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"If Numbers Are to Be Anything At All, They Must Be Intrinsically Something": Bertrand Russell and Mathematical Structuralism

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Russell's philosophy of mathematics is often opposed to structuralism for a number of reasons. First, Russell is a paradigm logicist (indeed, perhaps the most thoroughgoing and systematic defender of logicism ever), and structuralism is often defended as an alternative to logicism. Second, Russell's famous definition of cardinal numbers as classes of equinumerous classes has the very feature that structuralists deny is necessary: it goes beyond the "structural" features of numbers and attributes to them an intrinsic character (namely, as classes). Third, Russell forcefully defended his logicist definition of real numbers over Dedekind's, by accusing him of engaging in "theft over honest toil"—postulating the existence of objects that fulfill a certain structural description, without first proving that there are such objects (Russell 1919, 71).[1] In the century since Russell first wrote these words, this accusation has been a standard objection to at least some versions of structuralism, and overcoming this objection has been a source of ongoing work for many of structuralism's contemporary adherents.

Nevertheless, this chapter will show that Russell's relationship to structuralism is not entirely negative. Russell defended—and in some cases even introduced into philosophy—many ideas that were essential for the full articulation and defense of structuralism. (Indeed, some of Russell's ideas were explicitly appropriated in Ernst Cassirer's philosophical defense of structuralism.) Of course, Russell was a critic of mathematical structuralism—the most thoroughgoing and trenchant critic of structuralism in the early twentieth century. As this

[1] Russell is here discussing the definition of real numbers in terms of Dedekind cuts. He argues that Dedekind himself simply laid down an axiom that *postulates* that any segment of the series of rationals has a bound; he advocates instead for *constructing* the reals as sets of segments of the series of rationals.

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chapter will show, Russell's criticisms of structuralism are manifold and subtle, going well beyond the well-known ideas I mentioned in the opening paragraph.

This chapter has two parts. In the first part ("Russell's Positive Contribution to Structuralism," section 1), I identify three theses of Russell's philosophy of mathematics that could be—and indeed have—been employed as key parts of structuralism. In the second part ("Russell's Criticism of Dedekind's Structuralism," section 2) I show how Russell, between the years 1898 and 1901, returned again and again to the structuralist idea in Dedekind's philosophy of arithmetic, and developed four series of criticisms of this structuralism.

Two clarifications before we begin. First, the main topic of this chapter is Russell's relation to "non-eliminative" versions of structuralism, such as the version in Dedekind's *Was sind und was sollen die Zahlen?* (Dedekind [1888] 1963).[2] As other philosophers have made clear (Reck and Price 2000), the core idea of structuralism, that mathematics is about positions in structures, can be developed in multiple, incompatible ways. For non-eliminative structuralism, mathematical objects are just positions in structures: that is, all of the essential properties of, say, a particular natural number are irreducible relational properties between it and the other natural numbers. On this view, the positions in the structure are distinct from any of the systems of objects that have that structure. For example, the number 4 in the natural number series is an object in its own right, distinct from any particular things that have the fourth position in some system (e.g., the fourth planet in the solar system, or the fourth child of J. S. Bach). This clarification is necessary, since (as I will argue later) some of Russell's philosophy of mathematics is quite close to certain eliminative versions of structuralism. Second, beyond the quip about theft and honest toil in *Introduction to Mathematical Philosophy*, there is little substantial discussion of recognizably structuralist ideas in Russell's writings in the philosophy of mathematics after his 1903 *Principles of Mathematics* (*POM*). What's more, throughout *POM*, and in Russell's various papers and drafts that he wrote while composing *POM*, Russell returns to Dedekind's version of non-eliminative structuralism repeatedly. For this reason, my focus in this chapter will be on *POM* and Russell's papers in the years immediately preceding its publication.

1. Russell's Positive Contribution to Structuralism

In this section, I identify three theses of Russell's philosophy of mathematics that could be—and indeed, as I will show, have—been employed as key parts of a fully articulated structuralism. First, *the logic of relations makes it possible to conceive*

[2] On Dedekind as a non-eliminative structuralist, see Reck (2003).

structures abstractly, without any reference to space, time, or empirical properties. Second, *Russell is one of the first philosophers (if not the first) to explicitly separate pure from applied mathematics in such a way that all of the rival metric geometries become parts of pure mathematics.* Third, *Russell introduced the concept of a "relational type" and distinguished the various areas of pure mathematics according to the specific relational type that they study*—an approach that provides a concrete way of cashing out the idea that the various branches of pure mathematics concern distinct "structures." I take each of these theses in turn.

1.1. The Logic of Relations and Abstract Structures

The core idea of structuralism is that all the essential properties of mathematical objects are their relational properties to other mathematical objects within the structure. This core idea is incompatible with the view that spatial, temporal, intuitive, or empirical properties are essential properties of mathematical objects. Consider spatial properties (by which I mean properties of an object in relation to "physical" space, the space occupied by concrete bodies). Spatial properties involve essential relations to things in space, since it is the fact that physical space is occupied by concrete bodies that distinguishes it from, say, color space or abstract mathematical "spaces." A similar point holds for temporal, intuitive, and empirical properties: temporal properties involve relations to events in the physical world, intuitive properties involve relations to our sensibility, and empirical properties involve relations to empirical (and so non-mathematical) objects. Thus, structuralism requires that the concept of a structure does not depend conceptually on spatial, temporal, intuitive, or empirical concepts. In short, the objects of mathematics are *abstract* structures (or positions in abstract structures).

But is it possible to conceive structures abstractly, without any reference to space, time, intuitive, or empirical properties? Consider our paradigm structuralist theory, Dedekind's philosophy of arithmetic. Dedekind defines the natural number numbers by first defining a simply infinite system, or in Russell's language, a "progression." A progression is a structure with a distinguished element, 0, and a successor map that takes each position in the structure to the "next" position. But is this notion spatial, temporal, intuitive, or empirical? Certainly, the word "next" suggests such an origin. More generally, in chapter 31 of *POM*, Russell considers the following constellation of ideas, which he attributes to Leibniz and Meinong: progressions are a kind of *series; series* presupposes *order*, which in turn presupposes *distance*; distances are magnitudes, but *magnitude* is an empirical notion. This is a natural line of reasoning. After all, if, say, A, B, and C are ordered in such a way that B is between A and C, what else could this mean than that the distance from A to B is less than the distance from A to C? So, our objector concludes, the concept of a progression ultimately has an empirical origin.

Russell's reply to this objection depends on his definition of order, and ultimately on his new logic of relations. In chapter 24, he isolates six distinct ways of generating a series. For example, elements may be ordered into a series using the notion of *distance*, or the notion of *between*, or the notion of *separation.* In chapter 25, he argues in detail that these methods for generating a series can be reduced to one single method:

> The minimum ordinal proposition, which can always be made wherever there is an order at all, is of the form "*y* is between *x* and *z*"; and this proposition means: "There is some asymmetrical, transitive relation which holds between *x* and *y* and between *y* and *z.*" (§207)

(In the case of the natural numbers, this asymmetrical, transitive relation is $n < m$, and "*m* is between *n* and *o*" means "$n < m$ and $m < o$"). And so the objection is defeated, since the notion of order depends ultimately on the concept of an asymmetrical transitive relation—not on the notion of distance or magnitude. Russell concludes further that the concept of an asymmetrical transitive relation, being a *logical* notion, does not depend conceptually on any spatial, temporal, intuitive, or empirical concepts. And this is just what the defender of mathematical structuralism needed.[3]

Russell's analysis of the notion of series depends, then, on the concepts that he had developed in the logic of relations. Russell developed (independently of Frege) an original version of modern polyadic higher-order quantificational logic in the fall of 1900, and published his first version of it as "The Logic of Relations" (Russell 1901c). This paper (see also *POM* §§27–30; chap. 9) distinguishes kinds of relations—as say, transitive or intransitive, symmetrical, asymmetrical, or anti-symmetrical—in the now standard way, in many cases introducing the terms that we use today. Russell made the logic of relations independent of the theory of classes, thus avoiding the artificiality that beset the logic of relations done in the Boolean tradition by DeMorgan, Schröder, and Peirce. Unlike Frege, who thematized the function/argument analysis when arguing for the originality of his polyadic quantificational logic, Russell repeatedly pointed to the relational character of his logic to explain its originality and significance. And, most importantly for our purposes, he loudly proclaimed the centrality of

[3] I have spoken of the *conceptual independence* of the *concept* of an asymmetrical transitive relation. Russell would of course also held that certain abstract relations are *ontologically independent* of anything empirical, spatial, temporal, or intuitive.

the logic of relations for understanding mathematics: "the logic of relations has a more immediate bearing on mathematics than that of classes or propositions, and any theoretically correct and adequate expression of mathematical truths is only possible by its means" (*POM*, §27).

Of course, Russell himself was not a non-eliminative structuralist (see section 2). But a philosopher could draw on Russell's ideas to defend and elaborate structuralism. Not only *could* Russell's theory of relations be used to shore up structuralism, but in fact it *was* so used. Ernst Cassirer was, arguably, the first philosopher to give an explicit articulation and defense of a thoroughgoing non-eliminative mathematical structuralism (see Cassirer 1907, which is a very positive review of Russell's *POM*, and Cassirer [1910] 1923, chaps. 2 and 3). Though Cassirer finds the structuralist point of view paradigmatically in Dedekind's philosophy of arithmetic (Cassirer [1910] 1923, 39),[4] he self-consciously draws on ideas from Russell in this articulation and defense. In Cassirer 1907 (§II), Cassirer endorses Russell's idea that the reals, and more generally, continuity, can be defined entirely in terms of order; and that order, being definable using concepts from the logic of relations, does not presupposes space, distance, or magnitude. "One recognizes in this connection," Cassirer writes, "the value and necessity of the new foundation on which Russell is seeking to place logic. Mathematics in his treatment is nothing other than a special application of the general logic of relations" (Cassirer 1907, 7). Indeed, Cassirer claims, Russell's point of view is confirmed in Dedekind's structuralist philosophy of arithmetic (Cassirer 1907, 7).

1.2. Russell on Pure and Applied Geometry

According to structuralism, the objects of pure math are abstract structures. Concrete structures, then, are the concern of *applied* mathematics only (Parsons 2008, §14). Now, "physical" space, the space occupied by concrete bodies, is itself a concrete structure. And so, a thoroughgoing non-eliminative mathematical structuralism will have to identify some other subject matter for geometry besides physical space. The standard way for structuralists to address this issue is by distinguishing pure from applied geometry: only applied geometry is concerned with physical space; pure geometry concerns some family of abstract structures.

The pure/applied geometry distinction has played an important role in the emergence of mathematical structuralism through a more specific historical route. By the 1860s, mathematicians had proven that there are other consistent theories of metrical geometry besides classical Euclidean geometry. In the early

[4] On Cassirer's structuralism, see Erich Reck's chapter in this volume. On Cassirer's reception of Dedekind, see also Yap (2017).

1870s, Klein discovered deep interrelations between these non-Euclidean geometries and projective geometry and group theory.[5] In the 1880s, Poincaré used non-Euclidean geometry to prove some very important results in complex analysis. These results convinced mathematicians by the end of the 19th century that the non-Euclidean geometries were just as much a part of pure mathematics as classical Euclidean geometry. What, philosophically, could justify this attitude? How could mathematicians accept, as equally legitimate, contradictory theories of space? (In what follows, I'll call this "the puzzle of non-Euclidean geometry.") The structuralist has a ready answer: only applied geometry is concerned with physical space, and so whether it turns out to be Euclidean or not is a question for physics, not pure mathematics; pure geometry, on the other hand, concerns certain kinds of abstract structures, some of which are Euclidean and some of which are not.

Structuralism's ability to justify the mathematicians' attitude toward the rival metric geometries was a chief argument in its favor.[6] Once again, this argument was presented very clearly by Cassirer ([1910] 1923, chap. 3, sec. 4; [1921] 1923, 432), thereby extending the non-eliminative structuralism he found in Dedekind's philosophy of arithmetic to pure geometry (Schiemer 2018; Heis 2011). Structuralists such as Cassirer solve the philosophical puzzle posed by non-Euclidean geometry, then, in four steps: first, distinguish pure from applied geometry; second, argue that the question of the metric of physical space is a question for the latter only; third, conclude that therefore the subject matter of pure geometry is something other than physical space; and, fourth, propose abstract structures as the subject matter of pure geometry. The first three steps have now become standard in the philosophy of mathematics, even among those philosophers who do not take the final distinctively structuralist step. But it is essential to recognize that very few, if any, philosophers or mathematicians prior to Russell took these three steps.

In fact, the first philosopher to clearly take these first three steps, and thereby justify the equal legitimacy of the rival geometries as pure mathematical theories independent of physical space, was arguably Russell himself.[7] He first articulated the idea in Russell (1902), which was written around December 1898:

[5] On Klein, see Georg Schiemer's chapter in this volume.

[6] This historical point is presented in detail in Shapiro (1997), chap. 5, "How We Got Here", especially sections 2 and 3. Shapiro, unfortunately, does not mention Cassirer, who in fact presents this argument for structuralism very clearly.

[7] Russell was, as far as I know, the first *philosopher* to take these three steps. There were *mathematicians* before Russell who distinguished pure from applied geometry, and denied that physical space is the subject of pure geometry. These include Grassmann, Pieri, and Whitehead (Grassmann [1844] 1894, 23–24; Pieri 1898; Whitehead 1898, vii, 370).

> We have seen that there are a number of possible Geometries, each of which may be developed deductively with no appeal to actual facts. But no one of them, *per se*, throws any light on the nature of our space. Thus geometrical reasoning is assimilated to the reasoning of pure mathematics, while the investigation of actual space, on the contrary, is found to resemble all other empirical investigations as to what exists. There is thus a complete divorce between Geometry and the study of actual space. . . . It points out a whole series of possibilities, each of which contains a whole system of connected propositions; but it throws no more light upon the nature of our space than arithmetic throws upon the population of Great Britain. (Russell 1902, 503)

One year later (in Russell 1901a, written in December 1900 or January 1901), this solution to the puzzle of non-Euclidean geometry motivated[8] a new way of characterizing the distinction between pure and applied mathematics:

> Pure mathematics consists entirely of assertions to the effect that, if such and such a proposition is true of anything, then such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is, of which it is supposed to be true. Both these points would belong to applied mathematics. (Russell 1901a, 366)

On this view, the sentences of pure mathematics are all "formal implications," sentences of the form *for all* x, $\varphi(x) \supset \psi(x)$.[9] Thus, a sentence of Euclidean geometry, understood as a branch of pure mathematics,[10] would be *for all* $x_1, \ldots, x_n$, if the axioms of Euclidean geometry are true of $x_1, \ldots, x_n$, *then such and such is also true of* $x_1, \ldots, x_n$. Russell characterizes the antecedent of these generalized conditionals as definitions: in the case of Euclidean geometry, "$\varphi(x)$" would be the definition of a Euclidean space, and so a sentence of pure Euclidean geometry is equivalent to the sentence "ψ is true of every Euclidean space." In parallel passages in the following years,[11] Russell clarifies that "φ" and "ψ" contain only

[8] Russell cites the puzzle about non-Euclidean geometry as the decisive argument for his definition of pure mathematics in the introduction to the 1937 second edition of *POM* (vii) and earlier in a January 1902 letter to Couturat (Russell 2002, 220).

[9] Russell allows that the quantifiers in formal implications be higher order. On formal implications, see *POM*, §§40–45.

[10] I speak here of Euclidean geometry, understood as a branch of pure mathematics. However, there are passages in *POM* where Russell asserts that metric geometry is an empirical science and so "does not belong to pure mathematics" (*POM*, §411; cf. Gandon 2012, 72). These passages have led Gandon to conclude that there was no fundamental break in Russell's philosophy of geometry between Russell 1897 and *POM*, as I am claiming (2012, 53). Unfortunately, space considerations preclude the extended discussion that Gandon's claims merit.

[11] Draft of Part I of *Principles of Mathematics* (Russell 1901b, 185, 187), written in May 1901; Part I of *POM* (§1), which Russell composed in May 1902.

logical constants. A sentence of applied mathematics, then, results from a sentence of pure mathematics when the universal quantifier is instantiated by a constant that is not a logical constant (or analyzable into logical constants); when the antecedent of the conditional is asserted outright for some nonlogical constant; or when some new primitive, nonlogical vocabulary is added.

Once again, not only *could* Russell's use of the pure/applied mathematics distinction to solve the puzzle about non-Euclidean geometry be used to motivate a structuralist theory of pure geometry, in fact it *was* used in precisely this way. Cassirer, in the section of Cassirer ([1910] 1923) on non-Euclidean geometry, draws the pure/applied geometry distinction and solves the puzzle about non-Euclidean geometry in precisely Russell's way. The axioms of the various metric geometries, Cassirer says, simply pick out different "pure logico-mathematical forms" ([1910] 1923, 109). He criticizes other possible solutions to the puzzle, such as empiricist solutions or Poincaré's conventionalist solution. And of course we know that Cassirer had studied Russell's *POM* very closely just a few years earlier (Cassirer 1907). Furthermore, Carnap's *Der Raum*, which articulates a structuralist philosophy of pure geometry, explicitly points to Russell's distinction between pure and applied geometry for inspiration, and draws on Russell's characterization of pure geometry for his theory of "formal space."[12]

1.3. Relational Types

For a structuralist, it is not enough to characterize the sentences of mathematics as conditionals of the form "if axioms, then theorems": for a structuralist, the axioms characterize abstract structures. But what are abstract structures? How can we pick out the distinctly *structural* properties of a system of entities? Russell's logic of classes and relations provides a ready language for characterizing these structural properties. Moreover, the structuralist holds that the various areas of pure mathematics are distinguished from one another by the kind of structure they study: number theory studies the structure of progressions, analysis studies the structure of the continuum, etc. But how do we individuate structures? Once again, Russell's logic of relations and classes provides a means.

Russell picks out "structural" properties and distinguishes structures through his notion of a "relational type," which he defines in the following way:

[12] See Schiemer's chapter on Carnap for details. Carnap, like Cassirer (see section 12.1.1), also points to Russell's logic of relations to show that formal space, inasmuch as it is a "pure theory of relations," is "free of non-logical (intuitive or experiential) components" (Carnap 1922, 8).

> Now a *type* of relation is to mean, in this discussion, a class of relations characterized by the above formal identity of the deductions possible in regard to the various members of the class; and hence, a type of relations, as will appear more fully hereafter, if not already evident, is always a class definable in terms of logical constants. We may therefore define a type of relations as a class of relations defined by some property definable in terms of logical constants alone. (*POM*, §8; cf. §412)

In fact, Russell argues that the "true subject matter" of mathematics is relational types (§27), and he engages in a detailed program of analyzing the various branches of existing mathematics as each concerned with a different relational type.

An example will make Russell's analysis of mathematics vivid. In chapter 46 of *POM*, Russell gives an axiomatization of "descriptive geometry":

1. There is a class of relations *K*, whose field is defined to be the class *point*.
2. There is at least one point.

If *R* be any term of *K* we have

3. *R* is an aliorelative (i.e., for all *x*, $\sim Rxx$).
4. R^{-1} is a term of *K*.
5. $R^2 = R$ (i.e., for all *x*, *y*, *z*, if *Rxy* and *Ryz*, then *Rxz*).
6. The points in the domain or range of R^{-1} are also in the domain or range of *R*.
7. Between any two points there is one and only one relation of the class *K*.
8. If *a*, *b* be points in the domain or range of *R*, then either *aRb* or *bRa*.

Descriptive geometry, intuitively, is the geometry of directed line segments. "*Rxy*" means "*y* comes after *x* on the directed line segment *R*"; every relation *R* represents a directed line segment, R^{-1} is the same line segment directed in the opposite way. But note that this axiomatization does not make mention of lines or directions: it simply picks out various classes *K* of relations that have the specified logical properties. The only nonlogical word is "point," which is actually just a shorthand for "object in the domain or range of some relation *R* in some class *K* of relations satisfying the axioms." Any two classes of relations *K* and *K′* that each satisfy the axioms share a relational type, and descriptive geometry is the theory of this relational type. Russell summarizes his procedure in this way:

> We saw that the above method enabled us to content ourselves with one indefinable, namely the class of relations *K*. But we may go further, and dispense

> altogether with indefinables. The axioms concerning the class *K* were all capable of statement in terms of the logic of relations. Hence we can define a class *C* of classes of relations, such that every member of *C* is a class of relations satisfying our axioms. The axioms then become parts of a definition, and we have neither indefinables nor axioms. If *K* be any member of the class *C*, and *k* be the field of *K*, then *k* is a descriptive space, and every term of *k* is a descriptive point. . . . This affords a good instance of the emphasis which mathematics lays upon relations. To the mathematician, it is wholly irrelevant what his entities are, so long as they have relations of a specified type. It is plain, for example, that an instant is a very different thing from a point; but to the mathematician as such there is no relevant distinction between the instants of time and the points on a line. (§378)

This procedure is not exactly what a structuralist would adopt. For her, once the relational type of descriptive spaces has been identified, she would pick out (perhaps by an act of "Dedekind" abstraction) the *structure* exemplified by all descriptive spaces. This structure for the non-eliminative structuralist is an individual (as are positions in this structure), and is distinct from any concretum that has this structure. Russell does not seem to make this move: *POM* suggests two alternatives, neither of which would be palatable to the non-eliminative structuralist. On one alternative—which is suggested by his definition of pure mathematics—a sentence of descriptive geometry is just a universally quantified conditional: for all *K*, if *K* is a collection of relations that satisfies the axioms of a descriptive space, then $\psi(K)$. No individual is mentioned here and there is no object *the relational type of descriptive spaces*; instead we have the higher-order propositional function *x is a collection of relations that satisfies the axioms of a descriptive space.* In fact, this alternative is really a kind of eliminative structuralism. More precisely, it is a kind of modal eliminative structuralism, where the modal operator means "it is a logical truth that . . ." The modal character derives from Russell's insistence that the relational types be characterized using purely logical vocabulary, and that the sentences of pure mathematics be logical.[13]

The second alternative interpretation of relational types is suggested by his definition of a relational type at §8 and by §378, quoted earlier. On this alternative, a sentence of descriptive geometry expresses a relation between two

[13] For a reading of early Russell as an eliminative structuralist: Reck and Price (2000, 354–361). For a contemporary defense of modal eliminative structuralism, see Hellmann (1989). On the affinity between some of Russell's views and modal eliminative structuralism, see Hellman (2004, 564). Of course, the standard objection to a view like Russell's is that Russellian logic includes the theory of classes, which is no longer considered to be obviously logical. For contemporary readers, then, this view just collapses into set theoretic realism.

So-called if-thenism is closely related to eliminative structuralism. Reck and Price (2000) read Russell in *POM* as a kind of if-thenist, as does Musgrave (1977). Gandon (2012) argues at length that Russell in *POM* is not an if-thenist about pure geometry. Unfortunately, again space considerations preclude the extended discussion that Gandon's claims merit.

classes: The class of all classes of relations that satisfy the axioms of descriptive geometry is contained in the class of terms that are ψ. Thus, the relational type is a class. Since Russell never suggests a structuralist interpretation of the theory of classes, this alternative still does not provide what the non-eliminative structuralist would want. In fact, this alternative is really a kind of set-theoretic realism.

Interestingly, in the parts of *POM* that were written first in late 1900, such as part III (on quantity), Russell suggests a third reading of relational types that has a stronger structuralist flavor. When writing these sections, Russell endorsed a novel program using "abstraction" principles. By "abstraction" principles, Russell means principles, such as Frege's famous "Hume's Principle" (Frege 1884, §63), that analyze equivalence relations (say, among classes) into identity claims about some new entities (say, cardinal numbers). Thus, cardinal numbers are defined by the biconditional *The number of Fs = the number of Gs iff the Fs and the Gs are equinumerous*. Similarly, directions are defined by the biconditional *The direction of l = the direction of m iff l and m are parallel lines.* He makes free use of abstraction principles in these parts of *POM*. For instance, in §231 he defines the ordinal number ω by abstraction as the abstractum to which all progressions (which are themselves related by the equivalence relation of isomorphism) are related. Thus, when two collections of objects, classes, and relations both satisfy the same logically describable axiom system, they are related by an equivalence relation (*having the same relational type as*), and their common relational type is then defined by abstraction. At various places, he suggests that the entity defined by abstraction is "unanalyzable" and thus distinct from any class (see, e.g., §155 and §157 on magnitudes).[14] By spring 1901, Russell rejected definitions by abstraction (see §110, written in June 1901). However, if this program of late 1900 and very early 1901 had been carried out to completion, this would have been close to what non-eliminative structuralists would want. That is, a mathematical theory such as number theory would have as its object some abstract object, distinct from all concrete progressions and distinct from classes.

Just as in the case of his theory of relations and his pure/applied distinction, not only *could* Russell's notion of a relational type be employed in a structuralist account of mathematical objects, in fact it *was* used in precisely this way. In his review of *POM*, Cassirer emphasized Russell's project of identifying the various relational types that characterize the various branches of mathematics (Cassirer 1907, 5). Later, Cassirer systematically used Russell's logic of relations to identify

[14] Russell was not consistent on this point, even in late 1900: elsewhere Russell suggests that the abstracta picked out by definitions by abstraction are just classes of equivalent terms (see, e.g., §231).

Russell (1919, chap. 5) introduces what he calls a "relation-number," which is a class of "similar" (i.e., isomorphic) relations. This is clearly the descendant of *POM*'s relational type, now interpreted in this third way, where the equivalence relation is isomorphism and the abstracta are classes of isomorphic relations.

the relational type of some mathematical theories (e.g., Cassirer [1910] 1923, 37–39), before applying an act of abstraction to identify the "system of relations" (110), which constitutes the true object of pure mathematics. In fact, Cassirer's position is what one gets by taking the object *C* mentioned in §378, that is, *the relational type of all descriptive spaces*, considered not as a class, but as distinct kind of abstractum. Furthermore, Carnap self-consciously draws on Russell's notion of relational types in identifying structures in his structuralist "general axiomatics project" from the mid-1930s, and in his pre-*Syntax* period philosophy of mathematics. In many writings from these periods, Carnap follows Russell's procedure of axiomatizing a mathematical theory, removing all nonlogical vocabulary, and treating the resulting axioms as a definition of a higher-order propositional function that applies to tuples of objects, relations, etc. Indeed, Carnap at various points endorses all three of Russell's interpretations of relational types.[15]

2. Russell's Criticism of Dedekind's Structuralism

Although Russell's philosophy could furnish the raw materials for essential components of a worked out non-eliminative structuralism such as Cassirer's, Russell himself presented a sustained and multipronged attack on non-eliminative structuralism, in the form in which Dedekind had developed it. He returned to Dedekind's structuralism again and again in a series of writings, both published and unpublished, between 1898 and 1901.[16] In this section I present three groups of criticisms that Russell developed of Dedekind's non-eliminative structuralism in these years.

2.1. Russell's Earliest Criticisms: The Priority of Cardinals over Ordinals

Russell first read Dedekind's *Was sind und was sollen die Zahlen?* in April 1898.[17] Even on his first reading, Russell was alert to the non-eliminative structuralist aspect of Dedekind's work, and he found it untenable. In particular, from this

[15] See Schiemer's chapter on Carnap for details and references on the structuralist aspects of Carnap's "general axiomatics project" and his pre-*Syntax* philosophy of mathematics. Schiemer's chapter also clearly lays out Russell's influence on Carnap.

[16] Since many of Russell's criticisms can be adequately understood only in the context of the particular views and preoccupations he had at the time of their writing, I will discuss the chronology of Russell's criticisms of Dedekind's structuralism. However, I cannot here give a full defense of the chronology, nor can I give a complete account of Russell's rather complex history of reading and writing about Dedekind in this period. I hope to come back to these issues in more detail elsewhere.

[17] See "What Shall I Read?" (Russell [1891–1902] 1983).

first reading, he responded critically to the passage in *Was sind* where Dedekind presents his version of non-eliminative structuralism. The passage (§73) reads as follows:

> If in the consideration of a simply infinite system *N* set in order by a transformation φ we entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation φ, then are these elements called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called the *base-number* of the *number-series* N. With reference to this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind.

This act of "freeing the elements from every other content" is now often called "Dedekind abstraction." It purportedly allows one to move from a representation of a particular model of the Peano axioms to a new independent object—what we might call the "structure" shared by all models, or simply "the numbers."

From his earliest reading,[18] Russell highlighted two features of Dedekind's view. First, what Dedekind calls "natural numbers" or simply "the numbers" are finite ordinal numbers, not cardinals. Dedekind thus defines the finite ordinals independently of defining cardinal numbers, and in fact he defines the finite cardinals in terms of the ordinals. (That is, Dedekind shows that *there are n Fs* just in case the *F*s can all be paired off 1-1 with the ordinals from 1 to *n*. See Dedekind [1888] 1963, §161.) Second, Dedekind believes that the natural numbers are arrived at by what he calls "abstraction."

I'll say more about the second feature in the following two sections. Concerning the first feature, Russell argued in the following way.[19] To say of the *F*s and the *G*s that they have the same cardinal number requires only the notion of a "correlation," i.e., a 1-1, onto relation. Modifying Russell's terminology and symbols for readability, Russell suggests the following:

> The cardinal number of *F*s = the cardinal number of *G*s iff there is a 1-1, onto relation from the *Fs* to the *G*s.

On the other hand, to say of *x* (under some relation *R*) and *y* (under some relation *R′*) that they have the same ordinal number requires *both* the notion of a

[18] These two features are highlighted in a long marginal comment Russell made in April 1898 in his copy of *Was sind* next to §73. This copy is available at the Russell archives at McMaster University.

[19] This criticism was articulated in a set of notes from October 1900 (available at McMaster: RA 230.030870), and written out in prose in §232 of *POM*, which was written in November 1900.

correlation and the notion of a "serial relation." Again modifying Russell's terminology and symbols for readability, Russell holds:

> The ordinal number of x = the ordinal number of y iff x is in the co-domain, but not the domain[20] of the serial relation R, and similarly for y and the serial relation R', and there is a correlation S from the field of R to the field of R', such that for all x', y', x'', y'', if $x'Sx''$ and $y'Sy''$, then $x'Ry'$ iff $x''R'y''$.

Neither of these definitions presupposes the other. Thus, Russell holds, the ordinals need not be defined using the notion of a cardinal number, nor do the cardinals need to be defined using the notion of an ordinal number (as Dedekind in essence does). Nevertheless, since under Russell's proposed analysis of *the cardinal number of Fs = the cardinal number of Gs* and *the ordinal number of x = the ordinal number of y,* the first proposition requires only the notion of a correlation, and the second requires that same notion and a further one (namely, of a serial relation), the notion of a cardinal number is simpler than that of an ordinal number. Thus, the cardinal numbers are prior to the ordinals, when ordered by conceptual complexity.

The question of the relative priority of the notion of an ordinal and of a cardinal has been a mainstay of philosophical reflection on structuralism since the very beginning. Cassirer highlighted and defended Dedekind's view that the ordinals are conceptually prior to the cardinals, criticizing Frege's and Russell's alternative view (Cassirer 1950, 59ff.). Dummett, in his wide-ranging, probing, and highly influential critical discussion of Frege and Dedekind in his *Frege: Philosophy of Mathematics*, also highlights the issue of the conceptual priority of ordinals and cardinals (1991, 53, 293). Dummett criticizes Dedekind and other structuralists, who hold that the natural numbers are intrinsically ordinal, and defends the Fregean and Russellian view that numbers are intrinsically cardinal.[21] Charles Parsons has defended structuralism against this objection (2008, §14, 73ff.), as have W.W. Tait (1996, §§VI–VII) and Reck (2013, 159). Given this later history, it is very noteworthy that from his very first reading of Dedekind's book, Russell isolated the core philosophical issue of the priority of the cardinal and ordinals as a potential objection to Dedekind's non-eliminative structuralist theory of the natural numbers.

[20] A term that is in the co-domain but not the domain of a relation is a referent but not a relatum of the relation, as (for instance) the number 4 is in the finite ordinals up to 4 related by the successor relation. It is the "last" term in the series.

[21] For Dummett, the structuralist view of the natural numbers as intrinsically ordinal violates what has come to be called "Frege's constraint," that the definition of a mathematical object (e.g., a natural number) should make its canonical application obvious (e.g., its role in giving the cardinality of things). This argument was in fact given explicitly by Russell (1919, 9–10): "We want our numbers not merely to verify mathematical formula, but to apply in the right way to common objects. We want to have ten fingers and two eyes and one nose . . . and this requires that our numbers should have a *definite* meaning, not merely that they should have certain formal properties."

2.2. *Principles of Mathematics*, Chapter 30

As we've seen, from his very first reading of *Was sind*, Russell saw clearly the philosophical significance of the non-eliminative structuralist view suggested by Dedekind in §73, and focused on two issues: the alleged priority of ordinal over cardinal notions, and the philosophical tenability of "Dedekind abstraction." I discussed the first issue in the last section; in this section I turn to the second.

Russell addressed this second issue in earnest in a compressed and difficult-to-interpret passage that, though it was published in 1903 as chapter 30 ("Dedekind's Theory of Number") of *POM*, was actually written in November 1900. I believe that it is important to keep this date in mind, since the criticism of Dedekind abstraction in chapter 30 was written *before* Russell adopted his classic definition of cardinals as classes of equinumerous classes.[22]

In §241 of chapter 30, Russell quotes *Was sind*, §73, where Dedekind presents the natural numbers as abstractions from some simply infinite system. He objects as follows (I have numbered Russell's sentences to make later references easier, and italicized key phrases):

(1) Now it is impossible that this account should be quite correct. For it implies that the terms of all progressions other than the ordinals are complex, and that *the ordinals are elements in all such terms*, obtainable by abstraction. But this is plainly not the case. A progression can be formed of points or instants, or of transfinite ordinals, or of cardinals, in which, as we shall shortly see, the ordinals are not elements.

(2) Moreover it is impossible that the ordinals should be, as Dedekind suggests, nothing but the terms of such relations as constitute a progression. *If they are to be anything at all, they must be intrinsically something; they must differ from other entities as points from instants, or colours from sounds*.

(3) What Dedekind intended to indicate was probably a definition by means of *the principle of abstraction*, such as we attempted to give in the preceding chapter. But a definition so made always indicates some class of entities having (or being) a genuine nature of their own, and not logically dependent upon the manner in which they have been defined. The entities defined should be visible, at least to the mind's eye; *what the principle asserts is that, under certain conditions, there are such entities*, if only we knew where to look for them. But whether, when we have found them, they will be ordinals or cardinals, or even something quite different, is not to be decided off-hand.

[22] Russell adopted this definition sometime between March and June 1901. See Gregory Moore's introduction to Russell (1993, xxvii).

It will take a bit of unpacking to understand Russell's objections.[23] I will take the three objections in turn, starting with the second.

Objection (2) is directed against the metaphysical commitments that Russell finds in Dedekind's claim that the numbers "retain their distinguishability" despite having no "special character," standing only in relations to one another. Russell's objections draw on his own reflections on the metaphysics of relations. Since the time of his dissertation (in 1896), Russell had been preoccupied with an apparent paradox concerning points. Since each point is qualitatively indistinguishable from every other point, points must be distinguished by their relations to other points. If, for instance, there are two congruent triangles ABC and A′B′C′, A differs from A′ inasmuch as it stands in a certain relation to BC which A′ does not, and A′ in a certain relation to B′C′ that A does not. But what distinguishes BC from B′C′? A circularity or vicious regress threatens. Russell called this the "paradox of relativity": "a conception of difference without a difference of conception" (Russell [1898] 1983, 259; see Griffin 1991, 181ff., 317ff.; Galaugher 2013, 29ff.).

By 1900, Russell was keen to block this paradox. Russell's maneuver—which was articulated in a series of papers written in the summer of 1900, and incorporated into chapter 51 of *POM*, written in December 1900—was radical: though each point is qualitatively indistinguishable *to us*, he insisted that points are in fact all qualitatively different, even if we cannot detect these intrinsic properties.

> And more generally, two terms cannot be distinguished primitively by difference of relations to other terms; for difference of relation presupposes distinct terms, and cannot therefore be the reason why the two terms are distinct. Thus if there is any diversity at all, there must be immediate diversity, and this kind of diversity occurs between the various points of space. . . . As with people so with points: the impossibility of recognizing them must be attributed, not to the absence of individuality, but exclusively to our incapacity. (1900a, 255; cf. *POM* §428)

The supposed paradox of relativity concerning points in space, then, contravenes a principle that Russell believes holds generally: every term must have intrinsic properties peculiar to it, and no two terms can ever be distinguished by relational properties alone.

[23] There has been some discussion of Russell's criticisms of Dedekind in §241. Much of this literature, I believe, misinterprets Russell's meaning in various ways. For example, Shapiro (1997, 175), in a brief discussion, remarks only that Russell's objection "looks like Frege's Caesar problem." In fact, as I'll show, Russell's objections are quite different from the Caesar problem. See also Dummett (1991, 51–52); Tait (1996, §III); Hellman (2004, 570); Reck (2013, 145–147).

This reply to the paradox was surely fresh in his mind when he reread §73 of *Was sind* and formulated objection (2). He saw clearly that the structuralist view of the natural numbers, as intrinsically identical objects that differ only in their relational properties, was exactly like the paradoxical theory of points he rejected. Dedekind abstraction purports to take some particular progression, composed of terms with intrinsic properties, and form for us a new progression—*the* natural numbers, *the structure* common to all progressions—composed of terms that lack intrinsic properties. Russell rejects this move: "If they are to be anything at all, they must be intrinsically something."[24]

In objection (3), Russell argues that, even if Dedekind were correct in holding that the natural numbers are defined by "abstraction," it would not follow that the numbers have *only* the relational properties identified by this definition. This is because, on Russell's view, no definition (whether by abstraction, or otherwise) guarantees that the defined entities have *only* the properties that follow from the definition.

In formulating this objection, Russell interprets Dedekind abstraction in an idiosyncratic way: as an instance of what he calls definition by the "principle of abstraction." A definition by the "principle of abstraction" is a definition based on a principle, such as "Hume's Principle," that analyzes an equivalence relation into an identity claim about some new entities (see section 1.3).[25] In late 1900 and early 1901, Russell held that these definitions could be justified by a general principle, which he called the "principle of abstraction":

> This principle asserts that, whenever a relation, of which there are instances, has the two properties of being symmetrical and transitive, then the relation in question is not primitive, but is analyzable into sameness of relation to some other term; and that this common relation is such that there is only one term at most to which a given term can be so related, though many terms may be so related to a given term. (*POM*, §157)

[24] Although Russell does not point this out in *POM* §241, the paradox of relativity emerges in non-eliminativist structuralism in a more direct way. In symmetric structures, such as the integers together with addition, there is apparently no non-circular way to distinguish, say, −1 from 1. This paradox has been discussed in the contemporary literature on structuralism: e.g., Keränen (2001) and Parsons (2008, 107ff.). Contemporary philosophers have noted the affinity between this paradox and Kant's argument from incongruent counterparts; Russell had noted, a century earlier, an affinity between Kant's argument and the paradox of relativity (*POM*, §214n).

Of course, Euclidean 3-space is symmetric in uncountable ways, and so admits of uncountably many structure preserving nontrivial automorphisms. So the paradox discussed by Keränen and Parsons applies even more radically to space than to the integers. In this sense, this contemporary paradox is a special case of the more general paradox of relativity. Again, Russell's solution would be to deny the very possibility of objects with no distinguishing intrinsic properties.

[25] Russell in fact defines cardinal numbers in just this way in the first draft of "Logic of Relations" (Russell 1900b, §3, proposition 1.4).

According to this principle, the relation of equinumerosity (for example) between the class *F* and the class *G* is analyzable into a new relation, *is the number of*, that holds between both *F* and *G* and some new object, a cardinal number. Cardinal numbers are then defined as those objects to which equinumerous classes stand in the *is the number of* relation. Thus, when Russell was composing objection (3), he accepted definition by the "principle of abstraction" as an acceptable form of abstraction, and interpreted Dedekind abstraction accordingly.[26]

Russell's objection, then, is that though we define the numbers only in terms of the structural properties mentioned in the definition, it does not follow that the entities defined have *only* the properties that are mentioned in the definition. As Russell put it: "a definition so made always indicates some class of entities having (or being) a genuine nature of their own, and not logically dependent upon the manner in which they have been defined." Thus, though we make no mention of intrinsic properties in the definition, it does not follow that the defined entities themselves in fact lack intrinsic properties. A more pedestrian example will make this clear. If *A* and *B* are full siblings, then—in accordance with the principle of abstraction, since *is a full sibling with* is an equivalence relation—*A* and *B* stand in some common relation to some common third thing—in this case, a common set of parents. We can then define *the parents of A and B* by abstraction. But it surely does not follow that *A*'s and *B*'s parents have *only* the property of being parents—they are also intrinsically a certain height and weight. Each of them is an "an actual [person] with a tailor and a bank-account or a public-house," to repurpose a well-known Russellian passage (§56).

One possible reply to this objection would be to emphasize Dedekind's claim that the numbers are a "free creation of the human mind." On one possible interpretation of this phrase, Dedekind means that the mathematician, in performing Dedekind abstraction, *creates* a new set of objects.[27] These objects, plausibly, would fail to have nonstructural properties because the mathematician,

[26] Though Russell interprets Dedekind abstraction idiosyncratically as an instance of definition by the principle of abstraction, I do not believe that Russell's objection (3) depends on this interpretation. After all, Russell denies that definition by Dedekind abstraction picks out objects with only structural properties, not because of some specific feature of definition by the principle of abstraction, but because of a general feature of definitions in general: it never follows, from the fact that an object is defined as φ, that an object is only φ and therefore lacks properties that are not implied by the definition.

[27] A psychologistic reading of Dedekind is suggested by Dummett 1991; a non-psychologistic reading was first given by Cassirer (and by many others since: e.g., Reck 2013; Yap 2017). For Cassirer's non-psychologistic reading of Dedekind, see Reck's chapter on Cassirer.

Russell, at least in *POM* and earlier, does not read Dedekind psychologistically. (In this way, Russell's discussion of Dedekind's structuralism is both more sympathetic and more interesting than many later objections, e.g., by Dummett.) Russell's best reconstruction of Dedekind abstraction interprets it as definition from the principle of abstraction, which he took to be a candidate logical (not psychological) principle, motivated by a mind-independent metaphysical fact about equivalence. Indeed, none of the objections that are surveyed in this chapter depend on reading Dedekind in a psychologistic way.

in creating them, removed these intrinsic properties. Russell does not read Dedekind in this psychologistic way, and so does not formulate explicitly a response to this reply. However, it is clear that Russell would be deeply opposed to this way of thinking. We saw already, in his assertion that points do have intrinsic properties, even if they are indistinguishable *to us*, that Russell was deeply committed to the mind-independence of all entities, even mathematical entities. What is true is independent of the mind, both in its being, and in its being true. In the same vein, things do not come into being by being defined by us. The principle of abstraction does not bring new abstract objects into being. It is simply a true proposition about mind-independent reality: "what the principle asserts is that, under certain conditions, there are such entities." Furthermore, the defined entities are not under our control; it is emphatically not the case that they have only the properties that *we* give them.

Objection (1) draws on Russell's peculiar way of defending the "principle of abstraction." Russell in November 1900 motivated the principle on the grounds that it is an explication of the widespread philosophical intuition that equality and other relations akin to it (namely, equivalence relations) are "always constituted by possession of a common property" (§157). If two classes are equinumerous, they must have something in common (namely, the property of having n members); if two lines are parallel, the two lines must share some feature (namely, having such and such direction). However, Russell raises a worry about this defense. Plausibly, the intuition that equivalence relations are constituted by possession of a common property could be explicated in this way: for any relation R that is transitive, symmetrical, and non-empty,

$$(*)\ \exists S \text{ such that } \forall x, y\left(xRy \equiv \exists z\left(xSz \,\&\, ySz\right)\right).$$

This appears to be a perfectly correct explication of the intuition, where the "common property" is *being related by S to z*. However, on this explication, the right-hand side of the biconditional does not guarantee the transitivity of the relation R, for the following reason. Suppose A is equivalent under R to B, and A and B share property P, while B is equivalent under R to C, and B and C share property Q. It would therefore not follow that there is any property that A and C share. Thus, the fact that two equivalent terms share some property cannot be an *analysis* of what it is to stand in an equivalence relation, since *sharing a property*, in the sense of (*) guarantees only the symmetry, not the transitivity, of R.

Russell blocks this worry by insisting that, in the cases where we want to use a principle of abstraction to analyze an equivalence relation, the relation S is many-one: "In order that [the relation R] may be transitive, the relation [S] to the common property must be such that only one term at most can be the property of any given term" (§157). An example of a many-one relation is *x is the number* *of Fs*, which could be used to analyze by abstraction the relation *equinumerosity*; an example that is not many-one is *x is a parent of y*, which therefore could not be used to analyze by abstraction the relation *being a full sibling*. But what reason could be given, for a specific equivalence relation, that would guarantee that the relation that holds between the equivalent terms and the common property be many-one? Russell addresses this worry in the context of the relation *equality*, which he analyzes, through the principle of abstraction, in terms of abstract magnitudes: Two quantities (for instance, two material bodies A and B) are equal (say, in mass) if the body A *has* the magnitude *M* and body B *has* the magnitude *M*. Russell further claims that in this case, the abstractum (the magnitude; in our example, a magnitude of *M grams*) is an "element" of the concreta (the quantities; in our example, the two material bodies A and B) from which it can be abstracted (*POM*, §157). The relation between the quantity and the magnitude that it has is many-one, since, Russell argues, it is an "axiom" that only one magnitude can exist at a given spatiotemporal place. Thus, there cannot be *two* magnitudes of a given kind that both exist in the location where body A is located. That means that the troublesome case that I described in the previous paragraph cannot arise for equal quantities and their common properties, and transitivity is guaranteed after all.

The fact that a quantity has one and only one magnitude as its "element,"[28] then, explains why the principle of abstraction can be used to analyze the relation of equality, and magnitude can be defined by abstraction. Will the same be true in the case of the natural numbers, if they are defined by abstraction? We saw, in the case of objection (3), that Russell used his particular way of understanding definitions by abstraction to try to make sense of Dedekind's talk of "abstraction." I believe that this is true also of objection (1), and explains why he alleges that Dedekind's procedure can make sense only if ordinals are always "elements" of any terms arranged in a progression. Russell writes that Dedekind "implies that the terms of all progressions other than the ordinals are complex, and that the ordinals are *elements* in all such terms, obtainable by abstraction" (emphasis added).

Let me spell out in some more detail how Russell is interpreting Dedekind's "abstraction." Each progression, whether it be of natural numbers, points, or propositions, stands in an equivalence relation (namely, *being isomporphic to*) to every other progression. Similarly, each element in a progression stands in an equivalence relation to every corresponding element in some other progression. For example, 4, the fourth element of the natural number series, stands in an equivalence relation to D, the fourth element in the English alphabet: the

[28] By "element," Russell most likely here means what he calls a "part" in *POM*, chap. 16 ("Whole and Part").

relation is *having the same ordinal position in one's series as*.[29] Using the abstraction schema (*), the fact that 4 and D stand in the relation *R* (*having the same ordinal position in one's series as*) implies that there is some relation *S* between 4 and D and some abstract object *z*. The abstract object *z* is the "position" of 4 and D, and the relation *S* is the relation between 4 and the position that it occupies. But why think that the relation between 4 and its position is many-one?

Russell is probing what is plausibly a vulnerable commitment in Dedekind's picture: what guarantees that a definition by Dedekind abstraction will pick out a unique set of objects, *the* natural numbers?[30] For Russell, the only plausible reason is if the natural numbers are *elements* of all the objects that are ordered into progressions, just as (he claimed) magnitudes are elements in all quantities. Thus, Dedekind requires that "terms of all progressions other than the ordinals are complex, and that the ordinals are elements in all such terms." But, Russell alleges, this is plainly not the case. As Russell emphasizes strongly (§231), the position that a term occupies in a series is not intrinsic to the term itself, and there are infinitely many possible orderings of, say, the finite cardinals into a progression. In one series (1, 2, 3, 4, . . .) 4 is fourth, but in another (1, 3, 5, 7, 9, 2, 4, 6, 8, 10, . . .) 4 is seventh. So the cardinal number 4 must contain as an element both the ordinal 4 and the ordinal 7, and clearly an infinite number of other elements besides. But this is absurd.

2.3. *Principles of Mathematics*, Part II, Chapter 14

Russell returned to Dedekind's theory of the natural numbers seven months later, in June 1901, when he wrote Part II of *POM*, on cardinal numbers. In this part, he presents his classic definition of cardinal numbers as classes of equinumerous classes (§111), which he had developed sometime in March to June of 1901[31]—after he wrote the texts I discussed in sections 2.1 and 2.2 of this chapter. In *POM* Part II, Russell uses his definition of cardinal numbers to define (in chap. 14) the natural numbers (in essence, as classes of equinumerous finite classes). In defending this definition, he considers other definitions of the natural numbers by abstraction (§122). In this section, he poses the question: "Is any process of abstraction from all systems satisfying the five [Peano] axioms . . . logically possible?"

[29] More precisely: the series of numbers up to 4 is ordinally equivalent to the series of letters up to D. This is the notion of "likeness," which Russell defines in *POM*, §231.

[30] This objection is particularly pressing on psychologistic readings of Dedekind. Suppose I take some progression and freely create, by abstraction, a new system of objects, the numbers. Suppose you take the same progression and freely create a system: need it be the same system as the one I freely created? Or suppose I perform the act of abstraction a second time on the same progression: will I again get the same system of abstract objects? There needs to be some reason why the answer to these questions must be yes.

[31] See note 22.

He answers in the negative, giving a series of new objections to theories of abstraction such as Dedekind's.[32] In this section, I identify three such objections.

The first objection concerns the identity of the abstracta. Suppose Dedekind could identify the natural numbers as the unique elements of a progression that have merely structural properties. Even so, each of the progressions 0, 1, 2, . . . and 1, 2, 3, . . . satisfies Dedekind's definition of a progression, and each can make an equally good claim to be composed of elements with merely structural properties. So which progression is *the* numbers?[33] As Russell points out, if we consider the numbers with respect to their cardinal character, we can distinguish these two cases, but Dedekindian structuralists preclude this when they conceive of the numbers as having no features besides the structural features they have in virtue of being elements in a progression.[34] Perhaps, one might contend, the numbers are what one gets when one abstracts away the differences between the progression 0, 1, 2, . . . and the progression 1, 2, 3, ... But this is absurd, for then the products of that abstraction—the numbers themselves—would have to be distinct from the progressions from which they are abstracted: that is, they would have to be distinct from every progression of numbers.[35]

Russell considers, and rejects, one plausible escape from this objection. One might insist that the natural numbers are to be identified with neither 0, 1, 2, . . . nor 1, 2, 3, . . . since *the* natural numbers are that unique progression that has nothing but merely structural, and so no intrinsic, properties. Thus, the first element of the natural number progression is neither 0 nor 1, since it is not intrinsically anything other than the first element in the progression. But as we saw in objection (3) from section 2, Russell denies that possibility: "there is therefore no term of a class which has merely the properties defined by the class and no others" (§122). So there is no progression in the class of progressions that is merely a progression and nothing else.

[32] In §122, Russell specifically targets Peano's view that the natural numbers are defined by abstraction from what all progressions have in common. (On the use of definitions by abstraction in Peano and his school, see Mancosu 2016, chap. 2.2.1.) He clearly intended his criticisms to support his class-theoretic definition by undermining *every* definition of the natural numbers "by abstraction"—not just Peano's. Moreover, most of the objections leveled against Peano would, if valid, also apply to Dedekind's definition of the natural numbers by abstraction.

[33] This objection arises even on psychologistic readings of Dedekind. For suppose I create the numbers, and then pick them out ostensively as *the progression that I just created*. Still, each progression can also make an equally plausible claim to being *the progression that I just created*—since, if I create a new progression by abstraction and call it "the numbers," I would still be at a loss whether the first element is 0 or 1.

[34] One reply to this worry is to admit that the progression of numbers, defined by Dedekind abstraction from (0, N, S), cannot be identified with either series. However, when we bring in arithmetical operations and define the numbers by Dedekind abstraction from (0, N, S, +, ×), we expand the structure and definitively settle on one of the two alternatives. Russell does not consider this reply.

[35] *POM*, §122. More recently, this objection was directed against non-eliminative structuralists by Dummett (1991, 53). Parsons (2008, 76–78) provides a reply to Dummett, which to me at least is convincing. This objection is also articulated, and endorsed, though without reference to *POM* §122, in Hellman (2004, 572).

There is, however, one way that Russell identifies for Dedekind and other abstractionists to get around this objection: they could regard the symbols "0," "successor," and "number" as really variables:

> [One could] regard 0, number, and succession as a class of three ideas belonging to a certain class of trios defined by the five primitive propositions. It is very easy so to state the matter that the five primitive propositions become transformed into the nominal definition of a certain class of trios. There are then no longer any indefinables or indemonstrables in our theory, which has become a pure piece of Logic. But 0, number and succession become variables, since they are only determined as one of the class of trios. (§122)

This of course is the eliminative structuralism that we first encountered in section 1.3 in the context of Russell's discussion of relational types. On this view, a sentence of arithmetic is just a universally quantified conditional: for every x, class N, and relation S, if $\{x, N, S\}$ are an object, class, and relation that satisfy the axioms of arithmetic, then $\psi(x, N, S)$. This brings us to Russell's second objection: once this eliminative structuralist alternative is clearly articulated, Dedekind's non-eliminative structuralism becomes unmotivated. Dedekind insists that the intrinsic character of the numbers is irrelevant; but this insistence is satisfied by the eliminative procedure (whereby arithmetic is about all objects that form progressions, regardless of their intrinsic properties), just as much as it is satisfied by the non-eliminative procedure (whereby arithmetic is about some sui generis objects with no intrinsic properties).

Nevertheless, eliminative structuralism itself faces one last significant hurdle. Even if we construe the primitive symbols of arithmetic as variables, and treat every sentence of arithmetic as a claim about every class $\{x, N, S\}$ that satisfies the axioms of arithmetic, "nothing shows that there are such classes as the definition speaks of" (§123). Suppose the Dedekindian structuralist were able to evade objection (1) from section 2.2 of this chapter, by coming up with a principled reason why the relation S between the progression from which the numbers are abstracted and the numbers themselves is many-one. There is still a more fundamental worry, which even the eliminative structuralist must face. What justifies the claim that there is any relation S at all, or any abstract objects z? Surely, if definition by abstraction were creative, and the mathematician's act of abstraction produced the abstracta, these existence claims could be satisfied. But Russell rejects creative definitions.[36] Instead, Russell suggests that the existence claim can be justified only by explicitly constructing the numbers from classes. The class {0, N, successor}, defined in Russell's now well-known way in terms of

[36] Dedekind famously argued that his *Gedankenwelt* is an instance of a progression ([1888] 1963, §66). On Russell's reception of this argument, see Reck (2013, 147–149).

classes of equinumerous finite classes, proves the existence of trios that satisfy the Peano axioms. But, now, even eliminative structuralism is unmotivated. For once we've explicitly constructed in a class-theoretic way finite cardinals that satisfy the Peano axioms, the extra step of treating sentences of arithmetic in the eliminative structuralist way itself feels otiose. And the particular brand of logicism that Russell made famous in the published version of *POM,* and later in *Principia*, is left as the only plausible philosophy of mathematics.

This last objection to even eliminative structuralist is the very objection that Russell famously expressed, almost 20 years later, in his quip about theft and honest toil. The sentiments behind this quip have been well studied and elaborated in the century since it was written. Far less, unfortunately, has been devoted to the wealth of Russell's thinking that I have laid out in this chapter. I hope that this chapter has shown, though, that Russell's engagement with structuralist ideas was far deeper, more extensive, and more complex than a narrow focus on the virtues of honest toil would suggest.

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