.17}$$

$$r\eta r \leftrightarrow \neg(r\eta r) \tag{3.18}$$

In the step from (3.17) to (3.18), the witness to '$\exists x$' is used to instantiate '$\forall y$'. If the domain of '$\exists x$' extends that of '$\forall y$', the step becomes illicit.

[6] See, e.g., Parsons 1974a, 1974b; Glanzberg 2004, 2006.

This response brings to the fore the topic of absolute generality. Indeed, the response presupposes that the range of the quantifiers of the object language is not unrestricted and thus that a domain expansion is possible. The question of absolute generality is explored in Chapter 11, where we defend the permissibility of such generality. If we are right, then the response under discussion is unavailable. Still, since linguists are typically interested in ordinary discourse where absolute generality is not present, they can often bypass the argument.

Yet another response to the paradox of plurality is developed in Chapter 12, where we take a more critical stance towards plural comprehension. In particular, we block the paradox by developing a reason to reject instances of plural comprehension underlying the paradoxical reasoning, such as (3.14).

3.5 Plural Cantor: its significance

The paradox of plurality is closely related to a generalization of Cantor's theorem. Let us begin by reminding ourselves of the familiar set-theoretic version of Cantor's theorem, which can be formulated as follows.

> Cantor's theorem (informal)
>
> For any set A, the subsets of A are strictly more numerous than the elements of A.

The plural version of Cantor's theorem makes an analogous claim concerning pluralities.

> Plural Cantor (informal)
>
> For any plurality xx with two or more members, the subpluralities of xx are strictly more numerous than the members of xx.

There is only one tiny disanalogy: we need to assume that xx have two or more members, whereas no such assumption is required concerning A. The reason for this minor discrepancy is that pluralities, unlike sets, are required to be non-empty. If this requirement were lifted, the analogy between the set-theoretic and plural versions of the theorem would be perfect.

Of course, the cardinality comparisons involved in these two informal statements need to be explicated, and the resulting plural version of Cantor's theorem needs to be proved. But before we do so in the next section, we would like to explain the significance of Plural Cantor for our discussion.

The theorem can be seen as a diagnosis of the problem exploited by the paradox of plurality. Assume, as is done in traditional plural logic, that there is a "universal plurality" encompassing every object whatsoever. (This assumption will be challenged in Chapter 12.) Applied to this universal plurality, the theorem entails that there are more pluralities than objects. This implies that it is impossible to assign to each plurality a distinct object as its proxy. We now find ourselves in the "awkward dilemma" described in Section 3.4. Suppose we require that each plurality be assigned a unique proxy. Then, as we have just seen, we land in a contradiction. Alternatively, we may relax this requirement. But this means that some statements of $\mathcal{L}_O$ will receive an incorrect regimentation. For example, if distinct pluralities *xx* and *yy* are assigned the same proxy z, the true statement of $\mathcal{L}_O$ that these pluralities are not coextensive will be regimented as the contradictory statement that something does and does not bear η to z.

Can this dilemma be resisted? Once again, the question of absolute generality turns out to be central. Suppose that absolute generality is possible. Then, as we have observed, traditional plural logic yields an instance of Plural Cantor concerned with the universal plurality. We therefore obtain that there are more pluralities than objects. This means that each plural variable of the object language can have more possible values—namely each plurality—than there are objects or proxies. By contrast, suppose that absolute generality is not possible. Then the object language ranges over some plurality of objects *aa* which, when the domain is expanded, can be seen not to be universal. This makes it unproblematic that *aa* has more subpluralities than members. Each of these subpluralities can be represented by a distinct proxy—provided that most of these proxies are not among *aa* but are drawn from elsewhere. And there is no reason why such proxies should not be available when the object language has a restricted domain.

3.6 Plural Cantor: its statement and proof

We now turn to the task of explicating the cardinality comparison involved in Plural Cantor. As is turns out, there are various ways to do so, resulting in different versions of the theorem.

There are several ways to define what it means for one set Y to be "strictly more numerous than" another set X. One option is that there is no surjective function from X to Y; another is that there is no injective function from

Y to X.[7] In both cases, the notion of a function can be understood in the usual set-theoretic way.[8]

Consider now the cardinality comparison involved in Plural Cantor. What is it for the subpluralities of xx to be strictly more numerous than xx themselves? Let us try to imitate the answer given in the set-theoretic case. Suppose we add to our formalism variables of a new and primitive type for functions *from pluralities to objects*, that is, functions that take one or more objects as input and then output a single object as the value. We can then state that there is no injective function from subpluralities of xx to xx themselves by denying the existence of a function g from pluralities to single objects among xx such that:

$$\forall yy \forall zz (yy \preccurlyeq xx \land zz \preccurlyeq xx \rightarrow (g(yy) = g(zz) \rightarrow yy \approx zz)) \tag{3.19}$$

Alternatively, we might add variables of a new and primitive type for functions *from objects to pluralities*, that is, functions that take a single object as input and then output one or more objects as values. To state that there is no surjective function from xx to subpluralities of xx, we deny the existence of a function f from objects to pluralities such that:

$$\forall yy (yy \preccurlyeq xx \rightarrow \exists x (x \prec xx \land f(x) \approx yy)) \tag{3.20}$$

For each of these formulations, it is straightforward to prove the resulting formal version of Plural Cantor. The version using a function from objects to pluralities provides a good example. Assume, for contradiction, that there is a surjective function f of the relevant sort. We contend there is an $x \prec xx$ such that $x \not\prec f(x)$, as we shall prove shortly. Thus, plural comprehension allows us to define a subplurality $\delta\delta$ of xx such that:[9]

[7] Recall that a function f from X to Y is said to be *surjective* if and only if

$$(\forall y \in Y)(\exists x \in X) f(x) = y,$$

and *injective* if and only if $f(x) = f(x') \rightarrow x = x'$.

[8] More precisely, f is a function from X to Y if and only if (i) for every $x \in X$ there is a $y \in Y$ such that $\langle x, y \rangle \in f$, and (ii) if both $\langle x, y \rangle$ and $\langle x, y' \rangle$ are in f, then $y = y'$.

[9] Note that the instance of comprehension used is predicative; that is, the condition '$x \prec xx \land x \not\prec f(x)$' used to define the plurality does not itself quantify over pluralities. See Uzquiano 2015b, Section 3.1. Furthermore, note that this instance of plural comprehension is a case of what we will later (see Appendix 10.A and Section 12.5) call *plural separation*, namely, a comprehension axiom where a given plurality (in this case, xx) is cut down to a subplurality comprising the members of xx that satisfy some formula. The same applies to other instances of comprehension used in this and similar proofs. Thus, the proofs in question go through in the alternative system of critical plural logic that we defend in Chapter 12.

$$(3.21) \qquad \forall x(x \prec \delta\delta \leftrightarrow x \prec xx \wedge x \nprec f(x))$$

Since f is surjective, there is $\delta \prec xx$ such that $f(\delta) \approx \delta\delta$. By instantiating the quantifier of (3.21) with respect to δ, we easily derive

$$(3.22) \qquad \delta \prec \delta\delta \leftrightarrow \delta \nprec f(\delta)$$

which is inconsistent because $f(\delta) \approx \delta\delta$.[10]

It remains only to prove our contention that there is an $x \prec xx$ such that $x \nprec f(x)$.[11] Assume not. As already assumed in the statement of the theorem, there are at least two distinct objects, a and b, among xx. By the former assumption, f maps each of these objects to the corresponding singleton plurality. By the assumed surjectivity of f, there is an object c that f maps to the plurality of a and b. So c must be distinct from each of a and b, since we have established that these two objects are mapped by f to other pluralities. Since the members of $f(c)$ are a and b, this entails that $c \nprec f(c)$, which contradicts our assumption that there is no $x \prec xx$ such that $x \nprec f(x)$. This concludes our proof.

Of course, these formulations and proofs assume that we have variables of a new and primitive type, either for functions from objects to pluralities or for functions in the reverse direction. Fortunately, our proof requires no special assumptions concerning these new functions, only that quantification over them obeys the usual logical principles.[12] Even so, it is important to realize that the new type of function is not required. In Appendix 3.A, we provide some alternative formulations of the relevant cardinality comparisons. Some of these avoid the new type of function, thus making the theorem available also in systems that are less expressive. Other formulations achieve greater generality by regarding functions as just a special kind of relation. Moreover, by considering all these formulations side by side, we obtain a more complete picture of the assumptions that this important theorem requires.

[10] The version of Plural Cantor using a function from pluralities to objects is proved similarly. Assume there is an injective g of the relevant sort. Since g is injective, there is an inverse function g^{-1}. We can now use plural comprehension to define a subplurality $\delta\delta$ of xx such that:

$$(3.23) \qquad \forall x(x \prec \delta\delta \leftrightarrow x \prec xx \wedge x \nprec g^{-1}(x))$$

To obtain a contradiction, let $\delta = g(\delta\delta)$ and instantiate the quantifier of (3.23) with respect to δ.

[11] This proof is nearly identical with the one on p. 40.

[12] It might be objected that the proof mentioned in footnote 10 assumes the existence of an inverse function g^{-1}. But this assumption is easily eliminated in favor of an alternative use of (impredicative) plural comprehension. In particular, plural comprehension yields the existence of a subplurality $\delta\delta$ of xx such that:

$$\forall x(x \prec \delta\delta \leftrightarrow x \prec xx \wedge \forall yy(g(yy) = x \rightarrow x \nprec yy))$$

Since g is injective, a contradiction follows.

Just as we have generalized the ordinary version of Cantor's theorem to Plural Cantor, so Plural Cantor admits of further generalizations. Suppose there are "superpluralities", that is, pluralities of pluralities. Then, using resources analogous to those used for the proof of Plural Cantor, one can show that, given any domain with two or more objects, the superpluralities based on that domain are strictly more numerous than the pluralities based on the same domain. This is done by proving that, relative to the given domain, there is no surjective function from pluralities to superpluralities (and no injective function in the reverse direction).

3.7 Conclusion

We have considered four arguments against regimentation singularism: the substitution argument, the argument from unintended existential consequences, the paradox of plurality, and the argument based on Plural Cantor. Although the arguments differ in important respects, we also found some common themes.

The first three arguments turn on problematic forms of interaction between plurals and talk of proxies. These arguments can therefore be blunted by giving up the requirement that a fixed sort of proxies be used in all regimentations. Suppose this requirement is lifted. Then, for any given object language, it may well be possible to choose new proxies, that is, proxies that are not among the objects that this language can talk about. If new proxies can always be found, the problematic interactions can be avoided.

A central question is therefore whether new proxies are always available. In fact, their availability is called into doubt by the fourth argument against singularism, which uses a generalization of Cantor's theorem to argue that there are more pluralities than objects and thus *a fortiori* too many pluralities for each to be assigned a unique object as its proxy.

We found, however, that even this fourth argument relies on some assumptions that can be challenged, namely the possibility of absolute generality and the validity of traditional plural logic. These assumptions are discussed at length in Chapters 11 and 12. If either assumption fails, this will provide an additional and more definitive response to the third argument, that is, the one based on the paradox of plurality.

Overall, we conclude that the prospects for regimentation singularism are not nearly as bleak as many philosophers make them out to be. As we have seen, there are promising responses to the anti-singularist arguments. It is noteworthy that these responses are particularly strong in many of the cases that concern linguists. For their purposes, it is often unproblematic to assume that the proxies are new vis-à-vis the objects that the object language talks about. Moreover, linguists often have independent reasons to foresake the ambition of absolute generality (see Peters and Westerståhl 2006, 47–9). These considerations explain why linguists' singularist tendencies are less problematic than many philosophers and logicians claim.

As mentioned at the beginning of this chapter, there are several ways to talk about many objects simultaneously. In addition to using the primitive plurals available in many natural languages, we can add *sets* of objects already recognized, add *mereological sums* of such objects, or use *second-order logic* to quantify over all the ways for the objects already recognized to be. We therefore asked whether primitive plurals are necessary or even scientifically legitimate. While we grant that there is a presumption in favor of taking expressive resources available in natural language to be scientifically legitimate, it would be good to do better. So this chapter has discussed some very general anti-singularist arguments that purport to establish the need for primitive plurals. We have shown that these arguments make limited progress.

We will now change tack and undertake a detailed comparison of plural logic with each of the other ways to talk about many objects simultaneously. This is our agenda for Part II of the book. We will find that, although the four alternatives have some important structural similarities, there are also some significant philosophical and formal differences between them. Based on these differences, we defend the thesis that none of them should be eliminated in favor of any other. This yields, in particular, a more robust argument for the scientific legitimacy of primitive plurals than this chapter has produced.

Appendix

3.A Alternative formulations of Plural Cantor

Suppose we want to avoid primitive functions from pluralities to objects or from objects to pluralities, both of which we invoked in Section 3.6. We now outline two alternatives: one that uses higher-order relations, and another that "codes" these relations in terms of pluralities of ordered pairs. This yields several formulations of Plural Cantor.

A plural comprehension axiom

$$\exists x\, \varphi(x) \rightarrow \exists xx \forall y(y \prec xx \leftrightarrow \varphi(y))$$

is said to be *impredicative* if $\varphi(y)$ contains plural quantifiers, and *predicative* if not. In what follows, we pay close attention to the question of whether impredicative plural comprehension is needed to prove the different formulations. This question is theoretically important. Even by its defenders, impredicative comprehension is often regarded as a strong commitment (see Bernays 1935). While we are prepared to make this commitment, at least in our reasoning about plurals (see Appendix 10.A), it is important to keep track of when the commitment is needed. It should also be noted that our discussion of Plural Cantor carries over, with minor modifications, to a second-order version of Cantor's theorem. This says, loosely speaking, that there are more values of second-order variables based on any domain than objects in the domain. As a corollary of our discussion, one thus easily obtains results about when impredicative *second-order* comprehension is required for the proof of a second-order version of Cantor's theorem.

Suppose we wish to use relations to state that it is impossible to "tag" each subplurality of xx with a unique member of xx. So we consider relations of the form $R(x, yy)$, that is, dyadic relations whose first and second argument places are open to objects and pluralities, respectively. We can now state that there is no relation that effects the described "tagging" by saying that there is no R of the mentioned form such that:

- (R is functional) $\quad yy \preccurlyeq xx \wedge yy' \preccurlyeq xx \wedge R(x, yy) \wedge R(x, yy') \rightarrow yy \approx yy'$
- (R is surjective) $\quad (\forall yy \preccurlyeq xx)(\exists x \prec xx) R(x, yy)$

This provides a useful relational statement of Plural Cantor.

It is interesting to reformulate that statement in terms of the converse of R, that is, the relation $\bar{R}$ defined by $\forall x \forall yy(\bar{R}(yy, x) \leftrightarrow R(x, yy))$. It is easy to verify that the two mentioned requirements on R are logically equivalent to the following requirements on $\bar{R}$, respectively:

- ($\bar{R}$ is injective) $\quad yy \preccurlyeq xx \wedge yy' \preccurlyeq xx \wedge \bar{R}(yy, x) \wedge \bar{R}(yy', x) \rightarrow yy \approx yy'$
- ($\bar{R}$ is total) $\quad (\forall yy \preccurlyeq xx)(\exists x \prec xx)\bar{R}(yy, x)$

Thus, the statement that there is no relation specifying a surjective function from members of xx to subpluralities of xx is equivalent to the statement that there is no relation that associates subpluralities of xx with members of xx in a way that is injective and total. This equivalence relies only on the extremely weak (and obviously predicative) assumption that every relation has a converse. We have thus achieved a pleasing unification of the surjectivity-based and the injectivity-based characterizations of the cardinality comparison: the two characterizations are logically equivalent *modulo* an extremely weak assumption.[13]

The claim that there is no relation specifying an injective and total *function* from subpluralities of xx to members of xx is strictly stronger than our pleasing unification. For the mentioned claim adds a third requirement on $\bar{R}$, namely that $\bar{R}$ be functional; that is:

$$\forall x \forall x' \forall yy(\bar{R}(yy, x) \wedge \bar{R}(yy, x') \rightarrow x = x')$$

Let us now prove our relational statement of Plural Cantor. Suppose, for contradiction, that there is a relation R satisfying the conditions laid out above. We want to use plural comprehension to define a subplurality $\delta\delta$ of xx such that:

$$\forall x(x \prec \delta\delta \;\leftrightarrow\; x \prec xx \wedge \exists yy(R(x, yy) \wedge x \nprec yy)) \tag{3.24}$$

Of course, (3.24) is the consequent of a plural comprehension axiom whose antecedent is $\exists x(x \prec xx \wedge \exists yy(R(x, yy) \wedge x \nprec yy))$. It is easy to prove this antecedent, on the assumption that xx comprise at least two objects, by imitating our proof of an analogous claim in Section 3.6. So let us return to our proof of the relational statement of Plural Cantor. By the assumed surjectivity

[13] By contrast, Uzquiano (2019) sees a deeper difference between these two characterizations.

of R, there is thus a $\delta \prec xx$ such that $R(\delta, \delta\delta)$. We now ask whether $\delta \prec \delta\delta$. By standard Russellian reasoning, it is straightforward to derive that this holds if and only if it does not.

It is important to notice that this proof relies on an impredicative plural comprehension axiom. For the plurality $\delta\delta$ is defined by quantifying over all sub-pluralities of xx, to which the defined plurality itself belongs. In fact, the reliance on impredicative comprehension can be shown to be essential.[14]

So far, we have made use of primitive relations involving pluralities. An alternative is to "code" such relations by means of pluralities of ordered pairs. This alternative is available in systems without quantification over primitive functions or relations, as is the case for most systems of plural logic found in the literature. The basic idea is to represent the fact that a is related to the plurality xx by pairing a with each object in xx. The resulting ordered pairs represent that a is related to the plurality of objects with which a has been paired. A visual example will help.

$$\begin{array}{c|ccc}
c & \langle a, c\rangle & \langle b, c\rangle & \\
b & \langle a, b\rangle & & \langle c, b\rangle \\
a & & \langle b, a\rangle & \langle c, a\rangle \\
\hline
 & a & b & c
\end{array}$$

Consider the six ordered pairs displayed. This plurality codes a relation of objects with pluralities. Specifically, an object x is related to the plurality of objects that figure as second coordinates in pairs with x as its first coordinate. This can be read off by attending to each column. Visually, each column represents the fact that the object along the horizontal axis is related to the plurality of objects that figure as second coordinates in this column. Thus, the diagram above represents that a is related to the plurality of b and c, that b is related to a and c, and that c is related to a and b.

[14] This follows from the fact that Frege's "Basic Law V" is consistent in second-order logic with only predicative comprehension axioms (see Heck 1996). We begin by rewriting Basic Law V with plural variables in place of second-order ones:

$$\{xx\} = \{yy\} \leftrightarrow xx \approx yy$$

Now define '$R(x, yy)$' as '$x = \{yy\}$'. Heck's model can now be tweaked to produce a model of plural logic with predicative comprehension and the statement that R is functional and surjective.

Equipped with this notion of coding, we obtain a precise way of expressing the plural version of Cantor's theorem using only plural resources.

Plural Cantor (formal)

For any plurality xx with two or more members, there is no plurality that codes a functional and surjective relation of members of xx with subpluralities of xx.

The proof of this version of the theorem is based on the same idea as before, although with a subtle but important difference. Suppose, for contradiction, that there is a plurality rr of ordered pairs that code a relation of the mentioned sort. We want to define a diagonal plurality $\delta\delta$ of each and every object $x \prec xx$ such that x is not a member of the plurality with which x is related by the relation coded by rr. This requires some unpacking. The plurality of objects with which x is related in the mentioned way are all the y such that $\langle x, y \rangle \prec rr$. Thus, the claim that x is not a member of this plurality is just the claim that $\langle x, x \rangle \not\prec rr$. As before, it is easy to show that, if xx have two or more members, then there is at least one x that satisfies this condition. Thus, a plural comprehension axiom ensures the existence of our desired diagonal plurality $\delta\delta$. The advertised difference is that this comprehension axiom is fully predicative.[15] From this point on, the argument proceeds precisely as before. Since the coded relation is surjective, there is a δ that stands in this relation to $\delta\delta$. We now ask whether $\delta \prec \delta\delta$. Familiar Russellian reasoning enables us to prove that the answer is affirmative if and only it is negative.

The following table summarizes our findings concerning the need for impredicative plural comprehension:[16]

	primitive functions	higher-order relations	pluralities and pairs
no surjective function	predicative	impredicative	predicative
no injective total relation	—	impredicative	predicative
no injective function	impredicative (predicative if inverse functions are permitted)	impredicative	predicative

[15] We owe this surprising observation to Gabriel Uzquiano (see especially Uzquiano 2015b) and are grateful to him for discussion of its significance.

[16] In fact, every relevant instance of plural comprehension can be replaced by a corresponding instance of plural separation, as indicated in footnote 9 on p. 43.

This provides a richer and more interesting picture than would have been obtained had we focused solely on primitive functions. Our table raises the question of why predicative plural comprehension suffices to prove some formulations of the theorem, while others require impredicative comprehension. While this is not the place for a comprehensive assessment, we wish to make two remarks.

First, the resources needed to prove a formulation of Cantor's theorem are highly sensitive to the language in question. A striking example concerns the two formulations in terms of primitive functions. The "no surjective function" version uses a primitive function f from objects to pluralities. Assume f is surjective. Then, for any xx, there is x such that $f(x) \approx xx$. Generalizing, we establish the following equivalence:

$$\forall xx\, \varphi(xx) \leftrightarrow \forall x\, \varphi(f(x))$$

Using this equivalence, all plural quantification can be eliminated in favor of singular quantification. It is therefore unsurprising that predicative plural comprehension suffices for the proof. Since all plural quantification can be eliminated, it can obviously be avoided in the comprehension axioms. By contrast, no such elimination is available in the "no injective function" version, which uses a primitive function g from pluralities to objects.

Second, notice that all the plural versions of Cantor's theorem are negative existential claims to the effect that there isn't a function or relation that would establish that there are no more pluralities on a domain than objects in the domain. The strength of a negative existential claim obviously depends on the domain: the larger the pool of possible counterexamples, the stronger the negative existential. Compare the results recorded in the middle and right-hand columns of our table. The results in the middle column state that there isn't a counterexample in the large pool of all relations of the form $R(x, yy)$. By contrast, the results in the right-hand column state that there isn't a counterexample in what might prove to be a smaller pool of such relations *that can be coded by means of pluralities and ordered pairs alone*.

To investigate this possibility, let us compare the two pools of relations. Suppose that only predicative plural comprehension is accepted. Then there is no guarantee that every functional and surjective relation of objects to pluralities can be coded by means of a plurality of ordered pairs. To see this, consider a relation $R(x, yy)$ of the mentioned sort. If we had impredicative plural comprehension, we could establish that this relation is coded by means of the plurality of ordered pairs $\langle x, y\rangle$ defined by the impredicative condition $\exists yy(R(x, yy) \land y \prec yy)$. Without impredicative plural comprehension, however, this strategy for coding relations by means of pluralities of ordered pairs is unavailable and the two pools of relations will therefore differ in size. In fact, when only predicative comprehension is accepted, we cannot prove in general that all relations of the relevant type can be coded by means of a plurality of ordered pairs.[17]

Equipped with this observation, let us return to the difference between the middle and right-hand columns. We can now better understand the source of the difference. We found that, without impredicative plural comprehension, the middle column is concerned with a strictly larger pool of possible counterexamples than the right-hand column, namely the pool of all relations of the relevant type, not just those that can be coded by means of a plurality of ordered pairs. And it stands to reason that strictly stronger assumptions are needed to prove a negative existential claim when this claim is concerned with a strictly larger pool of possible counterexamples.

[17] The model construction described in footnote 14 on p. 49 provides an example of a relation that cannot be coded in this way: let $R(x, yy)$ be defined by $x = \{yy\}$.