10

Logic of Relations and Diagrammatic Reasoning: Structuralist Elements in the Work of Charles Sanders Peirce

Jessica Carter

1. Introduction

This chapter presents aspects of the work of Charles Sanders Peirce, illustrating how he adhered to a number of the pre-structuralist themes characterized in the introduction to this volume. I shall present aspects of his contributions to mathematics as well as his philosophy of mathematics in order to show that relations occupied an essential role. When writing about results in mathematics he often states that they are based on his "logic of relatives," and he refers to the reasoning of mathematics as "diagrammatic reasoning." Besides pointing to structural themes in Peirce's work, much of this exposition will be devoted to explaining what is meant by these two phrases.

In a recent article Christopher Hookway (2010) places Peirce as an *ante rem* structuralist.[1] In support of this claim Hookway refers to some of Peirce's writings on numbers (also to be treated here). In addition he spends some time analyzing what Peirce means by the phrase "the form of a relation." These considerations involve an in-depth knowledge of Peirce's categories and their metaphysical implications. In contrast I will focus on methodological aspects, in particular Peirce's writings on *reasoning* in mathematics, stressing that mathematics consists of the *activity* of drawing necessary inferences. This leads to a position that resembles methodological structuralism as it is characterized in Reck and Price (2000). Furthermore I find that Peirce's position is similar in spirit to the contemporary categorical structuralist views, in particular, as formulated by Steve Awodey (2004). Still I resist characterizing Peirce as a structuralist since I do not find that this label captures the richness of his views as presented here.

[1] But see Pietarinen (2010) arguing that Peirce's continuum cannot be a structure.

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To mention one point, besides claiming that mathematics is the science of necessary reasoning, Peirce has something to say about how this necessity is achieved.

The chapter consists of two main parts. The first documents Peirce's extensive knowledge of, and contribution to, the mathematics of his time. Areas include arithmetic, set theory, algebra, geometry (including non-Euclidean and topology)—and logic. Examples, together with indications of what drove his engagement with them, will be given from his work in geometry, arithmetic, set theory, and algebra. In relation to the pre-structuralist themes it can be mentioned that he presented different axiomatizations of the natural numbers. Furthermore his insistence on the inappropriateness of the characterization of mathematics as "the science of quantity" will be addressed. Finally we shall see that he draws a clear distinction between pure and applied mathematics—both in arithmetic and in geometry. I further note that Peirce's use of formal methods and his view of mathematics as an autonomous body places him as an early modernist according to the characterization given by Jeremy Gray (2008).

The second part is concerned with Peirce's philosophy of mathematics. It addresses Peirce's description of mathematical reasoning as diagrammatic reasoning. A diagram to Peirce is an iconic sign that represents rational relations. In order to explain what is contained in "diagrammatic reasoning" the chapter therefore includes a few relevant parts of Peirce's semiotics. In addition one example of a proof will be given in order to explain how mathematical, that is, necessary, reasoning proceeds by constructing and observing diagrams.

2. Mathematics and the Logic of Relations

A few biographic details are relevant.[2] Charles Sanders Peirce was born in Cambridge, Massachusetts, in 1839, as the son of Benjamin Peirce, a distinguished professor of mathematics at Harvard and a leading social figure. Benjamin Peirce taught his children mathematics and Charles certainly was very talented—as he was talented in so many fields. (Peirce had one sister and three brothers, of whom the oldest, James Peirce, became professor in mathematics at Harvard.) C. S. Peirce studied chemistry at Harvard and (in 1859), obtained a job at the US Coast and Geodetic Survey, and later, in 1879, was appointed a lecturer in logic at the Department of Mathematics at Johns Hopkins University. In 1884 his contract with Johns Hopkins was not renewed, and in 1891, due to disagreements, he was also forced to resign from his post at the US Coast and Geodetic Survey.

[2] This biographic information is based mainly on the introduction of Peirce (1976, vol. 1) and Gray (2008). I also recommend Brent (1998).

Peirce is often referred to as having a somewhat asocial behavior, something he admits and blames on his upbringing by his father focusing mainly on his formal training: "In this as in other respects I think he underrated the importance of the powers of dealing with individual men to those of dealing with ideas and with objects governed by exactly comprehensible ideas, with the result that I am today so destitute of tact and discretion that I cannot trust myself to transact the simplest matter of business that is not tied down to rigid forms" (NE IV, v).[3] Another peculiarity to mention is his habit of adopting his own terminology, e.g., calling relations "relatives" and writing "semeiotic" for "semiotics."

Two further things regarding Peirce's early years are worth mentioning here. First is Kant's influence on his thinking. Much of Peirce's thought is developed in reaction to the ideas of Kant; it is certainly the case that many of the ideas dealt with in this chapter are presented by Peirce with reference to Kant. Peirce writes (commenting on a text from 1867 introducing his categories) about his early influence by Kant, stating that he by 1860 "had been my revered master for three or four years" (CP 1.563). Second is Peirce's passion for logic. According to Peirce this passion was aroused by reading Whateley's *Logic*: "It must have been in the year 1851, when I must have been 12 years old, that I remember picking up Whateley's Logic in my elder brother's room and asking him what logic was. I see myself, after he told me, stretched on his carpet and poring over the book for the greater part of a week for I read it through. . . . From that day to this logic has been my passion although my training was chiefly in mathematics, physics and chemistry" (NE IV, vi).

There are two distinct periods in Peirce's contributions to logic (see Dipert 2004). The first is algebraic, using algebraic tools in order to formulate a calculus of the logic of relations with inspiration from (among others) Boole and de Morgan. A seminal paper in this period is his "Description of a Notation for the Logic of Relatives, Resulting from an Amplification of the Conceptions of Boole's Calculus of Logic" published in 1870 (reprinted in CP 3.45–148). The second and later period is characterized as "diagrammatic." In this period Peirce develops his existential graphs (see Roberts 1973 or Shin 2002).

An important part of Peirce's characterization of mathematics is his statement that mathematics is the science of necessary reasoning concerning hypothetical states of things. He attributes this claim to his father, writing: "It was Benjamin Peirce, whose son I boast myself, that in 1870 first defined mathematics as 'the

[3] Citations of Peirce follow traditional standards. (NE I, 3) refers to the collections *New Elements* edited by Carolyn Eisele (Peirce 1976) volume I, page 3. Similarly (CP 4.229) refers to the *Collected Papers of Peirce* edited by Hartshorne and Weiss (1931–1967) volume 4, paragraph 229. (EP 2, 7) refers to *Essential Peirce*, volume 2, page 7. I sometimes include a reference to the year the paper was written/published. This is available from R. Robin's catalog; see http://www.iupui.edu/~peirce/robin/robin_fm/toc_frm.htm.

science which draws necessary conclusions.' This was a hard saying at the time; but today, students of the philosophy of mathematics generally acknowledge its substantial correctness" (CP 4.229). The reference to 1870 is to *Linear Associative Algebra*, which opens with the statement C. S. Peirce quotes (B. Peirce [1870] 1881, 97). Peirce states at various places that the necessity of mathematical conclusions is obtainable precisely due to the hypothetical nature of mathematical statements, characterizing mathematics as the science "which frames and studies the consequences of hypotheses without concerning itself about whether there is anything in nature analogous to its hypotheses or not" (NE IV, 228). We shall return to these claims about mathematics throughout the chapter.[4]

2.1. Geometry

A good place to learn about the extent of his knowledge of geometry is his (unpublished) book *New Elements of Geometry Based on Benjamin Peirce's Works and Teaching,* which fills most of the second volume of the *New Elements of Mathematics* (Peirce 1976). As the title indicates, the book is an extension of his father's *Elementary Treatise on Geometry* (published in 1837), but it contains much more—apparently so much more that the publisher in the end refused to publish the book. When Peirce was forced to retire from his position in the US Coast and Geodetic Survey in 1891 he turned to textbook writing as a possible source of income. Ginn, the publisher of the American Book Company, made enquiries regarding an update of his father's book in 1894 (NE II, xiv). The introduction of NE II makes clear that Peirce worked for long on (versions of) the book while corresponding with the publisher, who did not see the need for publishing all the topics and sections Peirce wanted to include.[5] From the introduction it is possible to gain insight into Peirce's motivation for extending it as he wished to do. Given the developments of geometry during the 19th century, he found a substantial revision necessary. He lists a number of ways that geometry had "metamorphosed" since 1835: Given the acceptance of non-Euclidean geometries, Peirce claims, "geometry has two parts; the one deals with the *facts* about real space, the investigation of which is a physical, or perhaps metaphysical, problem, at any rate, outside of the purview of the mathematician, who

[4] Although Peirce claims that mathematics consists of the drawing of necessary conclusions, in some places he considers including the process of forming the hypotheses from which to reason as part of mathematics. See, for example, CP 4.238, where he praises the ingenuity of Riemann for developing the idea of a Riemann surface.

[5] See the correspondence between publisher, C. S. Peirce, and his brother, James (Jem) Peirce, professor of mathematics at Harvard (NE II, xiv–xxvii), also providing information about the different versions of the book.

accepts the generally admitted propositions about space, without question, as his *hypotheses*, that is, as the ideal truth whose consequences are deduced in the second, or mathematical part, of geometry" (NE II, 4). I return to this claim later. It is evident that Peirce was well informed about the various versions of non-Euclidean geometry formulated by Bolyai and Lobachewsky and even worked on both elliptic and hyperbolic geometry himself, claiming that space was hyperbolic (NE III, 710). To mention another thing, Peirce reviewed Halsted's[6] translation of Lobachevsky's geometry in *The Nation* (54, February 11, 1892), calling it an excellent translation. The next topic Peirce mentions among the areas that had not previously been included in his father's book is the new branch of geometry of Listing, named topology, which "deals with only a portion of the hypotheses accepted in other parts of geometry; and for that reason, as well as because its relative simplicity, it should be studied before the others."[7] The subsequent topic is what is today denoted as projective geometry. He then mentions "metrical geometry," which, he writes, was revolutionized after 1837 based on the contributions of Gauss's students Lobachewsky, Riemann, and Bolyai (building on the works of Lambert and Saccheri).[8] Finally, Peirce mentions the work of Cantor and others who "have succeeded in analyzing the conceptions of infinity and continuity, so as to render our reasonings concerning them far more exact than they had previously been" (NE II, 5).

Throughout his writings one finds explicit statements separating pure geometry, which traces the consequences of hypotheses, from "applied geometry," which makes enquiries about the properties of real space and so is a branch of physics.[9] At other places the distinction is implicit, as in the paper "Synthetical Propositions À Priori" (NE IV, 82–85). The aim of this paper is to show—opposing Kant—that mathematical propositions are not synthetic. He remarks that it is *possible* that the propositions of geometry could be regarded as statements concerning physical space, but consistent with his general claims

[6] Georg Bruce Halsted was a student of Sylvester's from John Hopkins University and became professor at the University of Texas in 1884. According to Eisele "Halsted was spearheading in his publications on the new geometry the effort to bring to mathematicians in America the awareness of the revolution in mathematical thought" (NE II, ix).

[7] Peirce also made contributions to topology (see Havenel 2010 for an account of this). Furthermore, Havenel notes that topology is "*par excellence* the mathematical doctrine that is incompatible with the widespread idea that mathematics is the science of nothing else than quantities, geometrical quantities, and numerical quantities, for the topological properties do not involve measurement" (Havenel 2010, 286).

[8] He makes references to Cayley (in 1854) and Klein (in 1873). In 1854 Cayley published a paper on finite groups, showing which multiplication tables are possible for a given number of elements of the group. Later, as Peirce indicates, Klein used the concept of a group and definitions of a metric (due to Cayley) to propose that the different geometries could be defined in terms of the invariance of properties of figures under a group of transformations, what is known as the Erlangen program.

[9] For explicit statements about the distinction between pure and applied geometry see NE IV, 359 and NE III, 703–709.

of mathematics, he concludes that to the mathematician they are simply held to be *hypotheses*: "nothing but ignorance of the logic of relatives has made another option possible" (NE IV, 82). He implicitly refers to the introduction of Riemann's *Über die Hypothesen, welche der Geometrie zu Grunde liegen* ([1854] 1892), calling it "Riemann's greatest memoir." According to Peirce, Riemann writes that geometrical propositions are matters of fact, and as such not necessary, but only empirically certain; they are hypotheses. Referring to the last part of this statement, Peirce comments: "This I substantially agree with. Considered as *pure* mathematics, they define an ideal space, with which the real space approximately agrees" (84–85).[10]

2.2. Foundations of Arithmetic

When Peirce writes about arithmetic, he distinguishes between different versions. The first is arithmetic as used in counting and calculations, which he denotes "vulgar" arithmetic (see NE I, xxxv and CP 1.291) or practical arithmetic. The other is pure arithmetic, concerning the abstract dealings with the properties of (the operations on) numbers. What is in particular worth mentioning in this context is that he bases both on axioms, and that proofs use what he denotes the "logic of relatives." (Note that the use of "axioms" here is my terminology. Peirce refers to them as "primary propositions" or "definitions." In general he is wary of using the label "axiom," which at the time referred to propositions held to be indubitably true.)[11] Note that one may find statements claiming that even practical arithmetic is based on (ideal) hypotheses: "2 and 3 is 5 is true of an idea only, and of real things so far as that idea is applicable to them. It is nothing but a form, and asserts no relation between outward experiences" (NE IV, xv).

Peirce's axiomatizations of (practical) arithmetic intend to prove that arithmetical propositions are logical consequences of a "few primary propositions," that is, countering Kant's view that arithmetical propositions are synthetic. In "The Logic of Quantity" from 1893 (CP 4.85–93) Peirce addresses Kant's

[10] See the article by J. Ferreirós (2006) for an interpretation of how Riemann understood "hypothesis" (and foundations). According to Ferreirós, Riemann used the word "hypothesis" instead of "axiom" precisely to emphasize that they are not evident.

[11] "The science which, next after logic, may be expected to throw the most light upon philosophy is mathematics. It is a historical fact, I believe, that it was the mathematicians Thales, Pythagoras, and Plato who created metaphysics, and that metaphysics has always been the ape of mathematics. Seeing how the propositions of geometry flowed demonstratively from a few postulates, men got the notion that the same must be true in philosophy. But of late mathematicians have fully agreed that the axioms of geometry (as they are wrongly called) are not by any means evidently true. Euclid, be it observed, never pretended they were evident; he does not reckon them among his κοιναὶ ἔννοιαι or things everybody knows, but among the αἴτηματα, postulates, or things the author must beg you to admit, because he is unable to prove them" (CP 1.130).

characterization of analytic judgments, finding that Kant's thought "is seriously inaccurate" (even calling it "Monstrous"). The distinction between analytic and synthetic judgments depends on whether a predicate is *involved* in the subject, or "whether a given thing is consistent with a hypothesis." Peirce accuses Kant of, due to insufficient knowledge of logic, confusing a question of logic with psychology when he writes that being *involved* in the conception of the subject is the same as being *thought* in it (CP 4.86). According to Peirce the question is easily resolved if one is familiar with the logic of relatives. Its solution does not depend on "a simple mental stare or strain of mental vision. It is by manipulating on paper, or in the fancy, formulae or other diagrams—experimenting on them, *experiencing* the thing" (CP 4.86). That is, whether a judgment is analytic or not can be determined by the use of logic and is an objective fact, not something depending on our thoughts. Note here also Peirce's formulation "experimenting on a diagram" and "experiencing the thing," which are central to his characterization of mathematical and diagrammatic reasoning, as I shall explain in the second part of the chapter. Concerning the status of arithmetic, he continues: "the whole of the theory of numbers belongs to logic; or rather it would do so, were it not, as pure mathematics, *prelogical*, that is, even more abstract than logic" (CP 4.90). Peirce holds that the different sciences can be ordered according to the generality of the objects they concern. In this philosophical system Peirce places mathematics at the top level, being the science that draws necessary conclusions, and logic as part of philosophy just below. Logic, according to Peirce, studies the drawing of necessary conclusions done in mathematics in order to formulate "laws of the stable establishment of beliefs" (CP 3.429). One may therefore note that Peirce, in contrast to, for example, G. Frege, who also took an interest in the foundations of arithmetic, did not claim that arithmetic is reducible to logic.[12] Another interesting point is Peirce's remark that there is no one unique way to found arithmetic on the logic of relations: There "are even more ways in which arithmetic may be conceived to connect itself with and spring out of logic" (CP 4.93). To document this claim Peirce refers to some of the texts presented in what follows.

Another motivation for providing axioms for arithmetic is to counter the empiricism of J. S. Mill, referred to in the introduction to the first paper presented in

[12] It should be noted, though, that Peirce in his 1881 paper introducing the numbers writes things that could be construed as approaching a logicist position. He first writes that the aim of the paper is to show that the truths of arithmetic are consequences of a few primary propositions. He states about these propositions (calling them definitions), "the question of their logical origin . . . would require a separate discussion" (CP 3.252). For a more precise formulation of the interrelation between logic and mathematics in terms of how the different subjects, i.e., mathematics, philosophy, logic, etc., relate in Peirce's system see Stjernfelt (2007, 11–12). Peirce writes, for example, the following about the relations between the subjects ("sciences") in his system: "The general rule is that the broader science [e.g., mathematics] furnishes the narrower with principles by which to interpret its observations while the narrower science furnishes the broader science with instances and suggestions" (NE IV, 227).

section 10.2.2.1, a position quite influential at the time. (Peirce implicitly refers to Mill as “a renowned English logician.”) In what follows three examples of characterizations of arithmetic are given, two of pure arithmetic and one of the “counting numbers.”

2.2.1. Basing Pure Arithmetic on the Logic of Relations

The first example comes from Peirce’s article “On the Logic of Number,” published in the *American Journal of Mathematics* 4 (1881).[13] (The paper is reproduced in CP 3.252–288.) The system presented here has been called the first successful axiom system for the natural numbers (see Mannoury 1909 and Shields 1997, 43).[14] In the introduction Peirce writes that the aim of the paper is “to show that [the elementary propositions concerning number] are strictly syllogistic consequences from a few primary propositions” (CP 3.252). Peirce remarks that the inferences drawn are not exactly like syllogistic consequences but that they are of the same nature.

The numbers, or as Peirce refers to them, “a system of quantities,” is introduced as a collection together with a particular relation defined on it. The natural numbers are defined as a totally ordered (discrete) set with a minimum element, fulfilling the axiom of induction. In Peirce’s terminology they are a *semi-infinite, discrete and simple system of quantity.* In contemporary terms a simple system of quantity is a totally ordered set (i.e., the relation defined on the set is transitive, reflexive, and anti-symmetric and fulfills trichotomy). Furthermore, “discrete” and “semi-infinite” mean the set has a minimum element (called a semi-limited system of quantity) and fulfills the axiom of induction. In what follows we shall see how Peirce defines these notions. His use of notation (or perhaps lack thereof) may be a bit confusing to a modern reader. He uses the expression “one thing is said to be r of another,” meaning that one thing is *related* to another. In contemporary technical terms we would instead write that ArB or $(A,B) \in r$ for A being “one thing” and B “another.” Peirce also uses the formulation that “the latter be r’d by the former.” Listing properties that hold for the relation “less than

[13] Peirce sketches a characterization of the natural numbers even earlier than 1881 in the paper “Upon the Logic of Mathematics,” dated 1867 and published in the *Proceedings of the American Academy of Arts and Science*, vol. 7 (CP 3.20-3.44). It is based on his modifications of “the logical calculus of Boole.” He defines, e.g., “logical identity” and “addition,” corresponding at first to operations on classes. That is, addition corresponds to taking the union of two classes. Identity between two classes states they consist of the same elements. Toward the end of the article Peirce notes that if one considers a kind of abstraction on classes—“numerical rank”—identity will play the role of equality and by considering the operation disjoint union one obtains the rules of arithmetic, for example, that $a+b=b+a$.

[14] Shields presents Peirce’s axioms formulated in a modern way and compares his axiomatization to Dedekind’s and Peano’s. One thing Shields points out as worthy of attention is that Peirce chose a transitive relation as the basic relation when defining the numbers instead of the successor relation. Peirce later formulates systems based on an equivalent of the successor relation.

or equal to" and using "r" and "q" to stand for this relation, he states the fundamental properties of a system of quantities as follows: "In a system in which r is transitive, let the q's of anything include that thing itself, and also every r of it which is not r'd by it. Then q q may be called a fundamental relative of quantity" (CP 3.253). That is, Peirce defines a system of quantity to be a collection Q on which there is defined a relation, q, which fulfills that q is transitive and reflexive, and for any A in the collection, AqB holds for all B s for which BqA is not the case. (If one thinks of the relation $\leq$ on the numbers, the last property states that for any two numbers A and B, if not $B \leq A$ then $A \leq B$.) Peirce continues to list the properties of q, stating that "it is transitive; second, that everything in the system is q of itself, and, third, that nothing is both q of and q'd by anything except itself." The last is anti-symmetry. A relation fulfilling these three properties defined on a set is usually called a partial order on that set. He defines a simple system (of quantity) to be one in which it is the case that for any two elements, A and B , it is the case that either ArB or BrA(i.e., trichotomy).[15] Simple systems can be discrete, which means "every quantity greater than another is next greater than some quantity (that is greater than without being greater than something greater than)" (CP 3.256). For simple and discrete systems of quantity he introduces semi-limited systems, i.e., systems that have a limit, often an absolute minimum element (which he calls "one"). Finally Peirce considers this class of quantities (that is, a simple, discrete system with a minimum element), noticing that "an infinite system may be defined as one in which from the fact that a certain proposition, if true of any number, is true of the next greater, it may inferred that that proposition if true of any number is true of every greater" (3.258). That is, Peirce notes that what is today called the induction axiom characterizes the natural numbers.[16] Elsewhere Peirce denotes this principle by Fermatian inference.

In the next paragraph Peirce continues to study "ordinary number," which can be defined as a semi-infinite (that is semi-limited and infinite), discrete, and simple system of quantity, defining addition and multiplication (using the notion of predecessor), and he shows how one may then prove a number of fundamental propositions of arithmetic by induction, e.g., associativity, commutativity of addition, and the distributive law.[17] I present one of Peirce's proofs that addition is

[15] In modern terminology a set on which there is defined a partial order fulfilling trichotomy is denoted a totally ordered set.

[16] Note the unfortunate choice of terminology calling a system of quantities for which the induction axiom holds an "infinite" system. The reason behind this might be that Peirce contrasts infinite systems with finite systems at the end of the paper.

[17] In a later paper from 1901–1904 (NE IV, 2–3) Peirce has introduced more notations, for example "G" denoting the successor function, but essentially maintains the same characterization of numbers. In this paper he shows that the associative law holds for numbers, where numbers are defined as an ordered system on which induction holds. Two numbers are defined to be equal in terms of the relation "greater than," where A = B means that A is at least as great as B and B is as least as great as A. Furthermore he notes that "as least as great as" is transitive and reflexive and if $N \geq M$, then

commutative; that is, using$x$ x and y x to denote natural numbers, $x + y = y + x$ (see CP 3.267). Addition is defined by the following two rules (here adding "s" to denote successor): $1 + y = s(y)$ and $x + y = s(x' + y)$, where x' denotes the predecessor of x. The proposition is proved using induction twice, and it employs the associative rule, $x + (y + z) = (x + y) + z$, that Peirce proves first. In the first step it is shown that the statement holds for $y = 1$, namely that $x + 1 = 1 + x$. I omit the details from this part. For the general proposition, one may now note that $x + y = y + x$ has been proven for $y = 1$. In order to conclude the statement by induction, it thus remains to show that if the statement holds for $y = n$, then it holds for $y = 1 + n$. We suppose that $x + n = n + x$ and consider $x + (1 + n)$. Calculating on—or manipulating—this expression, we obtain the following (which Peirce would refer to as a diagram):

$$
\begin{gathered}
x + (1 + n) = \\
(x + 1) + n = \\
(1 + x) + n = \\
1 + (x + n) = \\
1 + (n + x) = \\
(1 + n) + x.
\end{gathered}
$$

Here we have used associativity of addition, the result that $x + 1 = 1 + x$, , and the induction hypothesis. The diagram displays that if $x + n = n + x$, , then $x + (n + 1) = (n + 1) + x$ holds. Combining this with the fact that $x + 1 = 1 + x$ and using the principle of induction, the result follows.

Note that the (natural) numbers are defined as a *relational system*, that is, as a collection on which is defined a certain order relation. Peirce formulates properties of relations, e.g., transitive and "quantitative" relations, in his language of

$GN \geq GM$. Finally he formulates the axiom of induction: "whatever is true of zero and which if true of any number N, is also true of GN the ordinal number next greater than N, is true of all numbers" (NE IV 2). The addition of numbers is defined as follows: (i) 0 + 0 = 0, (ii) GM + N = G(M + N), (iii) M + GN = G(M + N). By successive use of these definitions and induction, he is able to prove the stated proposition.

the logic of relatives in many of his papers. To mention a couple of examples, he expresses formally a one-one relation in “On the Logic of Number” (1881) and properties of transitive relations in “The Logic of Quantity” (1893).

In a collection of papers called *Recreations in Reasoning* dated around 1897 Peirce defines the numbers by something very close to the Dedekind-Peano axioms. More precisely he adopts notation for what plays the role of a successor and states basic properties of this relation. These properties together with what Peirce derives as a consequence of them constitute what is now known as the Dedekind-Peano axioms. Another thing to mention about this system is that Peirce *derives* the principle of induction from this system, and thus calls it the *Fundamental Theorem of Pure Arithmetic* (CP 4.165). Peirce continues to define the relation “greater than” on this system and deduces some properties so that this system is comparable to the first-mentioned example of defining the natural numbers via an order relation.

2.2.2. Practical Arithmetic or Demonstration That Arithmetical Propositions Are Analytic

The fundamental theorem in practical arithmetic, serving as the foundation for counting, is denoted “The Fundamental Theorem of Arithmetic.”[18] It states that “if the count of a lot of things stops by the exhaustion of those things, every count of them will stop at the same number” (NE IV, 82). Peirce contrasts this to the Fundamental Theorem for Pure Arithmetic. In the text “Synthetical Propositions à Priori” (NE IV, 82–85) he demonstrates that “5 + 7 = 12” follows from the fundamental proposition of arithmetic. Therefore “5 + 7 = 12” is not “synthetical” (but corollarial, since it follows directly from the definitions). From the fundamental proposition Peirce deduces the principle of associativity (A + B) + C = A + (B + C) and then that the equality 5 + 7 = 12 follows. The article continues to prove the fundamental proposition using the language of the “logic of relatives” (demonstrating, according to Peirce, that it is not synthetic). The main steps are listed here. First a finite collection is defined. A collection, A, is finite, if whenever there is (here using modern notation) a one-to-one function $\lambda: A \rightarrow A$, then it is necessarily onto.[19] (That is, there is no one-to-one correspondence

[18] Peirce explicitly writes that the proposition 5 + 7 = 12 is analytic (NE IV, 84). He here explains that the proposition is analytic since it follows by necessity from the definitions. In contrast are propositions that he calls “theorematic,” to which we return in the second part of the chapter. See Levy (1997) for a discussion concerning the relation between the synthetic and analytic distinction and corollarial vs. theorematic proofs. See also Otte (1997) on the analytic-synthetic distinction in Peirce’s philosophy.

[19] In Peirce’s terminology it is expressed as “Suppose a lot of things, say the As, is such that whatever class of ordered pairs λ may signify, the following conclusion shall hold. Namely, if every A is a λ of an A, and if no A is λ’d by more than one A, then every A is λ’d by an A. If that necessarily follows, I term the collection of As a *finite* class” (NE IV, 83).

between A and a proper subset of A.) In the next step it is shown that if a collection is counted, it is finite. To count a collection, according to Peirce, means to establish a one-to-one correspondence with the objects in the collection (taken in some order) and an initial segment of the natural number sequence. Finally, in the last step, it is demonstrated that if one assumes that a count of a sequence results in two different numbers, and the relation "next followed in the counting by" is employed, then there will be no least number in the sequence. This is a contradiction.[20]

2.3. Foundations: Set Theory

It is clear that underlying Peirce's conception of the various systems of numbers is a form of naive set theory. The same holds for his work in logic. Following Boole, Peirce's algebraic logic deals with classes. As an example one may point to his early paper "On an Improvement of Boole's Calculus of Logic" (1867), where Peirce uses letters to refer to classes of things or occurrences. It is then possible to define operations corresponding to addition and multiplication (and their inverses) on these classes (addition corresponding to, at first, union, and later to disjoint union). Such classes then form the basis for formulating the laws of arithmetic, as noted in note 13.

In later papers Peirce develops what can be denoted as versions of transfinite set theory along the lines of Cantor (and Dedekind)[21]—although he disagrees with Cantor on certain points. He often mentions the theory of multitudes, which is how he refers to cardinal numbers, and is well aware of Cantor's work on set theory, but developed some ideas independently. In particular Peirce claims credit for two results. One is the proof of the theorem that "there is no largest multitude," which in contemporary terms is that the cardinality of a set is strictly less than the cardinality of its power set. I return to this result later. The second is the definition of a finite collection as a collection for which the syllogism of transposed quantity holds.[22] In connection with his work on logic Peirce realized that the validity of inference rules depends on the size of the collection they are applied to (see "On the Logic of Number" from 1881 or the letter to Cantor in NE

[20] In the paper "On the Logic of number" (1881), referred to in section 10.2.2.1, a similar, although more complicated, proof is made.

[21] Peirce notes that his approach is closer to Cantor's since they both start with cardinal numbers, whereas Dedekind is concerned with ordinals.

[22] In his later years Peirce expresses his frustration (e.g., in CP 4.331) that he has not received more credit for his original ideas. For one thing, he accuses Dedekind of not giving him credit for his definition of a finite collection. Peirce writes in 1905 that he sent his 1881 paper, where he defines a finite collection, to Dedekind. There is no evidence, however, that Peirce's definition served as inspiration for Dedekind since he formulated his definition of an infinite set as early as 1872 (see Ferreirós 2007, 109).

III(2), 772). The syllogism of transposed quantity is the following—using one of Peirce's own examples:

Every Texan kills a Texan.
Nobody is killed but by one person.

Every Texan is killed by a Texan.

This syllogism is only valid when applied to finite collections (of Texans), so a finite collection may be defined as a collection for which this syllogism is valid. If one translates the premises and conclusion to expressions using functions, it states the same as the definition given earlier. The first premise is that there is a function, $k: \text{Texans} \to \text{Texans}$, the second that this is one-to-one. The conclusion is that the function is onto.

A further thing to note is that Peirce, like others at the time, struggled to find a proper definition of a collection.[23] Such a characterization could serve as a hypothesis from which the properties of sets would follow, similar to what he had accomplished for the numbers. One definition offered is the following: "We may say that a collection is an object distinguished from everything which is not a collection by the circumstance that its existence, if it did exist, would consist in the existence of certain other individual objects, called its members, in the existence of these, and not in that of any others; and which is distinguished from every other collection by some individual being member of the one and not a member of the other; and furthermore every fact concerning a collection will consist in a fact concerning whatever members it may have" (NE IV, 9).

The paper "Multitude and Number," dated 1897, presents in some detail Peirce's contribution to the theory of multitudes (see CP 4.170–226). These notes start out by defining a relation "being a constituent unit of" that can be regarded as a membership relation. Via this relation he defines a collection, as "anything which is *u*'d by whatever has a certain quality or general description and by nothing else" (CP 4.171). Having defined collections, he defines the notion of multitude to "denote that character of a collection by virtue of which it is greater than some . . . others, provided the collection is discrete" (CP 4.175). A collection is discrete if its constitutive units are or may be distinct as opposed to a continuous collection. Equality of collections is defined in terms of one-to-one relations: That the "collection of M's and the collection of N's are equal is to say: There is a one-to-one relation, c, such that every M is c to an N; and there is a one-to-one

[23] See Dipert (1997) for a discussion of Peirce's philosophical conception of sets. Noting the difficulty of providing a characterization of a set, Dipert furthermore presents Peirce's subtle criticism of Dedekind's definition of an infinite collection. For this criticism see (CP 3.564).

relation, d, such that every N is d to an M" (CP 4.177). Before dealing with the different types of multitudes, Peirce addresses a question of which kinds of relations are meaningful on collections, mentioning in particular what we would denote as trichotomy. Peirce's classification of multitudes can be compared to Cantor's treatment of cardinal numbers (Peirce also refers to his papers, e.g., in CP 4.196). But Peirce disagrees with his names, calling them enumerable (finite), denumerable (countable), primipostnumeral (first uncountable), secundopostnumeral, etc. When dealing with the countable collections he shows standard propositions, e.g., that the product of two denumerable multitudes is a denumerable multitude. Furthermore he uses Cantor's notation for cardinal numbers, i.e., the alef, $\aleph$. Whenever moving on to the next multitude, Peirce writes that the problem is to determine the smallest multitude exceeding the previous (CP 4.200). For example, the section on the "primipostnumeral" begins: "Let us now enquire, what is the smallest multitude which exceeds the denumerable multitude?" Interestingly he finds that a way to obtain a primipostnumeral collection is by taking the collection of subsets of a denumerable set, and so he implicitly accepts the Continuum Hypothesis. He shows that this has the same multitude as, e.g., the collection of quantities between zero and one. He also argues that the size of this is $2^{\aleph}$ and that in general larger multitudes can be obtained by taking further powers.

Taking the collection of subsets as a larger collection corresponds to Peirce's theorem, namely that there is no largest multitude. Peirce seems to be particularly fond of this theorem as he presents many different proofs of it. The proofs are often used to illustrate various points: In the "Prolegomena for an Apology to Pragmaticism" the proof serves as an example of "diagrammatic reasoning." In other places it is given as an example of "theorematic" reasoning, something Peirce contrasts with "corollarial" reasoning. I return to these notions in the last part of the chapter.

In addition to studying multitudes, Peirce engages himself with a characterization of the continuum that he in the paper just treated argues cannot be a multitude. The reason is the stated property, that there is no greatest multitude. For one thing Peirce finds that it is possible "in the world of non-contradictory ideas" to consider the aggregate of all postnumeral multitudes and that this aggregate cannot be a multitude. It must instead be a continuous collection. There are both mathematical and philosophical angles to Peirce's thoughts on the continuum. Here I will make a few remarks pertaining to the mathematical ones.[24] First, one

[24] Scholars have explained how "continuity" is fundamental to Peirce's mature philosophy; see Hookway (1985), Stjernfelt (2007) and Zalamea (2010). Moore (2015) evaluates Peirce's description from a mathematical point of view. Dauben (1982) presents in some detail Peirce's conception of the continuum from the point of view of set theory.

may mention that Peirce's conception of the continuum has little to do with the project of rigorization of analysis, which led, e.g., Weierstrass and Dedekind to formulate their versions of the mathematical continuum, although he is aware of these developments. He is critical of the replacement of infinitesimals with the "cumbrous" method of limits pointing to the odd formulations mathematicians made, such as defining a limit "as a point that can 'never' be reached," stating that "This is a violation not merely of formal rhetoric but of formal grammar" (CP 4.118). Furthermore he objects to the characterization of the continuous line as composed of points and mentions topology and projective geometry as areas where continuous quantity in this sense does not enter at all (see CP 4.218–225, 3.526).

Peirce's and Cantor's motivations for engaging in set theory are thus quite different, and both had motives and sources of inspiration besides the mathematical. It is usually said that Cantor's initial inspiration came from analysis, where he worked on the conditions for unique representation of functions by trigonometric series. Peirce, on the other hand, was first influenced by his work in logic and later his interest in mathematics in general. He also had a philosophical motive, and doubly so, since mathematics (and logic) served as a foundation for his philosophical system.

2.4. Algebra

In this section I address another theme from what can be denoted Peirce's use of the axiomatic method. More importantly, the examples from Peirce's writings on algebra illustrate his emphasis on the inadequateness of the claim that mathematics is the "science of quantity." He writes things like "To this day, one will find metaphysicians repeating the phrase that mathematics is the science of quantity,—a phrase which is a reminiscence of a long past age when the three words 'mathematics,' 'science,' and 'quantity' bore entirely different meanings from those now remembered. No mathematicians competent to discuss the fundamentals of their subject any longer suppose it to be limited to quantity. They know very well that it is not so" (NE IV, 228–229). Furthermore, I will note his characterization of algebra as a system of symbols functioning as a calculus, i.e., a language to reason in. Part of Peirce's knowledge of algebra stemmed from his father, Benjamin Peirce, including his monograph *Linear Associative Algebra*, first published in 1870, the same year as Peirce's remarkable paper on the logic of relatives (that is, his "Description of a Notation for the Logic of Relatives, Resulting from an Amplification of the Conceptions of Boole's Calculus of Logic," CP 3.45–149). Peirce remarks that he and his father discussed the contents of both with each other, writing: "There was no collaboration, but there were frequent conversations on the allied subjects, especially about the algebra" (NE III, 526). Inspiration for this work clearly comes from the British algebraists and the emerging way of designing algebras by detaching symbols of their traditional meaning (denoting numbers), and simply focusing on the rules of combinations. This work had a boost from Hamilton's introduction of the quaternions, where it turned out that multiplication is not commutative. In a sense B. Peirce's *Linear Associative Algebra* can be seen as a generalization of the work of Hamilton, dealing in general with systems—or algebras—of expressions formed as linear combinations of a given number of elements. In C. S. Peirce's writings there are numerous examples from linear associative algebra, but the examples to be considered here concern the imaginary numbers and permutation groups.

In a section of the paper "The Logic of Quantity" Peirce discusses the imaginary number i. This (long) paper starts out with the criticism of Kant's claim that mathematical propositions are synthetic, as referred to in section 10.2.2.[25] Peirce starts by praising Cauchy for giving the first "correct logic of imaginaries," but regrets that the rule-of-thumbists "do not understand it to this day" (CP 4.132). They object that there cannot be a quantity that is neither positive nor negative and that the square of a quantity is always positive. Despite this Peirce explains how it is possible to introduce a quantity whose square is negative. The mathematician "would reason indirectly: that is the mathematician's recipe for everything" (CP 4.132). The algebraist simply states that he needs a quantity whose square root is -1, noting: "there is no such thing in the universe: clearly then, I must import it from abroad" (CP 4.132). Peirce's explanation displays his use of the axiomatic method. He lists the fundamental properties of numbers[26] (quantities), stating that "If there is one of those laws which requires a quantity to be either positive or negative, find out which it is and delete it. If you have a system of laws which is self-consistent, it will not be less so when one is wiped out." Peirce deduces that the property "(16) $x > 0$ or $x < 0$ or $x = 0$" is required in order to prove that the square of all (nonzero) numbers is positive. The conclusion is that if this property is deleted, one may introduce the hypothesis that there is a quantity, i, defined as the square root of -1. The symbols so introduced have no other meaning than given by the hypotheses,[27] i.e., the meaning of i is that $i^2 + 1 = 0$

[25] "The Logic of Quantity" is dated 1893. It was supposed to be included in Peirce's book *The Grand Logic*. It is a long paper starting out with criticisms of the positions of Kant and Mill on mathematics. The ensuing sections deal with the logic of quantity, that is, expressing properties of quantitative relations in his language of relations and deriving their consequences. Toward the end are sections treating the imaginary quantities, quaternions, and a section of measurement and infinitesimals.

[26] The listed properties of quantities include, for example, the commutative and associative properties of addition and multiplication and properties of the relation "less than."

[27] In CP 4.314 a similar statement is made, i.e., that symbols have no meaning other than that we give them. The example in this case concerns developing an algebra of three elements.

: "the meaning of a sign is the sign it has to be translated into" (CP 4.132). In this way the system of symbols of algebra becomes a calculus; "that is to say, it is a language to *reason in*" (CP 4.133). He continues: "To say that algebra means anything else than just its own forms is to mistake an *application* of algebra with the *meaning* of it" (CP 4.133).

In order to define a complex number, reference to numbers (and so quantities) is required. But a complex number goes beyond quantities since relations must be introduced that do not fulfill the properties of relations defining quantities, e.g., transitivity and the like: "[It] is readily seen that what is called an imaginary quantity or a complex quantity is not purely quantity" (NE IV, 229). To show that there are examples from mathematics that have nothing to do with quantity whatsoever, Peirce presents the notion of a group: "By a 'group,' mathematicians mean the system of all the relations that result from compounding certain relations which are fully defined in respect to how they are compounded" (NE IV, 229). As an example of a group where these relations have nothing to do with quantity he presents what is essentially a group of permutations, where elements are permutations on the four letters A, B, C, and D. The group is presented as containing relations that, composed by themselves four times, give the identity. That is, if l represents such a relation, it fulfills that $l^4 = \mathrm{Id}$. One such relation is $D:A+C:B+A:C+B:D$, meaning that that D maps to A, A to C, C to B, and finally B to D. Today this could be written in cyclic notation as $(DACB)$. He notes there are more such relations, 24 in total, that they have converses (i.e., inverses), and refers to the product of such relations—which he notes has a logical meaning having nothing to do with quantity (similarly to the use of "+" above in the presentation of the permutation). The totality of these 24 relations thus forms a group. Furthermore he talks about smaller sub-collections of the 24 elements that will also form a group (NE IV, 227–234, 1905–6).

2.5. Conclusions: Peirce, Pre-structuralist Themes, and Relations

Summing up on Peirce's adherence to a number of pre-structuralist views, I have noted Peirce's distinctions between physical geometry and mathematical geometry on the one hand and practical and pure arithmetic on the other. Regarding the first distinction, he remarks that the hypotheses in pure geometry are studied irrespective of whether they apply to the real world or not. I showed that it is possible to find similar comments referring even to practical arithmetic. In the writings on algebra I also noted that Peirce several times explicitly rejects the characterization of mathematics as the science of quantity, producing examples that have nothing to do with quantity.

Furthermore I have shown that Peirce uses formal methods in arithmetic to determine which hypotheses are sufficient in order to derive the properties in question. One aim was to argue that the statements of arithmetic are logical consequences of certain definitions, or hypotheses. His use of the "axiomatic method" in arithmetic can be likened to the process described by Hilbert ([1918] 1996) where, given a collection of propositions, a certain collection of axioms can be identified so that the given propositions can be derived from them—what Hilbert calls "deepening of foundations." This seems to fit well with Peirce's procedure. His method thus has two interrelated aims. Focusing on reasoning and inference rules, the point is on the one hand to formulate "a few primary propositions" of the numbers so that properties of them follow by necessity. On the other hand, focusing on the propositions, the aim is to determine the postulates sufficient for deriving the propositions of arithmetic. Peirce's discussions of the meaning of "postulates" and "hypotheses" reflect these concerns: "For what is a postulate? It is the formulation of a material fact which we are not entitled to assume as a premiss, but the truth of which is requisite to the validity of an inference" (CP 6.41). A further similarity to Hilbert's method is Peirce's claim that there are multiple ways of organizing the propositions of arithmetic (cf. CP 4.93). One could take as basic the propositions defining the numbers via the successor function or the definition of numbers as a certain ordered collection.

In the case of the imaginary quantity, I indicated how Peirce traces out the consequences of a body of fundamental properties of the numbers, in order to determine which of these contradicts a desired property (i.e., that the square of a quantity is negative). In this case, he mentions the property of a collection of axioms of "being internally consistent." It does not seem, however, that he is concerned with further metamathematical considerations such as consistency in general, independence, and completeness. He appears to be quite confident in the mathematical method, writing in numerous places "in mathematics there are no mistakes and no (deep) disagreement" (CP 3.426).

Peirce's use of formal methods as well as his distinction between pure and applied versions of mathematics places him as an early modernist, characterized by J. Gray (2008) as "an autonomous body of ideas, having little or no outward reference, placing considerable emphasis on formal aspects of the work and maintaining a complicated—indeed anxious—rather than a naïve relationship with the day-to-day world, which is the de facto view of a coherent group of people, such as a professional or discipline based group that has a high sense of what it tries to achieve" (1).

After the many of examples of the mathematics of Peirce we may better understand what is meant when stating that a result or theory is based on the logic of relations. The first thing to note is that Peirce finds that relations of various sorts play a key role in the definition of mathematical objects. Having seen the examples presented here, we must concur. We have seen that, for example, an order relation is used to define the numbers and a bijective correspondence is used to define multitudes as well as "a count." The properties of these can be formulated in his language of the logic of relations. Second, to Peirce the main activity of mathematics is reasoning, that is, the practice of drawing necessary conclusions. Logic, according to Peirce, includes the study of (the methods of) such inferences. Peirce notes that he together with other logicians like de Morgan (NE IV, 1) early realized that the previous versions of logic came up short when trying to capture the structure of the statements of mathematics.[28] To formulate definitions as well as statements in mathematics thus requires reference to relations, so reasoning in mathematics must take into account how one draws inferences from statements involving relations.

3. Philosophy: Diagrammatic Reasoning

I now turn to focus on how Peirce proposes the necessity of reasoning is achieved, namely through diagrammatic reasoning. The description given here draws mainly on Peirce's 1906 paper "Prolegomena for an Apology to Pragmatism" (PAP), published in *The Monist* (reprinted in CP 4.530–582), and a draft of this (NE IV, 313–330).[29] But others of Peirce's writings will also be referred to. My presentation focuses on how diagrammatic reasoning applies to mathematics. It thus complements the contributions of Stjernfelt (2007) on diagrammatic reasoning in general and Shin's (2002) account of his existential graphs. I also refer to Marietti (2010) for a more detailed account than I am able to give here.

There are two key points to bear in mind when addressing "diagrammatic reasoning." The first is that Peirce thinks of a diagram as a certain type of sign. An important property of this sign, the diagram, is that it is *observable*. Peirce explains that the necessity of the conclusion of a proposition is established because it can be perceived in the diagram. The second key point is that his definition of a "diagram" applies to objects that one would not normally count as diagrams. I mention three possible sources of inspiration for Peirce's view of reasoning as linked to observing diagrams: First, Peirce's work on logic contributed to this view. I will return to this point at the end of this section. Second, the reasoning based on

[28] In Peirce's early papers on logic (see, e.g., volume 3 of CP) there are sections on the Aristotelian syllogisms. But these are not used when he turns to his algebra of logic. One may also find comments as to the shortcomings of the syllogisms; see CP 4.426 in relation to Euclid's *Elements*.

[29] The last part of PAP consists of a presentation of the existential graphs. The paper also includes an explanation of which types of signs these graphs are. The iconic existential graphs were supposedly meant to pave the way for a proof of his pragmaticism: "For by means of this, I shall be able almost immediately to deduce some important truths of logic, little understood hitherto, and closely connected with the truth of pragmaticism" (CP 4.534). See also EP 2, xxvvii–xxix and Shin (2002).

diagrams in Euclid's *Elements*—a source Peirce is familiar with and often cites from—proceeds in a way that is compatible with the description of diagrammatic reasoning. Third, Peirce explicitly mentions Kant in connection with the characterization of mathematical reasoning. According to Kant reasoning in mathematics proceeds by constructions, or the drawing of diagrams, formed in intuition. Peirce remarks that this view is partially correct, since it focuses on the method of mathematics rather than stating what mathematics is about, and he agrees that mathematics deals with constructions—but not in intuition (CP 3.556, 1898). Peirce claims the necessity of mathematical reasoning is due to the procedure of constructing "a diagram, or visual array of characters or lines. Such a construction is formed according to a precept furnished by the hypothesis. Being formed, the construction is submitted to the scrutiny of observation, and new relations are discovered among its parts, not stated in the precept by which it was formed, and are found, by a little mental experimentation, to be such that they will always be present in such a construction" (CP 3.560). That is, although he agrees with Kant that reasoning is done by constructions, as I have noted, he disagrees with Kant that this construction invokes intuition and depends on "thought"—although a diagram might be considered in one's imagination. As noted, it is essential for Peirce that the relations discovered are observed.[30]

When Peirce refers to a "diagram" he does not only understand it in its common sense, that is, as a figure mainly composed of points, lines, and circles, since he also describes it as a "visual array of characters or lines." To Peirce "diagram" refers to a sign that represents (intelligible) relations: "a Diagram is an Icon of a set of rationally related objects . . . the Diagram not only represents the related correlates, but also, and much more definitely represents the relations between them" (NE IV, 316–317, 1906). Mentioning an "icon," he refers to his semiotics. The next section therefore extracts a few points from his theory of signs. This introduction will be followed by an example of a proof together with a further elaboration of how to understand his characterization of necessary reasoning as diagrammatic reasoning.

3.1. Signs: Tokens and Types; Icons, Indices, and Symbols

Early on Peirce attached importance to signs, conceiving of them as the vehicles of thought. His theory of signs is interrelated with his categories (at first developed as a response to Kant's 12 categories, see, for example, CP 1.545–567 from 1867). According to Peirce there are only three types of categories. The categories consist

[30] That relations are seen to hold because they are observed brings mathematics on a par with natural science. See Marietti (2010) for an elaboration of this point.

of feeling, reaction, and law—or as he also called them, possibility, existence, and habit.[31] One way Peirce arrives at these categories is in terms of his logic of relations. Any given relation applies to a fixed number of relata, and so a relation may be monadic, dyadic, or triadic, and so on. Peirce claimed that he could prove that higher-order relations are reducible to relations taking only one, two, or three relata.[32] The monadic relations (predicates) correspond to the first category (feeling or quality), dyadic to the second (reaction or existence), and irreducible triadic relations to the third (law or habit). Later Peirce referred to the categories more abstractly in his phaneroscopy as firstness, secondness, and thirdness.

A sign, according to Peirce, is an irreducible triadic relation (corresponding to the three categories): it relates the sign, the object that is represented by the sign, and the interpretant of the sign. The last is important, in that Peirce holds that a sign is not a sign unless it is interpreted as such: "a sign (stretching that word to its widest limits), as *anything which, being determined by an object, determines an interpretation to determination, through it, by the same object*)" (PAP CP 4.531). In Peirce's early classification of signs, each of these three, that is, the sign, the relation between the sign and object, and the interpretant, is considered in terms of the previously mentioned three categories: possibility, existence, and law.[33] I only mention two of these here. Peirce's first division concerns the nature of the sign itself. This division includes the well-known notions of a *token* and a *type*: "A common mode of estimating the amount of matter in a MS. or printed book is to count the number of words. There will ordinarily be about twenty *the*'s on a page, and of course they count as twenty words. In another sense of the word 'word,' however, there is but one word 'the' in the English language; and it is impossible that this word should lie visibly on a page or be heard in any voice, for the reason that it is not a Single thing or Single event. It does not exist; it only determines things that do exist. Such a definitely significant Form, I propose to term a *Type*. A Single event which happens once and whose identity is limited to that one happening or a Single object or thing which is in some single place at any one instant

[31] The paper "What Is a Sign" (Peirce 1894, EP 2, 4–10) explains the three categories in terms of possible ways experience can be had: The first, most immediate, is *feeling*, e.g., thinking about the color red. Second is *reaction*, as when we are startled by a loud noise and try to figure out its origin. The second category thus requires "two things acting on each other" (EP 2, 5). Third is thought, or *reasoning*, formulating a law based on our immediate experiences and actions. This is described as "going through a process by which a phenomenon is found to be governed by a general rule" (EP 2, 5). Note also that the third category mediates between the other two. See also Hoopes (1991).

[32] See Misak (2004, 21), Burch (1997), and the paper "Detached Ideas Continued and the Dispute between Nominalists and Realists" (NE IV, 338–339).

[33] Around 1903 (see *Syllabus* 1903, published in EP 2, 289–299) Peirce presents his classification of signs into 10 different classes. Later, after introducing a more elaborate theory of interpretants and a distinction between the immediate and the dynamic object, he is able to produce 66 classes of signs. Peirce refers to both of these additions in PAP. In addition to PAP, see Hoopes (1991) and Short (2007) for an elaboration of the development of Peirce's semeiotics. Bellucci and Pietarinen (n.d.) give an account in relation to logic and Carter (2014) in relation to use in mathematics.

of time, such event or thing being significant only as occurring just when and where it does, such as this or that word on a single line of a single page of a single copy of a book, I will venture to call a *Token*" (CP 4.537). The sign corresponding to the first category is named a *quality* or a *tone*. A diagram is to be taken as a *type*, but a type can only be shown through a replica of it, that is, a token.

The second division is his division of signs into *icons, indices*, and *symbols*. They appear as answers to the question: In what capacity does the sign represent the object? The sign may represent because of similarities (likeness) between the object and the sign, in which case the sign is an icon: "Anything whatever, be it quality, existent individual, or law, is an icon of anything, insofar as it is like that thing and used as a sign of it" (EP 2, 291). Simple examples of icons used in mathematics are geometric objects, such as drawn triangles and circles. Icons do not only represent by visual resemblance; an important, and a characterizing, property of the icon is that it reveals new facts about the object that it represents. As such they are essential to mathematics: "The reasoning of mathematicians will be found to turn chiefly upon the use of likenesses, which are the very hinges of the gates of their science. The utility of likenesses to mathematicians consists in their suggesting, in a very precise way, new aspects of supposed states of things" (Peirce 1894, 6). As will be shown below, icons may represent relations. Note also that most icons used in mathematics involve conventional (symbolic) elements.[34] If I wish to prove something about an odd number, I could represent it iconically as "$2 \cdot k + 1$," for some number k, using the symbols "$\cdot$" and "+". Subsequently I will represent the statement that "a number divides another number" by the icon "$p \cdot k = a$."

The index represents its object because of some existent (causal) relation between the two. Peirce mentions as an example a weathercock, which, as a result of the wind blowing, tells us about the direction of the wind, so that the weathercock becomes an index of the direction of the wind. The type of index just mentioned represents due to some causal relation between the sign and the object. A pure index represents because of some purposeful association of it with what it represents, as one does in mathematics. Peirce mentions the geometers assigning of letters to geometric figures, naming places on such figures, so that one may reason about these places, points, lines, etc., via these letters.[35] This is obviously done in mathematics in general, as will be noted in the examples to follow.

[34] Peirce (CP 3.363) refers to the shading in Venn diagrams as a symbolic, or conventional element. See Carter (2018) for further examples of iconic representations in mathematics.

[35] In a paper published in 1885 Peirce characterizes an index as follows: "the sign [index] signifies its object solely by virtue of being really connected with it. Of this nature are all natural signs and physical symptoms. I call such a sign an *index*. . . . The index asserts nothing; it only says 'There!' It takes hold of our eyes, as it were, and forcibly directs them to a particular object, and there it stops. Demonstrative and relative pronouns are nearly pure indices, because they denote things without describing them; so are the letters on a geometric diagram, and the subscript numbers which in algebra distinguish one value from another without saying what those values are" (CP 3.361).

Finally, the sign could represent by virtue of a law, or a habit, stating that the particular sign refers to a certain kind of object. These are symbols. Examples of symbols are words; in mathematics we use symbols like "+," "π," etc.

3.2. Diagrammatic Reasoning

I now return to Peirce's description of the process of reasoning in mathematics. Reasoning consists of three steps: "following the precepts," (1) one constructs a diagram representing the conditions of a proposition and (2) one "experiments" on it until (3) one is able to read off the conclusion from the resulting diagram. This description seems to fit well (part of) the proof procedure in Euclid's *Elements*. Take, for example, proposition I.32, where it is proved that the sum of angles in a triangle is equal to two right angles. In order to prove this, a triangle ABC is drawn. In the next step, "experimenting on it," one extends the base line, say AB, and, from the starting point of this extended line, B, one draws a line parallel to AC. Reasoning in this diagram, one comes to see that the conclusion holds. What is remarkable is that Peirce finds that the above characterization also holds for mathematics in general, where the notion of "diagram" extends according to the preceding usage: "for even in algebra, the great purpose which the symbolism subserves is to bring a skeleton representation of the relations concerned in the problem before the mind's eye in a schematic shape, which can be studied much as a geometric figure is studied" (CP 3.556). (See also NE IV, 158.) The example of diagrammatic reasoning given by Peirce in PAP is the proof of the above-mentioned theorem that there is no largest multitude.[36] I present instead a (simpler) algebraic proof, proving that "if an integer divides two other integers, then this integer divides any linear combination of the two."[37] Introducing indices, a, b and p standing for the numbers and the symbol "|" to denote "divides,"[38] the proposition can be expressed as

For p, a *and* b *being integers, if* $p \mid a$ *and* $p \mid b$ *then* $p \mid sa + tb$ *for any integers* s *and* t.

[36] Peirce has a number of different formulations of this theorem in PAP, for example, "the single members of no collection or plural, are as many as are the collections it includes, each reckoned as an single object" (CP 4.532).

[37] Note that this example is not taken from Peirce. It is introduced by the author in order to explain "diagrammatic reasoning."

[38] $n|m$ means that there exists a number k such that $kn = m$. Using this notation it is for example the case that $2|8$, $-2|8$, and $3|-39$.

In order to prove this theorem we follow the three steps given previously. First we have to "form a diagram according to a precept of the hypothesis." That is, considering the antecedent of the proposition and translating the definition(s) used, we write down, in this particular case, the relations stated to hold between the numbers p,a and p,b respectively (cf. "calculating with a system of algebraic symbols"). The diagram thus obtained is that there exist numbers k and l such that

$$kp = a \text{ and } lp = b.$$

In the second step this diagram is experimented on; the signs are manipulated by using relevant (and valid) algebraic formulas:

$$\text{If } kp = a \text{ and } lp = b \text{ then } skp = sa \text{ and } tlp = tb\ .$$

Combining (adding) the last two we see that

$$sa + tb = skp + tlp = (sk + tl)p.$$

Noting that $sk + tl$ must be an integer since s, k, t, and l are all integers, one is able to observe that p divides the linear combination of a and b. It is thus possible to read off the conclusion of the proposition in the final line—corresponding to the third step.

Combining the preceding and leaving out the explanatory text so that it is in fact a "visual array of characters" makes it easier to appreciate why Peirce insists on calling it a diagram. p | a and p | b is represented as

kp = a and lp = b.
$kp = a$ and $lp = b$ implies that $skp = sa$ and $tlp = tb$.
$sa + tb = skp + tlp = (sk + tl)p$.

Observation of the last line tells us that p divides the linear combination, which is the conclusion.

A further, and most important, point is that by going through this diagram[39] one should be able to see that the conclusion follows by necessity from the stated condition. Relations referred to thus subsist on two different levels, as indicated by the following explanation: "a Diagram is an Icon of a set of rationally related objects . . . the Diagram not only represents the related correlates,

[39] In fact Peirce urges the reader to construct a diagram herself while following the instructions of the proof.

but also, *and much more definitely represents the relations between them*" (NE IV, 316–317, 1906, my emphasis). In the first stage of constructing the diagram relations referred to are relations that hold between numbers, the main relation used being the relation of "a number dividing another." At the second level are what can be denoted *logical* relations. Recall that a major interest for Peirce when studying mathematics was to extract the principles of drawing necessary conclusions. The stated purpose of the "Prolegomena" is precisely to argue that all necessary reasoning is diagrammatic reasoning, assuming that mathematical reasoning is necessary reasoning. What is achieved by the process of diagrammatic reasoning is that one comes to *see* the necessary relation that holds between the hypothesis and the conclusion of the proposition, that is, what I here refer to as a logical relation. In support of this view, in a passage telling us how to do proofs in mathematics (again referring to this as an activity) by constructing a diagram, making alterations to it, and comparing these two diagrams, Peirce writes that finally "the book . . . will make it quite plain and evident to you that the relation *always will* hold exactly" (NE IV, 200). This last use of "relation" refers to the logical relation in question.[40] Recall also the proof given in section 2.2 that addition is commutative. I remarked that the signs produced constituted a diagram. The purpose of that diagram was to allow us to see (or deduce) that if $x+n=n+x$ then $x+(1+n)=(1+n)+x$ follows by necessity.

In the different versions of PAP, Peirce analyses which type of sign is involved in diagrammatic reasoning in order to address a number of issues, such as how the necessity of reasoning, and generality of the conclusions, are obtainable.[41] In these papers Peirce mentions his extended theory of interpretants.[42] According to Peirce the drawn diagram is a sort of hybrid sign. He stresses that a diagram is an icon, but of a special kind. A diagram *shows* that a consequence follows "and more marvellous yet, that it *would* follow under all varieties of circumstances accompanying the premisses" (NE IV, 318). Peirce explains that this is achieved since diagrams are *schemas*. Being drawn and so capable of being perceived, they are tokens. But they are at the same time representations of symbolic statements (actually the interpretant of a symbol) and so general: the diagrams

[40] As further support of this claim, the paragraphs CP 4.227–240 link Peirce's characterization of mathematics as the science that draws necessary conclusions with a description of diagrammatic reasoning.

[41] See Stjernefelt (2007, chap. 4) for a more elaborate explanation of these issues.

[42] The extension made by Peirce includes different interpretants, in PAP named the *immediate, dynamic*, and *final* interpretant. The immediate interpretant is how it "is revealed in the right understanding"—the meaning of the sign; the dynamic interpretant is the actual effect the sign has on some interpretant. The final interpretant is "the manner in which the Sign tends to represent itself to be related to its Object" (CP 4.536). Another addition is that all of these can partake in either firstness, "emotional," secondness, "energetic," or thirdness, "logical" or "thought" (CP 4.536).

are representations of (symbolic) statements like "The sum of the angles of a triangle is equal to two right angles" or "If a number divides two numbers, then it will divide any linear combination of those two numbers." In Peirce's words: "the Iconic Diagram and its Initial Symbolic Interpretant taken together constitute what we shall not too much wrench Kant's term in calling it a *Schema*, which is on the one side an object capable of being observed while on the other side it is General" (NE IV, 318).

Referring to "experimenting on a diagram" brings us to Peirce's distinction between corollarial and theorematic reasoning. In corollarial reasoning, the consequences of the hypotheses can be read off directly from the constructed diagram. Furthermore the proof only makes use of the definitions of the concepts presented in the proposition, whereas this is not the case for theorematic reasoning. Corollarial reasoning "consists merely in carefully taking account of the definitions of the terms occurring in the thesis to be proved. It is plain enough that this theorematic proof we have considered differs from a corollarial proof from a methodeutic point of view, in as much as it requires the invention of an idea not at all forced upon us by the terms of the thesis" (NE IV, 8). The theorematic proof referred to is a proof of his theorem that the cardinality of a collection is less than the cardinality of its power set. Another example of a theorematic proof is the proof of Euclid I.32, since additional lines have to be drawn.[43] The deductions of the properties of numbers are corollarial proofs (as well as the example mentioned in note 18).

I finally note that Peirce also worked with diagrams (closer to the ordinary meaning of diagram) in relation to logic. In several places Peirce notes the schematic shape of the presentation of arguments (as in the syllogism of the transposed quantity). As early as 1885 Peirce refers to syllogisms as "diagrams," stating that their purpose is to make it possible to observe the relations among the parts (CP 3.363). The reason this has not been noticed before, Peirce assumes, is that the constructions of logic are so simple that they are overlooked: "Why do the logicians like to state a syllogism by writing the major premiss on one line and the minor below it, with letters substituted for the subject and predicates . . . he has such a diagram or a construction in his mind's eye" (CP 3.560, 1898). Later, in Peirce's so-called diagrammatic period in logic,[44] the representations of logical propositions and inferences *were* diagrams, that is, figures composed of lines. See Figure 1 for an example of such a diagram (that is also an example of an existential graph).[45]

[43] Various interpretations have been proposed regarding the distinction between theorematic and corollarial reasoning; see Hintikka (1980) for a logical interpretation and Levy (1997) for a specific interpretation concerning the theorem that there is no largest multitude.

[44] See Dipert (2004).

[45] In this period Peirce studied the well-known diagrams of Euler and Venn making it possible to visualize the validity of arguments and used these as inspiration for developing his own systems

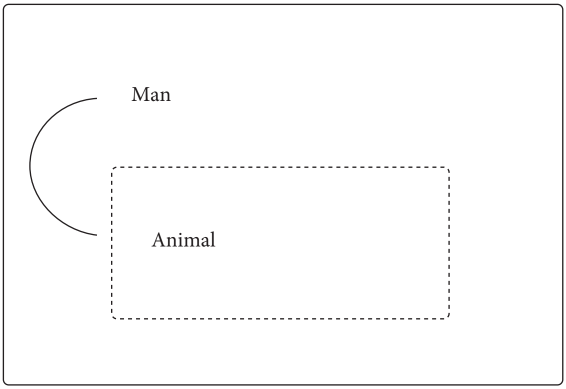

Figure 1 This diagram represents the statement "Any man would be an animal" or that nothing is both a man and not an animal. A box around p means "not p." A line joining p and q means p is related to q, in the sense that "some p is q."

In conclusion note the following. When writing down proofs in mathematics something *like* a diagram is formed. They are not exactly like the diagrams used in the *Elements*, but (relations between) concepts are represented in a schematic form that allows us to do things to them so that new relations become visible. Furthermore when Peirce managed to formulate his systems of logic, as in Figure 1, the representations *are* composed of letters and lines. In PAP Peirce comments that if all steps of a proof were to be spelled out, they would be reproducible by his graphs (NE IV, 319). The conclusion is that reasoning in mathematics involves (representations of) relations and that in the existential graphs these are displayed using diagrams so mathematical reasoning (which is necessary reasoning) is diagrammatic reasoning.

4. Structuralist Elements

In the introduction I proposed that Peirce could be interpreted as a methodological structuralist. Reck and Price (2000) characterize such a position by two principles. The first states that mathematicians "study the *structural features* of"

of logic, his existential graphs (see, e.g., CP 3.456–498, CP 4.347–371, and Bellucci and Pietarinen, n.d.). Besides Euler and Venn diagrams, other visual tools used in mathematics, chemistry, and their combination served as inspiration for these systems. In the mid-19th century "diagrammatic" notation was being developed and used both in chemistry and in graph theory. It was even proposed by Sylvester (a colleague of Peirce at Johns Hopkins) and Clifford to combine work in chemistry and the algebra of graphs around 1877 (see Biggs et al. 1976). Peirce was aware of the developments in both areas as well as the proposed link.

the entities assumed in their everyday practices, such as the various number systems, algebraic structures, various spaces, etc. Second, "it is (or should be) of *no* real concern in mathematics what the *intrinsic nature* of these entities is, beyond their structural features" (Reck and Price 2000, 45). Besides the emphasis on structure and structural features, this description resonates well with Peirce's emphasis that a mathematician only cares about deriving the consequences of her hypotheses. I have furthermore shown that the hypotheses, or definitions, formed by Peirce often characterize objects (e.g., the number systems) as relational systems. But I have also stressed, in particularly referring to the numbers, that Peirce found that there are different ways to define them, that is, there are multiple ways to logically organize the theory of numbers.

I have noted that Peirce did not seem to be interested in the foundations of mathematics, being convinced of the rigorousness of the reasoning of mathematicians and placing mathematics at the top of his philosophical system. These three elements, i.e., an "anti-foundationalist" view of mathematics,[46] the methodological structuralism, and the (relativism of) logical structure can also be found in the contemporary categorical structuralist view of Steve Awodey (2004). One component of Awodey's position is to "avoid the whole business of 'foundations'" (Awodey 2004, 55). Categorical structuralism rejects the idea of having a foundational system consisting of enough objects of some type, e.g.. sets, from which all mathematical objects may be built, and a collection of "laws, inference rules, and axioms to warrant all of the usual inferences and arguments made in mathematics about these things" (Awodey 2004, 56). In contrast structuralists advocate the "idea of specifying, for a given theorem or theory only the required or relevant degree of information or structure . . . for the purpose at hand, without assuming some ultimate knowledge, specification, or determination of the 'objects' involved. The laws, rules, and axioms involved in a particular piece of reasoning, or a field of mathematics, may vary from one to the next, or even from one mathematician or epoch to another" (Awodey 2004, 56). Awodey illustrates this top-town, or schematic, approach by the following example. Say one wishes to prove that if $x^2 = -1$ then $x^5 = x$.. The result follows in a field and a consequence is that $i^5 = -1$. A proof can also be found based on the axioms of a ring. Assuming even less, it can be proved in a semi-ring with identity that $x^2 + x + 1 = x$ implies $x^5 = x$. From a foundational (bottom up) point of view, one has to presuppose that the construction of the complex numbers as well as rings and semi-rings have been made in order to state these propositions. From a structuralist perspective the propositions are schematic statements about any structure (ring or semi-ring) fulfilling the appropriate conditions. There are also differences between Awodey's categorical structuralism and Peirce's position.

[46] I borrow this term from Pietarinen (2010).

To mention one, the basic entity of categorical structuralism is the morphism, whereas Peirce still refers to relations and their relata. Another is Peirce's study of mathematics in order to extract, for logic, its method of drawing valid inferences. It thus appears that he believes in the objectivity and reality of such inference rules. He would presumably not, as does Awodey, accept the arbitrariness of inference rules.

5. Conclusion

In this chapter I have documented Peirce's impressive knowledge of and contributions to the mathematics of his time. Examples of his contributions to geometry, set theory, and the foundations of arithmetic and his discussions on algebra have been given. These examples also served to illustrate a number of pre-structuralist themes, such as Peirce's distinction between pure and applied mathematics, e.g., his claim that applied geometry does not belong to mathematics. In addition I mentioned his objection to the characterization of mathematics as the science of quantity.

In a number of papers Peirce characterizes mathematics as the science that draws necessary conclusions from stated hypotheses. In the case of arithmetic we saw that he was able to deduce the properties of numbers from a system of axioms or, as he referred to them, "a few primary propositions." A key element of Peirce's position was to acknowledge the role of relations in mathematics both as used in the definition of mathematical objects (such as the numbers) and when formulating mathematical statements in general. We saw, e.g., that he defines the natural numbers as a relational system, and I noted that he formulates the properties of relations in his language of the logic of relatives. I have also presented Peirce's notion of "diagrammatic reasoning," that is, his explanation of how the necessity of reasoning is achieved by constructing, experimenting on, and observing diagrams. In this connection I proposed that these diagrams allow us to see the necessary relation, that is, a logical relation, holding between the antecedent and conclusion of a proposition.

In the final section I identified two structuralist positions that have some common elements with Peirce's views as presented here. Peirce defined a system of quantity as a relational system, that is, as collections on which is defined a specific order relation. His motive was to show that the properties of numbers follow by necessity from this characterization. That is, in structural terms, one may say that they are structural properties. In this way Peirce may be construed as a methodological structuralist. Furthermore I find that his anti-foundationalism, the claim that there are multiple ways to organize a mathematical theory and his insistence that mathematics concerns hypotheses, led to a view that is similar in spirit to categorical structuralism.

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