15

Carnap's Structuralist Thesis

Georg Schiemer

1. Introduction

Rudolf Carnap's philosophy of mathematics of the 1920s and 1930s is usually identified with his work on Fregean or Russellian logicism and with the principle of logical tolerance first formulated in his *Logical Syntax of Language* (Carnap 1934).[1] However, recent scholarly work has shown that Carnap also made significant contributions to the logical analysis of modern axiomatics and its (meta-)theory, in particular in his unpublished manuscript *Untersuchungen zur allgemeinen Axiomatik*, written between 1927 and 1929. While the early metalogical work presented there has been investigated in detail (e.g., Awodey and Carus 2001; Reck 2007), no closer attention has so far been paid to the structuralist account of mathematics underlying Carnap's "general axiomatics" project.

This chapter investigates Carnap's mathematical structuralism in his work on formal axiomatics as well as in related contributions from the time. As will be shown, his account is based on a genuinely structuralist assumption, namely that axiomatic theories describe abstract structures or the structural properties of the objects in their domains. A central motivation for his work in the 1920s and early 1930s was to give a logical analysis and explication of this structural content of theories. I will dub this assumption Carnap's *structuralist thesis*.

The aim in the present chapter is twofold: first, to show that Carnap, in his various contributions to the philosophy of mathematics from the time, proposed different ways to characterize the notion of mathematical structure. Three approaches will be analyzed in detail here. According to the first one, structure is what can be specified axiomatically, that is, in terms of "implicit definitions" expressed in a formal axiom system. Second, mathematical structures are also characterized in Carnap's work in terms of "logical constructions," more specifically, in terms of explicit definitions in a purely logical type-theoretical language.

[1] See, e.g., Carnap (1930, 1931), as well as Bohnert (1975) for a detailed study of Carnap's account of logicism. Compare, e.g., Friedman (1999) and Wagner (2009) for surveys of Carnap's contributions in his *Logical Syntax*.

Georg Schiemer, *Carnap's Structuralist Thesis* In: *The Prehistory of Mathematical Structuralism*. Edited by: Erich H. Reck and Georg Schiemer, Oxford University Press (2020).
DOI:10.1093/oso/9780190641221.003.0015

Finally, again in the context of his general axiomatics project, Carnap proposes a way to think about the structures shared by isomorphic models of a given theory in terms of the notion of structural abstraction. Thus, so-called model structures are explicitly specified in *Untersuchungen* as isomorphism types that can be specified by means of "definitions by abstraction." The chapter will survey Carnap's different approaches to characterize the structuralist thesis and point out several connections between them.

The second aim is to re-evaluate Carnap's early contributions to the philosophy of mathematics in light of current work on mathematical structuralism. Specifically, I will discuss two connections between his approaches to characterize mathematical structures and present philosophical debates. The first point of contact concerns his attempt to specify structures in terms of definitions by abstraction, or equivalently, by abstraction principles. The general idea here is to specify an identity criterion for structures based on the notion of isomorphism between mathematical systems that instantiate these structures. As we will see, different versions of this type of structural abstraction have also been introduced in recent work on structuralism.[2]

Another point of contact with the present debate concerns the notion of "structural properties" of mathematical objects. Informally speaking, structural properties are characterized as those properties not involving the intrinsic nature of objects, but rather their interrelations with other objects in a given system. In Carnap's work from the late 1920s, one can find two suggestions how to specify such properties, namely (i) in terms of the notion of logical definability and (ii) in terms of the notion of invariance under isomorphic transformations of a given system. As will be shown, a similar *duality* between two ways to think about structural properties is also discussed in contemporary work on structural mathematics.

The chapter is organized as follows: section 2 will provide a brief overview of Carnap's work on the philosophy of mathematics before the publication of his *Logical Syntax*. Section 3 will then focus on Carnap's structuralist account of mathematics, in particular on three ways to characterize the structuralist thesis, namely in terms of axiomatic definitions (section 3.1), logical constructions (section 3.2), and definitions by abstraction (section 3.3). Given this, section 4 will then compare Carnap's position with modern mathematical structuralism. The comparison will focus on the notion of structure abstraction (section 4.1) and the duality between definability-and invariance-based approaches to thinking about structures (section 4.2). Section 5 will contain a brief summary.

[2] See, in particular, Linnebo and Pettigrew (2014), Leach-Krouse (2017), and Reck (2018).

2. Pre-*Syntax* Philosophy of Mathematics

Carnap made central contributions to the philosophy of mathematics throughout his intellectual career. For the purpose of the present chapter, it makes sense to distinguish between two phases in his engagement with modern mathematics, namely a "structuralist" phase in his work from the 1920s and early 1930s and the subsequent turn to a "syntactic" period leading to the publication of his *Logical Syntax of Language* in 1934.[3] Our focus here will be limited to Carnap's pre-*Syntax* work on the philosophy of mathematics.[4]

His research on mathematics from this period mainly focuses on three areas that, on closer inspection, are connected with each other in interesting ways. Carnap's most well-known work is foundational in character and concerns a Fregean or Russellian logicism, that is, the reduction of mathematics to higher-order logic. Carnap was a central proponent of the logicist program and published a number of articles on the topic.[5] Logicism is described in these works as based on two main assumptions, namely (i) that all mathematical primitive terms can be explicitly defined in a purely logical language and (ii) that all mathematical axioms (such as the axioms of Peano arithmetic) can be derived from purely logical principles (see, e.g., Carnap 1931).

While Carnap does not specify in detail the logical system to be used for such a logicist reduction in his work, he indicates at several places that it should be a simplified version of Russell and Whitehead's type theory (henceforth TT) first presented in *Principia Mathematica* (Russell and Whitehead 1962). Logicism is thus understood as the general project of interpreting mathematical theories in a logical type theory. More specifically, according to Carnap, the logicist thesis consists of several interpretability results according to which the language of a given mathematical theory (such as Peano arithmetic) can be translated into a purely logical language such that all mathematical axioms and theorems become deducible from certain definitions and the logical principles of TT alone.[6]

[3] Compare Awodey (2017) for a similar distinction in Carnap's work on the philosophy of mathematics. See Awodey and Carus (2007) for a study of Carnap's transition to a purely syntactical approach in the philosophy of logic and mathematics. Compare the articles contained in Part 2 of Wagner (2009) for more detailed discussions of Carnap's account of mathematics in his *Logical Syntax*.

[4] This should not suggest that that Carnap has made no interesting contributions on the topic in *Logical Syntax* or in later work. The present focus on Carnap's early work is due to the fact that his structuralist understanding of mathematics is formulated in most detail here. See, for instance, Goldfarb and Ricketts (1992) for a more general discussion of Carnap's philosophy of mathematics. Compare also Awodey (2007) for a study of Carnap's post-syntactic philosophy of mathematics and logic developed in his later work on semantics.

[5] See, in particular, Carnap (1930, 1931), and also Carnap (1929).

[6] Compare Carnap on this understanding of the logicist thesis: "Every provable mathematical statement can be translated into a statement that consists only of logical primitive signs and that is provable in logic" (1931, 95).

Given this type-theoretic version of classical logicism, it should be noted that Carnap's approach differed in several respects from earlier accounts of the logicist thesis. Most importantly, logicism was not developed by him in strong opposition to other foundational programs such as Hilbert's formalism. Rather, at least from 1931 onward, his work on the foundations of mathematics can be seen as the attempt to "reconcile" Frege's logicism with the emphasis on the formal axiomatic method in the Hilbert school.[7]

A second field of Carnap's research on mathematics concerns the foundations of geometry and the nature of space. While this work precedes his contributions to type-theoretic logicism by a number of years, one can nevertheless identify several interesting thematic connections or continuities between the two fields. One such connection will be discussed in detail in section 3.2. It concerns Carnap's use of logical or set-theoretic constructions (also of central importance for the logicist program) in the representation of geometrical structures such as the structure of topological or projective space.

Carnap's central contribution in this respect is *Der Raum: Ein Beitrag zur Wissenschaftslehre* of 1922, a monograph based on his 1920 dissertation written under the supervision of the neo-Kantian Bruno Bauch (Carnap 1922). Carnap's aim in the book is to settle the long-term debate on the nature of space by distinguishing between three types of geometrical space, namely *formal, intuitive*, and *physical* space, and by studying their respective interrelations. These different notions of space can be investigated by different types of geometrical theories: formal space presents an abstract "order-configuration" whose properties can be specified in terms of a formal axiomatic theory. Intuitive space, in turn, is described by geometrical principles grounded in some form of a priori intuition (or a Husserlian *Wesenserschauung*). Physical space is described in applied or physical geometry, based on conventions concerning its metrical properties.[8]

This novel philosophical analysis of geometrical space was clearly motivated by several fundamental developments in 19th-century geometry as well as by a long-standing debate on the status of geometrical axioms. The immediate mathematical background of Carnap's book includes Grassmann's *Ausdehnungslehre*, Riemann's theory of formal manifolds presented in his *Habilitationsschrift*, Klein's and Sophus Lie's algebraic study of different geometries in terms of transformation groups, and Hilbert's axiomatization of Euclidean geometry presented in his *Grundlagen der Geometrie* of 1899, to name only some. On the philosophical side, Carnap's book engages with work on the reception of the Kantian

[7] See Awodey and Carus (2001), Reck (2004), and Schiemer (2012b) on Carnap's attempted synthesis of the different foundational approaches in mathematics.

[8] See Carus (2007) for a detailed discussion of the neo-Kantian background of Carnap (1922). See Friedman (1999) and Mormann (2007) for different analyses of Carnap's philosophy of geometry, in particular, on his account of the role of conventions in the book.

account of geometrical knowledge in works by Natorp and Cassirer, Poincare's geometrical conventionalism, as well as with contributions by Helmholtz and others on the status of geometrical axioms.

Carnap's philosophical investigation of the nature of space and its axiomatic description in 1922 is closely connected to a third area of research, namely his subsequent work on formal axiomatics. The axiomatic method in mathematics is investigated in detail in several publications from the late 1920s and early 1930s. A main contribution is the second part of his logic textbook *Abriss der Logistik* (Carnap 1928) entitled "Applied Logicistic." Carnap discusses here the logic of axiomatic definitions as well as the formalization of different axiomatic theories (including arithmetic, set theory, projective geometry, and topology).

A second important source is Carnap's already-mentioned *Untersuchungen zur allgemeinen Axiomatik*.[9] In this unpublished manuscript, he develops a general study of the methodology of axiomatic mathematics and a logical explication of several metatheoretic concepts. This includes different notions of (relative) consistency, independence, and completeness of axioms or axiom systems that were discussed informally in preceding mathematical work. Carnap's immediate mathematical background comprises work by Hilbert, the Italian "Peanists," the American postulate theorists, as well as Richard Dedekind's proto-axiomatic study of arithmetic (Dedekind 1888). Moreover, regarding the study of different completeness properties of axiom systems, Carnap frequently refers to Fraenkel's influential *Einleitung in die Mengenlehre* (1928) as an important background for his own more systematic contributions.[10]

Given these thematic fields in Carnap's early philosophy of mathematics, one comment concerning his general structuralist thesis is in order here. His analysis of the nature of formal geometry in 1922 and, more importantly, of general axiomatics from the late 1920s clearly shows that Carnap was not only a "foundationalist," but also an early proponent of a version of philosophical structuralism. Interestingly, his structuralism was not an isolated position at that time, but shared by several other prominent philosophers, including Russell, Cassirer, and Quine.[11] What clearly distinguishes Carnap's account from that of his contemporaries is that the structuralist thesis for him was not just an informal position regarding the nature of mathematics. On the contrary, a central motivation

[9] Related articles written by Carnap on modern axiomatics are Carnap (1929, 1934), and Carnap and Bachmann (1936).

[10] See Awodey and Carus (2001) and Schiemer, Zach, and Reck (2017) for surveys of Carnap's early metatheoretic work. Compare, in particular, Awodey and Reck (2002) for a detailed study of the development of metatheoretic notions in 19th- and early 20th-century mathematics.

[11] See the articles on these philosophers in the present volume for detailed studies of their respective structuralist accounts of mathematics.

underlying his work was to characterize in logical terms the *structural content* of formal theories. So what precisely is Carnap's mathematical structuralism?

3. Three Structuralist Ideas

Carnap's work on the philosophy of mathematics from the 1920s and 1930s contains three distinct but interrelated proposals on how to characterize the structuralist thesis, that is, how to specify the structural content of mathematics:

(i) *Structures via axiomatic definitions*: a mathematical structure is what can be defined in terms of an axiom system.
(ii) *Structures via logical constructions*: a mathematical structure is what is logically constructible in terms of explicit definitions in a purely logical language.
(iii) *Structures via definitions by abstraction*: a mathematical structure is what can be specified in terms of definitions by abstraction (or by abstraction principles).

In the following section, I will give a more detailed discussion of these approaches as well as of Carnap's understanding of their relations. Moreover, I will also discuss how the different methods of thinking about mathematical structure are connected to his generalized logicism.

3.1. Formal Axiomatics

Carnap's early writings on the philosophy of mathematics are strongly motivated by the development of modern axiomatics in work by Hilbert, Dedekind, the Peanists, and the American postulate theorists (among others).[12] What characterizes their contributions is a novel conception of the nature of mathematical theories. Axiomatized theories were no longer understood descriptively, that is, as organizing our knowledge about a pre-theoretically given system such as physical space or the natural numbers. Rather, they came to be understood prescriptively, as definitions of abstract mathematical structures.[13]

[12] Compare, e.g., Torretti (1978), Grattan-Guinness (2000), and Gray (2008) for historical accounts of the development of modern axiomatics.

[13] See Schlimm (2013) for a more detailed discussion of this development and the distinction between a *descriptive* and *prescriptive* account of axiomatic theories.

Interestingly, this new account of the axiomatic method was applied not only in the case of algebraic theories such as the theory of groups, but also to theories traditionally viewed as descriptive in character. Hilbert's axiom system for Euclidean geometry in his *Grundlagen der Geometrie* (1899) is a case in point here. Compare Paul Bernays's apt characterization of the abstract character of Hilbert's approach:

> A main feature of Hilbert's axiomatization of geometry is that the axiomatic method is presented and practiced in the spirit of the abstract conception of mathematics that arose at the end of the nineteenth century and which has generally been adopted in modern mathematics. It consists in abstracting from the intuitive meaning of the terms . . . and in understanding the assertions (theorems) of the axiomatized theory in a hypothetical sense, that is, as holding true for any interpretation . . . for which the axioms are satisfied. Thus, an axiom system is regarded not as a system of statements about a subject matter but as a system of conditions for what might be called a relational structure. (Bernays 1967, 497)

Two issues are particularly noteworthy about Bernays's account of the "abstract conception of mathematics" characteristic of modern axiomatics. (As we will see, both issues also play a significant role in Carnap's own work on the topic.) The first one is a methodological point: the meaning of primitive mathematical terms is not supposed to be specified independently of the axiomatic theory, for instance, by reference to some form of empirical or a priori intuition. Instead, their meaning is determined solely through their occurrence in the axioms in terms of *implicit definitions*. Second, this change is related to a new understanding of the very subject matter of an axiomatic theory. As is highlighted by Bernays, the axiomatic approach of Hilbert is characterized by the assumption that relational structures form the real content of mathematical theories.[14]

The idea that axiomatic theories deal with abstract structures also forms a central assumption in Carnap's work on the philosophy of mathematics. One of his earliest works on the topic, *Der Raum* of 1922, already contains a specification of this structuralist account of theories. As mentioned previously, Carnap distinguishes here between three different concepts of space, namely "formal," "intuitive," and "physical space." The former type of space is the one investigated in pure or formal geometry. It is characterized by Carnap in the introduction to the book in terms of the concept of an "order system" (*Ordnungsgefüge*):

[14] Compare Torretti (1978) as well as the articles on Hilbert, Bernays, Dedekind, and Cassirer contained in the present volume for more detailed accounts of this structuralist understanding of modern axiomatics.

> Formal space is a general order-system of a certain kind. By "general order-system" we mean a system of relations—not between certain objects of a sensible or nonsensible domain, but between entirely indeterminate relata about which we only need to know that one kind of link entails a different kind of link in the same domain. So formal space deals not with the figures usually considered spatial, such as triangles or circles, but with meaningless relata whose place may be taken by an enormous variety of things (numbers, colors, degrees of kinship, judgments, people, etc.). (Carnap 1922, 5–6)

Notice the emphasis on the purely relational character of a formal space and on the fact that the nature of the primitive elements is irrelevant for its geometrical study. In fact, as Carnap points out, these objects are left "indeterminate" in the sense that only their interrelations to other objects are specified by the theory in question.

The first section of the book contains a closer specification of the characteristic properties of a formal space. It is here that the background of Carnap's understanding, namely modern axiomatics in the spirit of Hilbert's work, becomes most explicit. Compare the following remark on the role of axiomatic definitions:

> Only relations among the elements . . . are specified by the axioms. . . . Theorems are then derived from the axioms with no regard whatever for the intuitive meaning of these elements and relations. . . . If we think of all the theorems as put into this more general form, then instead of geometry proper (that of points, lines, and planes) we have a "pure theory of relations" or "theory of orders," i.e., a theory of indefinite objects and of the equally indefinite relations holding among them. (Carnap 1922, 7–8)

An axiom system (such as Hilbert's axiomatization of Euclidean geometry) is described here as a "pure theory of relations," that is, roughly as a formal theory in the modern sense of the term. The primitive terms of a theory are not interpreted but understood schematically. Axioms and theorems derived from the former are, in turn, not assertoric statements about a concrete space, but reinterpretable relative to different systems of the specified structure.

Interestingly, formal space itself is identified by Carnap with this abstract structure shared by the different models of the theory in question. Compare again Carnap on this characterization of the subject matter of formal geometrical theories:

> The object of this discipline is not space, i.e., the system of points, lines, and planes determined by *geometrical* axioms (which we call "intuitive space" to distinguish it), but a "relational or order system" [*Beziehungs- oder*

> *Ordnungsgefüge*] determined by the *formal* axioms. As this represents the formal design of the spatial system, and turns into the spatial system again when spatial elements are substituted for indeterminate relata, it too will be called "space": "*formal space*." (Carnap 1922, 8)

Notice that, in 1922, Carnap does not yet use the term "structure" to label such abstract forms or order systems. This use of terminology changes in the course of the 1920s, however, and Carnap eventually comes to introduce the notion of structure in his work on axiomatics. An early instance of this can be found in Carnap's lecture notes for a course entitled "Philosophy of Space; Foundations of Geometry" held at the mathematical department of the University of Vienna in 1928 and 1930 as well as in Prague in 1932.[15] The subject matter of a formal axiom system of pure geometry is sketched here as follows:

> The AS [axiom system] is about undetermined objects. It determines only a relational structure between them. . . .
>
> Implicit definition: but more precisely: definition of a class of systems of objects, that is the shared "structure" of these systems. . . .
>
> An AS determines (defines) one (or several) structure[s] of a relational system, the "theorems" [*Lehrsatze*] determine structural properties of that system that follow from this definition, the AS; therefore analytic. (RC 089-62-02)

These brief comments highlight Carnap's general conception of axiomatics at the time: a theory can define one or several abstract structures shared by different relational systems satisfying the axiom system in question. How is the notion of an axiomatically defined structure understood here?

This issue as well as the method of implicit definition is first addressed in closer detail in Carnap's "Eigentliche und uneigentliche Begriffe" of 1927 as well as in his logic textbook *Abriss der Logistik* of 1929. The article contains a number of interesting observations regarding the axiomatic method, in particular on so-called definitions through axiom systems. Carnap illustrates this type of definition based on the example of a theory of basic arithmetic.[16] According to him, this theory can either be understood as describing the properties of the intended

[15] See documents RC 089-62-02 of the Rudolf Carnap Papers at the Archive of Scientific Philosophy (Hillman Library, University of Pittsburgh).

[16] The axiom system presented here is based on Russell's theory of arithmetical progressions presented in Russell (1919). Compare section 15.3.3 for a more detailed discussion of the theory. A second paradigmatic example discussed in the text is again Hilbert's axiomatization of Euclidean geometry in Hilbert (1899).

or standard model of the natural numbers. Alternatively, it can also be viewed as a formal theory in the following sense:

> We take the words “number” and “successor” as new terms that have not yet been given a meaning, and we stipulate that they are to refer to those concepts with the character specified by the AS. Thus here the AS makes no initial assumptions, but rather only through it is a class determined, which will then be called “the numbers,” and a relation, which will be called “successor.” In contrast to the determination of a concept by explicit definition, as discussed earlier, here the new concepts are not connected to old ones, but are specified by the formal characteristics they inherently possess; hence the terminology “implicit definition” for the determination of a concept by an AS. (Carnap 1927, 360–361)

As Carnap makes clear, this account of the implicit definition of primitive terms implies that theories so construed can be interpreted relative to different models. As will be shown in section 3.3, Carnap developed a detailed account of the model theory of axiomatic theories in his manuscript *Untersuchungen zur allgemeinen Axiomatik*, also written around the same time.

More important in the present context is how these models are related to the general structure defined by an axiom system. In the case of elementary arithmetic, this relation between the possible interpretations of the axiom system (including the standard or intended model) and their shared structure is described as follows:

> The first model, the sequence of cardinal numbers, is that for the sake of which the AS was set up. As we see, however, the AS, and therefore the implicit definition it expresses, applies not only to that case, but also to infinitely many others, namely all those that agree with it with respect to the specified formal properties, i.e., the structure. In the theory of relations, the sequences with these properties are called “progressions.” . . . The implicit definition of the sequence of numbers therefore does not uniquely determine the number sequence, but only the unique class of all progressions. (Carnap 1927, 362)

Given this model-theoretic account of axiom systems and their interpretations, what does Carnap mean by the structure of a theory? In addressing this issue, his distinction between “improper” and “proper” concepts plays a central role. Briefly put, an axiom system provides an implicit definition of several improper concepts whose meaning remains indeterminate. In the case of arithmetic, these are the concepts expressed by the primitive terms “natural number,” “zero,” and “successor” respectively. In addition, an axiom system can also be understood as an explicit definition of a proper concept whose meaning is in turn fully determined by the definition.

According to Carnap, this "explicit concept" of an axiom system closely corresponds to the class of models or realizations satisfying the axioms in question. In fact, in the case of elementary arithmetic discussed in his 1927 article, the relevant explicit concept (i.e., the "Peano number concept") is simply defined as the "class" of all arithmetical progressions (see Carnap 1927, 368). This insight that an axiom system not only provides an implicit definition of its primitive terms, but also an explicit definition of a higher-level mathematical concept, was not new at the time. In fact, it is likely that Carnap adopted the idea from his teacher Frege and the latter's critical discussion of Hilbert's work.[17]

Carnap's reformulation of the Fregean understanding of axiom systems as definitions of higher-level concepts is left informal in the 1927 article. This changes in Carnap's *Abriss der Logistik* of 1929, where the topic is taken up again. In Part II of the book, titled "Applied Logistic," Carnap gives a type-theoretic explication of the notion of axiomatic theories and their content. Roughly put, axiom systems are formalized here in a language of simple type theory in the following way: the primitive terms of a theory are expressed by free variables (of a given order and type) $X_1, \ldots, X_n$. Axioms and theorems are expressed as propositional functions $\Phi(X_1, \ldots, X_n)$, that is, as open formulas in the modern sense of the term.[18]

Given this formalization of mathematical theories, Carnap reiterates the point that an axiom system not only provides an implicit definition of the primitive terms occurring in the axioms, but also an explicit definition of a higher-order concept, the "explicit concept" of an axiom system. He gives the following formal account of the notion in the *Abriss*:

> For instance, if $x, y, \ldots \alpha, \beta, \ldots P, Q, \ldots$ are the primitive variables of the *AS* and if we name the conjunction of axioms (that is a propositional function) $\mathrm{AS}(x, y, \ldots \alpha, \beta, \ldots P, Q, \ldots)$, then the definition of the explicit concept of this *AS* is
>
> $\hat{x}, \hat{y}, \ldots \hat{\alpha}, \beta, \ldots \hat{P}, \hat{Q}, \{AS\,(x, y, \ldots \alpha, \beta, \ldots P, Q, \ldots)\}$ (Carnap 1929, 72)

[17] Frege's view of formal axiom systems as definitions of higher-level concepts is first expressed in his famous exchange with Hilbert. It is also presented in Frege's lecture "Logic in Mathematics" presented in Jena in 1914. Compare Carnap's notes of the lecture as well as Gottfried Gabriel's introduction, both published in Awodey and Reck (2004).

[18] This convention to express primitive mathematical terms as variables and axioms as propositional functions has a rich mathematical prehistory and is discussed more extensively in Carnap (1927).

The formula in this passage stands for a class of n-tuples of possible interpretations of the primitive variables of a given axiom system *AS*. Put in modern terms, an explicit concept is thus understood purely extensionally here, as determined by the class of models defined by the theory.[19] Carnap's notion of the explicit concept of an axiom system can thus be reconstructed in modern terms as a genuinely model-theoretic notion, namely as the model class of a given theory. Regarding the previous example of elementary arithmetic, Carnap holds that

> The explicit concept of Peano's AS of the numbers, e.g., is the class of number sequences that satisfy the AS; this is the logical concept *prog* (class of the progressions). (Carnap 1929, 72)

The central point to note here is that this notion of explicit concepts can be understood as Carnap's first attempt of a formal specification of the informal notion of "structure" (or "order system") used previously to describe the subject matter of a theory. To put it in Howard Stein's words, "A Fregean 'second-level concept' simply is the concept of a species of structure" (1988, 254).

3.2. Logical Construction

A significant part of Carnap's pre-*Syntax* work on the philosophy of mathematics was dedicated to foundational issues, in particular, to the further articulation of Frege's and Russell's logicist program. In the relevant publications on this topic, Carnap's understanding of concept formation in mathematics seems to be at odds with his structuralist thesis.[20] In particular, he states a strong preference here for the "logical construction" of mathematical concepts based on explicit definitions compared to the mere "postulation" of them in terms of axiomatic conditions. This clearly echoes Russell's preceding discussion of the genetic and the axiomatic method and his well-known remark on "theft over honest toil" in Russell (1919).

Logicism for Carnap too is based on a constructivist account of mathematics that distinguishes it from the axiomatic tradition of Hilbert and Dedekind. Compare Carnap on this general difference in his discussion of impredicative definitions:

[19] One should add here that, strictly speaking, the explicit concept of a theory cannot be identified with its class of models. Rather, what Carnap seems to suggest here is more of a "methodological identification" in the sense that one can study the one by studying the other. I would like to thank Erich Reck for emphasizing this point to me.

[20] See, e.g., Carnap (1930) and Carnap (1931).

> The essential point of this method of introducing the real numbers is that they are *not postulated but constructed*. The logicist does not establish the existence of structures that have the properties of the real numbers by laying down axioms or postulates; rather, through explicit definitions, he produces logical constructions that have, by virtue of these definitions, the usual properties of the real numbers. As there are no "creative definitions," definition is not creation but only name-giving to something whose existence has already been established. . . . This "constructivist method" forms part of the very texture of logicism. (Carnap 1931, 94)

The logicist approach to the formation of concepts in analysis (as well as in other mathematical fields) stated here is clearly incompatible with Hilbert's understanding of axiom systems as implicit definitions of the primitive terms of a theory.[21] How did Carnap address the apparent conflict between the two foundational approaches, namely logicism and formal axiomatics?

Interestingly, the two traditions are usually not treated separately in his work. In fact, Carnap's writings from the time have been described as a systematic attempt to "reconcile" Frege's logicist constructivism with Hilbert's structuralist understanding of mathematics.[22] One approach relevant here has to do with Carnap's own characterization of the structuralist thesis. According to him, mathematical structures can be specified not only through axiomatic definitions, but also as those entities characterizable in *purely* logical terms. Thus, a principled way to think about mathematical structures for Carnap is to say that structure is what is logically definable in higher-order logic (where higher-order logic is usually taken to be a system of simple type theory).

A closer look at his writings from the 1920s helps to see how this "logicist" account of the structuralist thesis and its relation to the axiomatic approach were understood by him. A first formulation of the former approach can be found already in *Der Raum*. In the first chapter of the book and based on the discussion of Hilbert's axiomatic approach, Carnap introduces a second way to specify a formal space (understood again as an abstract "order system"):

> The construction of formal space can also be undertaken by a different path, however, not just by the above way of setting up certain axioms about classes and relations: by deriving (ordered) series and, as a special case, continuous

[21] Compare again Carnap on the constructivism underlying Frege's logicism: "A concept may not be introduced axiomatically but must be constructed from undefined, primitive concepts step by step through explicit definitions" (Carnap 1931, 105).

[22] Compare Awodey and Carus (2001), Reck (2004), and Schiemer (2012a) for more detailed discussions of this point.

series from *formal logic*, the general theory of classes and relations. (Carnap 1922, 8)

This logical construction of formal space is specified as follows: based on work by Russell (in particular Russell 1903), Carnap first introduces the notion of order relations and order systems, so-called *series*. Special types of such order systems are series of the natural numbers, that is, arithmetical progressions in the sense specified in the previous section as well as continuous series of the real numbers. Given the latter, Carnap argues, one can set-theoretically construct continuous series of higher levels, that is, sets of ordered tuples of real numbers. A formal space (of n dimensions) is then defined as a "continuous series of n-th level (a series of series)" (Carnap 1922, 14). Put in modern terms, this is a manifold of n-ary tuples of real numbers.

Given this general notion of a formal space—also called a topological space R_{nt} here—one can construct other spaces such as projective space or different metrical spaces by imposing "more restrictive conditions on the order relations in these series" (Carnap 1922, 14). Now, Carnap does not specify in detail how these restrictive conditions are to be understood. It becomes clear from his remarks, however, that they should not be identified with axiomatic conditions.[23] More important to note here is that each of the resulting spaces remain formal in the sense specified above. Compare again Carnap on this point:

> We are here still dealing with merely formal relations, without any assumptions about what sort of objects have these relations to each other. The different *R*'s are therefore also called systems of order-relations (systems of ordinal relations), briefly, order-systems. (Carnap 1922, 17)

Given this set-theoretical construction of formal spaces as manifolds of real numbers, two points of commentary are in order here. The first point concerns the relation between Carnap's logicist account of geometrical structures and the axiomatic approach discussed in the previous section. How precisely does the specification of structure in terms of entities characterizable in purely logical terms correspond to the one in terms of axiomatic definitions?

[23] In fact, in an interesting passage, Carnap mentions the axiomatic method as an alternative approach to the specification of such a formal space: "Now, it has emerged that the resulting order-structures (e.g., R_{3p}), if they are to be investigated on their own (i.e., without reference to R_{3t} or R_{nt}), are simpler to construct if they are presented directly as structures of certain simple relations whose formal properties are given—rather than taking the circuitous route by way of continuous series of the first, and then of the third level subject to certain limiting conditions" (Carnap 1922, 15). See Mormann (2007) and Carus (2007) for more detailed discussions of Carnap's approach.

Interestingly, we saw that at least in Carnap (1922), Carnap viewed the two approaches as essentially equivalent ways to think about formal space. More specifically, as pointed out by Friedman in his editorial notes in Carnap (2019), the axiomatic approach gives implicit definitions of the primitive terms of a theory, whereas the logicist approach consists in "*explicitly* defining a model for such an axiom system within . . . set theory." One could therefore think of the connection between the axiomatic and the logicist approach in the following way: a formal space, conceived of as an "order system," is treated here as a concrete model of an axiomatic theory that is representable in set theory.[24] It thus forms a particular instance falling under the higher-level "explicit concept" defined by the theory.

It should be noted however that, strictly speaking, Carnap does not identify the subject matter of a formal geometry with a particular order system (conceived as a set-theoretic model of the theory) in 1922. Given the notion of number series, Carnap introduces the notion of a similarity between such systems. This corresponds roughly to the modern notion of an isomorphism between two ordered sets.[25] An "order type" of a particular series is then defined as the concept holding of all series similar to it. In the case of progressions, this is the order type ω; in case of continuous number series, this is order type λ. Compare again Carnap on this point:

> To express more briefly what holds for these mutually similar series, we assert it of a single formal representative of them that we construct for this purpose. . . . Strictly speaking, this representative of the progressions is nothing other than their concept (in our sense of the word). (Carnap 1922, 13)

Applied to Carnap's account of formal spaces sketched previously, it follows that a (topological, projective, or metrical) space should not be understood as a particular order system. Rather, it presents an order type, that is, a higher-level similarity concept or, put in purely extensional terms, a similarity class of such a system. Thus, both in Carnap's axiomatic approach and in the set-theoretic approach, mathematical structures are identified with higher-level concepts. We will return to his conception of structures as similarity (or isomorphism) types in the next section.

Turning to the second point, one immediate consequence of Carnap's approach in 1922 is that formal geometry itself becomes a part of logic or set theory. This fact was clearly intended and led him to formulate a *generalized* logicism

[24] The latter approach, to think about structure in terms of logically definable models, can be found also in subsequent work by Carnap, in particular in his *Untersuchungen* manuscript. See Schiemer (2012b) for a closer discussion of this point.

[25] Compare section 15.3.3 for a closer discussion of Russell's notion of the similarity of relations and Carnap's later generalization of it.

not limited to number theory and analysis.[26] The view that formal space (as the subject matter of pure geometry) is essentially a logical concept is expressed at different stages in his work on the foundations of geometry. An early formulation of the idea is contained in his dissertation manuscript of 1920, which formed the basis for *Der Raum*:

> An [abstract space] is a logical system of relations among indefinite elements. It says: in case certain relations, specified purely formally, hold among the elements of a set, then certain theories hold for this system. (Unpublished manuscript, Quoted from Carus 2007, 110)

Compare also a related remark concerning the status of pure geometry in Carnap's lectures notes of 1928:

> (Mathematical) geometry is essentially relation theory (theory of relations, of structures, of order systems) a branch of formal logic, therefore analytic. (RC 089-62-02)[27]

Pure geometry forms a part of logic because its subject matter, namely abstract space, can be represented in terms of sets of real number tuples that, given Frege's thesis, are effectively reducible to arithmetical and thus to purely logical notions.

A different but related account of the logical nature of geometry can be identified in Carnap's subsequent work on axiomatics. Returning again to his *Abriss der Logistik*, we saw that an axiom system not only gives an implicit definition of its primitive terms, but also an explicit definition of a higher-level concept applying to all models of the theory in question. Carnap discusses a number of mathematical examples to illustrate this Fregean account, including Peano arithmetic, Zermelo-Freankel set theory, projective geometry, and topology (among others).

For instance, Carnap presents the following formalization of Hausdorff's neighborhood axioms for topological spaces: the theory describes one primitive binary relation, namely $\{\alpha \mathbf{U} x\}$ standing for "α is a neighborhood set of x." The class of points is defined as the range of relation **U**, that is, as $\mathbf{pu} := Ran(\mathbf{U})$.

[26] This geometrical logicism, i.e., the fact that pure space is constructable in pure logic, essentially goes back to Russell's extensive discussion of different geometries in his *Principles of Mathematics* (1903). Carnap frequently refers to this book, as well as to Russell and Whitehead's *Principia Mathematica* as the primary sources for his own discussion of formal space. See, in particular, Gandon (2009) and Gandon (2012) for further details on Russell's approach.

[27] I leave open the issue here how the concept of analyticity used here was understood by Carnap in his pre-syntactical work.

The theory of neighborhoods is given by the following axioms (in slightly modernized form):

Ax1a: $Dom(\boldsymbol{U}) \subset \wp(\boldsymbol{pu})$ (Neighborhoods are classes of points.)

Ax1b: $\boldsymbol{U} \subset Kon(\in)$ (A point belongs to each of its neighborhoods.)

Ax2: $\forall \alpha, \beta, x(\alpha \boldsymbol{U} x \wedge \beta \boldsymbol{U} x \rightarrow \exists y(y \boldsymbol{U} x \wedge y \subset \alpha \cap \beta))$ (The intersection of two neighborhoods of a point contains a neighborhood.)

Ax3: $\forall \alpha, y(\alpha \in Dom(\boldsymbol{U}) \wedge y \in \alpha \rightarrow \exists \gamma(\gamma \boldsymbol{U} y \wedge \gamma \subset \alpha))$ (For every point of a neighborhood α, a subclass of α is also a neighborhood.)

Ax4: $\forall x, y(x, y \in \boldsymbol{pu} \wedge x \neq y \rightarrow \exists \alpha, \beta(\alpha \boldsymbol{U} x \wedge \beta \boldsymbol{U} y \wedge \alpha \cap \beta = \emptyset))$ (For two distinct points, there exist two corresponding neighborhoods with no points in common.)

Given this axiomatization, it seems natural to say that the explicit concept "*hausd*" represents the structure defined by Axioms 1–5, i.e., the structure shared by all concrete models satisfying the theory. Moreover, given the fact that in Carnap's formalization of the theory, the only primitive term, **U** (standing for the neighborhood sets), is symbolized as a relation variable, it follows that the concept *hausd* turn out to be purely logical in character. Compare Carnap on this point:

> The explicit concept of a geometrical AS . . . presents the logical concept of the relevant type of space (e.g., the concept "projective space"). In this sense geometry can also be represented as a branch of logistic itself (as arithmetic) instead of being a case of application of logistics to a nonlogical domain. (Carnap 1929, 72)

Concerning the specific example of Hausdorff topology, he goes on to add:

> The explicit concept of the AS is the class of the "Hausdorffian neighborhood systems" (*hausd*), a purely logical concept. (Carnap 1929, 76)

These passages illustrate Carnap's attempt to reconcile the logicist's emphasis on explicit definitions with structural axiomatics. The resulting version of the logicist thesis does not amount to the claim that the individual models of an axiomatic theory are logically constructible. Rather, Carnap adopts the Fregean strategy to represent the structural content of a mathematical theory in terms of a *higher-level* concept defined by the theory's axioms. Since such explicit concepts of theories (such as hausd) are definable in a language of *pure* type theory, it follows that the represented mathematical content is also purely logical, and the axiomatic theory thus "a branch of logic."

3.3. Model Structures

An important characteristic of modern axiomatics is the new focus on metatheoretic properties of theories and their interpretations. As a consequence of this "metatheoretic turn" at the end of the 19th and early 20th century, axiom systems themselves became an object of (meta)mathematical investigation. Moreover, mathematicians working in geometry, number theory, and other disciplines started to investigate systematically the content of theories in terms of structure-preserving mappings between their models.[28]

This metatheoretic approach in modern axiomatics is usually characterized today in structuralist terms, that is, by referring to the structures or structural properties defined by an axiom system. More specifically, it is usually held that one can investigate the logical structure of a given theory not only by deriving theorems, but also by analyzing how particular axioms contribute to the specification of this content, how the structure is changed if particular axioms are added or omitted from the system, and so on.[29]

Interestingly, a similar approach to expressing the metatheoretic properties of theories in structuralist terms can be identified in Carnap's work on general axiomatics from the late 1920s. We saw in section 3.1 that Carnap, from his *Der Raum* onward, defended the view that an axiom system defines a structure (or an "order system") that in turn can be instantiated by different "formal models" or physical "realizations." While he does not discuss models and their properties in published work in closer detail, the model-theoretic account of theories is developed in his project on "general axiomatics," in particular, in his *Untersuchungen zur allgemeinen Axiomatik*. The manuscript contains a detailed discussion of the logical formalization of axiomatic theories that is similar to the account presented in *Abriss der Logistik* (see again section 3.1 for details). In addition, Carnap's manuscript also contains a logical explication of several genuinely metatheoretical concepts (such as the notions of logical consequence, truth in

[28] This line of research includes Dedekind's categoricity result for arithmetic in *Was sind und was sollen die Zahlen* (1888), Hilbert's consistency and independence proofs in his *Grundlagen* (1899), as well as the formulation of different notions of completeness in subsequent work by the postulate theorists. See Awodey and Reck (2002) for a rich study of early metatheoretic work in modern axiomatics. Compare also the articles on Hilbert and Dedekind in the present volume for further details.

[29] Compare Hintikka for a characterization of this general approach: "An axiom system is also calculated to serve also as an object for a metatheoretical study. . . . For the purpose of reaching such a metatheoretical overview, it is crucial to grasp the logical structure of the theory in question, in the sense of seeing what the different independent assumptions of the theory are, of seeing which theorems depend on which of these basic assumptions and so on. For this purpose, the axiomatic method is eminently appropriate" (Hintikka 2011, 72–73).

a model, etc.), as well as several metatheorems on the relation between different notions of completeness.[30]

A central concept defined in this context is that of a "model isomorphism," that is, a mapping between two models of a given theory that preserves their relational structure. The isomorphism relation (or, in Carnap's terms, the "isomorphism correlation") between two models is defined roughly in the modern sense as a bijective function between the respective individual domains that induces correlations between the higher-order domains and thus preserves the relations in the models.[31] Based on this notion, Carnap specifies several completeness properties that turn out to be crucial for the understanding of the "logical structure" of theories, including the notions of *non-forkability* and *monomorphicity* (or, in modern terminology, of semantic completeness and categoricity).[32]

How does Carnap specify the structural content of axiomatic theories in *Untersuchungen*? In contrast to previous work, he argues here that an axiom system does not only define an "explicit concept" (conceived of as the class of its models), but possibly also several more fine-grained structures, so-called *model structures* (conceived of as subclasses of its model class). Roughly put, a model structure is the structure shared by isomorphic models of a given theory. As Carnap points out, such structures are to be identified with the classes of isomorphic models:

> In logistic, one tends to define structures, including also the cardinalities, in terms of isomorphism classes. (Carnap 2000, 72)

In the related article (Carnap and Bachmann 1936), a more detailed specification of model structures in terms of the notion of a "complete isomorphism" is given:

> Since the complete isomorphism between *n*-place models (i.e., sequences with *n* members) is a 2*n*-ary equivalence relation, *n*-place relations can be defined over the field of this relation . . . such that the *n*-place relations have the following properties: for each model there exists exactly one such relation which is satisfied by the constituents of the model and is satisfied by the constituents of two different models if and only if the models are completely isomorphic. The

30 See Carnap (2000). Several of the concepts introduced here were published later on in Carnap and Bachmann (1936). Compare Awodey and Carus (2001), Reck (2004), and Schiemer, Zach, and Reck (2017) for further details on Carnap's axiomatics project.

31 Carnap's definition of "model isomorphism" is actually more complex than this since it takes into account a mapping between "inhomogeneous models," that is, models with relations of different types and orders. See Carnap (2000) and also Carnap and Bachmann (1936) for further details. See also Carnap (1929) for a simplified definition.

32 Compare, in particular, Awodey and Carus (2001) and Schiemer, Zach, and Reck (2017) for assessments of Carnap's early metatheory and of the limitations of his approach.

> relations so determined we will call *structures* and will say that model M_1 *has* structure S_1 if "$S_1(M_1)$" is analytic. (Carnap and Bachmann 1981, 74)

Structures are specified here as unary relations that hold between any two models of a given theory in case there exists an isomorphism between them. In an attached footnote to the passage, Carnap goes on to add that structures in this sense are relations introduced by a "definition through formation of abstraction classes" or simply by "definition through abstraction" (Carnap and Bachmann 1936, 171).[33]

Translated into modern terminology, the idea expressed here is to treat structures as particular equivalence classes, namely as "isomorphism classes" of models. Let K be the class of models defined by a theory T. Let $K/_{\cong}$ be the partition of class K induced by a suitable isomorphism relation $\cong$ between the objects in this class. For a model $M \in K$, the relevant model structure is simply the isomorphism class $[M]_{\cong} := \{N \mid N \cong M\}$. Each model structure of T is a cell of the partition of K induced by $\cong$. Moreover, given that $K/_{\cong}$ forms a partition, for any two different model structures we have the following two results: (i) $[M]_{\cong} \cap [N]_{\cong} = \varnothing$ and (ii) $\bigcup_{M \in K}[M]_{\cong} = K/_{\cong}$.[34]

Given this approach, two further points of commentary are in order here. First, it should be noted that Carnap's approach to thinking about mathematical structures in terms of definitions by abstraction was not new, but fairly conventional at the time. In fact, in his *Abriss* and in other publications, Carnap refers to Frege's famous definition of cardinal numbers in terms of an abstraction principle as well as to work by Couturat and Weyl for further details on the method. Concerning the notion of mathematical structure, Carnap's central background is Russell's logical work on the general theory of relations. In fact, the notion of "model structures" outlined in *Untersuchungen* present a straightforward generalization of the notion of "relational structures" previously introduced by Russell in his *Introduction to Mathematical Philosophy* (Russell 1919).[35]

In chapter 6 of his book of 1919, Russell first defines what he calls a "similarity relation" between relations: two n-ary relations R, S are similar if there exists a monotone, that is, a structure-preserving function f: $R \rightarrow S$ such that $x_1, \ldots, x_n \in R$ iff $f(x_1), \ldots, f(x_n) \in S$ (Russell 1919, 52–55). The "relation-number" of a given relation is then defined as "the class of all those relations that are similar to the given relation" (Russell 1919, 56). Based on this, Russell then introduces the

[33] This notion of structure based on the method of definition by abstraction but restricted to a single relation is discussed also in Carnap's *Abriss*. See section 15.4.1 for further details.

[34] A natural way to think about the kind of structural abstraction from isomorphic models underlying this approach is in terms of abstraction principles. I will return to this point in section 15.4.1.

[35] See the article by Heis in the present volume for a more detailed study of Russell's structuralist views.

notion of "structure" in the sense that two similar relations "have the same structure." More explicitly, he holds that

> two relations have the same structure when they have likeness, *i.e.* when they have the same relation-number. Thus what we defined as the "relation-number" is the very same thing as is obscurely intended by the word "structure"—a word which, important as it is, is never (so far as we know) defined in precise terms by those who use it. (Russell 1919, 61)

This passage shows how strongly Carnap's account of structures in his general axiomatics project is influenced by Russell's preceding ideas. In particular, in *Untersuchungen* and also in Carnap and Bachmann (1936), Russell's notion of similarity is generalized to apply also to "non-homogenous" relations as well as to models understood as ordered sequences of such relations. Similarly, the Russellian account of structures as "relation numbers," that is, as similarity classes of relations, is adopted in Carnap's work to apply also to formal models of different arities and of more complex type levels.

The second point to emphasize here is that Carnap's motivation for the introduction of model structures was clearly metatheoretic in spirit. Talk of such structures allowed him to develop a more refined account of the subject matter of axiomatic theories than in previous work. More specifically, instead of identifying the structural content of a theory with a single "explicit concept," Carnap proposes a classification of axiomatic theories here based on the number and type of model structures they describe. In the completed first part of *Untersuchungen* (published as Carnap 2000), he introduces the notion of the "structure number" of theories as the number of isomorphism classes they describe. Categorical theories such as second-order Peano arithmetic have number 1; noncategorical theories such as group theory or Hausdorff topology have structure numbers greater than 1.

In the projected but unfinished second part of the manuscript (RC 081-01-01 to 081-0133) as well as in Carnap and Bachmann (1936), a further specification of the structural content of theories is given based on the notion of so-called extremal structures.[36] The fundamental idea here is that the content of a theory is not only determined by the number of its isomorphism classes of models, but also by possible relations between them. A central notion introduced by Carnap for the study of such interrelations between structures is that of a "proper

[36] In the following, I refer mainly to the published results in Carnap and Bachmann (1936). For a closer discussion of the differences between the 1936 paper and the existing notes on Part 2 of *Untersuchungen* see Schiemer (2013).

structure extension" (or "proper substructure"). Carnap proposes the following definition of this notion in his article with Friedrich Bachmann of 1936:

> We call a structure *S* a proper substructure of a second structure *T*, if *S* and *T* are distinct and every model having the structure *S* is isomorphic to a proper part of every model having the structure *T*. (Carnap and Bachmann 1936, 175)

Put differently, given a theory **T** and two model structures *S*, *T* described by it, we say that *S* is a proper substructure of *T*, in symbols $S \sqsubset T$, if and only if (i) $S \neq T$ and (ii) for every model *M* with structure *S* and for every model *N* with structure *T*, there exists a mapping that *embeds* (in the model-theoretic sense of the term) *M* into *N*. Notice that the relata of this substructure-relation are the structures (conceived as isomorphism classes) themselves and not the models instantiating them.

Based on this notion of substructure, defined in terms of isomorphisms and embeddings between models, Carnap suggests an ordering of the class of model structures of a given axiomatic theory in terms of their extremal structures. The extremal structures consist of "initial structures," "end structures," and "isolated structures," defined in the following way. Given the class of structures defined by a theory **T**, we say that

1. S is an "*initial structure*" iff there exists no *T* of theory **T** such that $T \sqsubset S$;
2. S is an "*end structure*" iff there is no *T* of theory **T** such that $S \sqsubset T$;
3. S is an "*isolated structure*" iff there is no *T* of theory **T** such that $S \sqsubset T$ or $S \sqsubset T$.[37]

Put less formally, the initial structures (taken together with the isolated structures) represent the structures of minimal models of the theory in question, that is, of models that do not contain isomorphic copies of other models as submodels. Similarly, end structures (taken together with the isolated structures) represent structures of maximal models, that is, models not embeddable in other models. Isolated structures stand for models without any embeddings to other, non-isomorphic models.

This framework of extremal structures was explicitly introduced by Carnap to further analyze the structural content of axiomatic theories.[38] In particular, according to him, each theory can be assigned a "structure diagram," that is, a

[37] Compare Carnap's slightly different definition of these extremal structures in terms of the domain and range of the substructure relation (Carnap and Bachmann 1936, 176).

[38] We refer the reader to Schiemer (2012a) for a closer discussion of this theory of extremal structures and the limitations of Carnap's approach.

(possibly infinite) directed graph where the nodes represent model structures defined by the theory and the edges represent the proper substructure relation (and thus the embedding properties between models of different structures). Such a structure diagram of a theory can thus be viewed as a graphical representation of its structural content.

To see how Carnap thought about this structural content in terms of his newly introduced terminology, let us briefly look at one of his mathematical examples discussed in this context, namely the theory of elementary arithmetic. This is essentially a version of Russell's theory of arithmetical progressions with a single primitive relation *R(x,y)* (standing for a successor relation) and based on four axioms:

*b*1 $\forall x \forall y \big(R(x,y) \rightarrow \exists z(R(y,z))\big)$

b2 $\forall x \forall y \forall z \Big(\big(R(x,y) \wedge R(x,z) \rightarrow y = z\big) \wedge \big(R(x,y) \wedge R(z,y) \rightarrow x = z\big)\Big)$

*b*3 $\exists ! x \big(x \in Dom(R) \wedge x \notin Ran(R)\big)$

*b*4 $Min_S \left(b1 - b3; R\right)$ (Carnap and Bachmann 1936, 179)

Axiom *b*1 states that relation *R* is endless. Axiom *b*2 states that *R* is an injective function. Axiom *b*3 states that there exists a base element in the progression. Axiom 4 is a so-called minimal axiom similar in effect to an induction axiom. It effectively imposes that all models satisfying axioms *b*1–*b*3 belong to minimal structures in the sense specified. What is particularly interesting about Carnap's discussion of this mathematical axiom system is the way in which he characterizes its structural content by analyzing the corresponding structure diagrams of its subtheories. Consider the two graphs in Figure 1, presenting the possible structures of models satisfying axiom systems *b*1–*b*2 and *b*1–*b*3 respectively.

The structures described by subtheory *b*1–*b*2(*R*) include the intended natural number structure, i.e., all isomorphic models of the form of a "progression" as well as infinitely many cycles of order 1 up to infinity. These structures, as well as the possible combinations of them, are presented by the nodes in the diagram on the right-hand side.

By adding axiom *b*3 to the system, the structural content is significantly restricted. In particular, as is illustrated in the diagram on the left-hand side, adding *b*3 to the base theory will have the effect that all structures of isolated cycles will be eliminated. The model class of the theory now contains the models of the intended structure *P* (i.e., progressions) as well as unintended

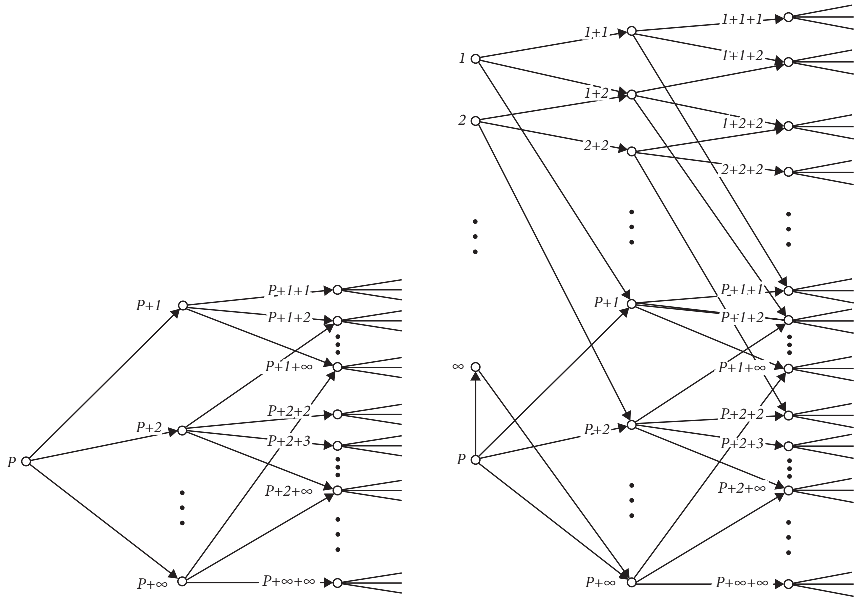

Figure 1. Structure diagrams of theories b1-b3(R) and b1-b2(R).

models consisting of combinations of progressions and cycles. Adding the minimal axiom *b*4 finally has the effect that all further unintended models are ruled out and the only remaining structure defined by the theory is that of an arithmetical progression *P*. In other words, adding axiom *b*4 to the system *b*1–*b*3 will render the resulting theory categorical or, in Carnap's own terminology, monomorphic.

4. Points of Contact with Modern Structuralism

The previous section has shown that one can identify several proposals in Carnap's early philosophy of mathematics on how to characterize the structuralist thesis. Interestingly, not only does his general structuralism connect his work with that of several of his contemporaries, including Russell, Husserl, Cassirer, and Quine, but one can also find several parallels between Carnap's early views on the structural nature of mathematical theories and contemporary structuralism. In this section, we will focus on two specific points of contact with the present philosophical debate.

4.1. Structural Abstraction

Carnap's treatment of model structures in his work on general axiomatics is based on the notion of abstraction. Specifically, we saw that the structure of a model of a given theory was identified with its isomorphism type, that is, with the class of models isomorphic to it. The main philosophical background for his approach was clearly Russell's work, in particular, the extensive treatment of abstraction principles in Russell (1903) and the subsequent discussion in Russell (1919). Interestingly, Carnap's abstraction-based approach is also closely connected to much more recent debates on mathematical structuralism.

Present research on the topic is based on a general distinction between two ways to think about the nature of mathematical structures. According to "eliminativist" structuralists, the mathematicians' talk about abstract structures should be understood merely as an abbreviation for generalizing over all models of a given theory. In contrast, "non-eliminative" structuralists such as Parsons, Shapiro, and others are realists about mathematical structures. For them, abstract entities such as the structure of the natural numbers exist in addition to the particular (set-theoretic) systems satisfying a theory.[39]

In the literature on non-eliminative structuralism, a further distinction is usually made between forms of *ante rem* and *in re* structuralism.[40] Briefly put, *ante rem* structuralists hold that abstract structures are bona fide objects that exist independently of their instantiating systems. Thus, the structure of the natural numbers exists irrespectively of whether there are particular number systems satisfying the axioms of Peano arithmetic. In contrast, *in re* structuralists usually argue that such higher-order entities are conceptually or ontologically dependent on their instantiating systems. Thus, according to this position, the natural number structure shared by all models of second-order Peano arithmetic exists only insofar as there are concrete models of the theory that instantiate the structure.

Carnap's own account of model structures outlined in *Untersuchungen* can be understood as an early formulation of *in re* structuralism about mathematics. In particular, his method of introducing structures by definitions by abstraction, i.e., by taking equivalence classes of isomorphic models, can be considered as one way to specify the conceptual dependency between structures and particular systems. Structures—conceived of as isomorphism types or classes—exist only if there are models of the axiomatic theory in question.[41] Comparable accounts

[39] See Reck and Price (2000) for an overview of the different accounts of mathematical structuralism.

[40] See, in particular, Shapiro (1997) on this distinction.

[41] One should add here that in Carnap's understanding of structures as isomorphism classes, the conceptual dependency between structures and systems is only given under the assumption that the classes in question are non-empty.

of such an abstraction-based structuralism can also be found in the current literature on the topic. Linnebo and Pettigrew have recently introduced a version of non-eliminative structuralism based on Fregean abstraction principles that determines this kind of abstraction from concrete systems to pure abstract structures (Linnebo and Pettigrew 2014).[42] The motivating idea underlying their approach is described as follows:

> A pure structure is the result of some operation of abstraction on a class of systems that are pairwise isomorphic. (Linnebo and Pettigrew 2014, 270)

Pure structures such as the structure of the natural numbers or of complete ordered fields can be introduced by abstracting away all nonessential or nonstructural properties of the objects in such systems. Such properties are identified here with properties not shared by isomorphic systems. The corresponding principle of structural abstraction has the form

$$[S] = [S'] \Leftrightarrow S \cong S' \tag{SA}$$

where S, S' represent relational systems of the same signature, $\cong$ symbolizes the isomorphism relation between such systems, and $[S], [S']$ express the structures of S and S' respectively. The principle (SA) specifies an identity condition for abstract structures: for any two systems of a given signature, one can say that they share the same abstract structure just in case they are isomorphic.[43]

From a methodological point of view, this abstraction-based account of structuralism (as developed by Linnebo, Pettigrew, and Reck) is clearly similar to Carnap's position from the late 1920s. Mathematical structures are specified here and there as general forms shared by isomorphic models or systems. Moreover, even though Carnap does not explicitly introduce a structural abstraction principle of the form of (SA) in his work on axiomatics, a similar principle can be found in his *Abriss der Logistik* of 1929. In §22, in the context of his discussion of relations, the structure (or relation number) of a relation is identified with the "class of its isomorphic relations." Theorem L 22-24 then states an abstraction principle very similar to the one given above (Carnap 1929, 90):

$$P\, Smor\, Q\, . \equiv .\, Nr'\, P = Nr'Q$$

[42] A related account of mathematical structuralism based on a notion of "Dedekind-Cantor abstraction" has recently been developed in Reck (2018).

[43] Linnebo and Pettigrew also formulate structural abstraction principles for positions and relations in such abstract structures. See Linnebo and Pettigrew (2014) for further details.

where P, Q are relations of a given type and order, Smor stands for the isomorphism relation between them and $Nr'P$, $Nr'Q$ stand for the structure of P and Q respectively.[44]

Despite the obvious similarity between Carnap's and the contemporary accounts, there are also important differences concerning the very notion of structural abstraction. With respect to abstraction principles such as (SA), this relates to the question how the abstraction operator used on the left-hand side of the equivalence statement is understood. Such operators are usually treated as functions from a domain consisting of relational systems to a codomain of abstract structures. How can the codomain of the structural abstraction operator be understood?

In addressing this question, it is interesting to compare recent contributions to abstraction-based structuralism with different uses of abstraction principles (and definitions by abstraction) in 19th- and early 20th-century mathematics. In his recent study of this topic, Mancosu has shown that one can distinguish between at least three ways in which the operator in abstraction principles was understood in mathematics (Mancosu 2016). The values of abstraction functions were either taken to be (i) (canonical) representatives of the equivalence cells determined by an equivalence relation between mathematical objects or (ii) the equivalence classes themselves. Alternatively, the values of a given abstraction operator were also sometimes thought of (iii) as newly introduced *abstracta*, that is, as a type of "new object not coinciding with the equivalence class or one of its representatives" (Mancosu 2016, 87).

Mancosu's taxonomy of the possible values of abstraction functions corresponds closely to the different ways in which structural abstraction is described in the literature on structuralism. We saw that in Carnap's case, structures of models are identified with their isomorphism types.[45] Similar versions of this understanding of mathematical structures as equivalence classes can also be found in the more recent literature. Compare, for instance, how the nature of abstract structures in the case of basic arithmetic is described by Benacerraf in his influential article of 1965:

[44] Carnap refers to Russell (1919) for further discussion of the notion of structure in this section. Compare Heis's article in the present volume for a detailed discussion of similar structural abstraction principles in Russell's work.

[45] It should be noted here that a corresponding structural abstraction principle of the form (SA) can lead to inconsistency in case the structures on the right-hand side of the biconditional can also be inserted as models on the left-hand side. This fact is related to the Burali-Forti Paradox and has been discussed in the (neo-)logicist literature and in philosophy of mathematics more generally. See, in particular, Linnebo and Pettigrew (2014) on this point. Notice that this danger of yielding an inconsistent account of structural abstraction is excluded in Carnap's type-theoretic framework given the fact that model structures are required to be of a higher type than their instantiating models.

> If we identify an abstract structure with a system of relations (in intension, of course, or else with the set of all relations in extension isomorphic to a given system of relations), we get arithmetic elaborating the properties of . . . all systems of objects (that is, *concrete* structures) exhibiting that abstract structure. (Benacerraf 1965, 70)

While Benacerraf does not address the issue of structural abstraction from systems to abstract structures here, he explicitly mentions the possibility of identifying such structures as isomorphism classes of a given system.[46]

A different view of structural abstraction is presented in a recent paper by Leach-Krouse (2017). Leach-Krouse discusses different "structural" abstraction principles for models of axiomatic theories in the context of a neologicist approach to mathematics. The principles introduced here are similar in logical form to the structural abstraction principles already mentioned. However, the abstraction operators are understood neither in Carnap's nor in Linnebo and Pettigrew's sense.[47] Instead, Leach-Krouse's account follows an "approach to abstraction favored by Georg Cantor and Richard Dedekind, on which abstraction serves to introduce the isomorphism type of a mathematical structure as a first-class citizen of the mathematical universe" (Leach-Krouse 2017, 3).

More specifically, given a finitely axiomatizable theory T expressed in a second-order language with signature $\Sigma = \{R_1, \ldots, R_n\}$, a structural abstraction principle A_T for T is characterized here as a second-order sentence of the form

$$(\forall \overline{X}_n)(\forall \overline{Y}_n)[\S_T(\overline{X}_n) = \S_T(\overline{Y}_n) \leftrightarrow \overline{X}_n E_T \overline{Y}_n] \qquad (A_T)$$

The (sequences of) variables $\overline{X_n}$ and $\overline{Y_n}$ present model variables in Carnap's understanding of the term, that is, ordered sequences of relation or function variables substituted for the primitive terms of the theory. The binary relation E_T presents an isomorphism relation between models of theory T. The terms $\S_T(\overline{X_n})$ and $\S_T(\overline{Y_n})$ present the structures of models $\overline{X_n}$ and $\overline{Y_n}$ respectively. Thus, in a sense comparable to (SA), this principle states that any two models of T

[46] In his 1965 article, Benacerraf does not mention Carnap as an early proponent of such an account of mathematical structures.

[47] Leach-Krouse explicitly mentions the possibility of identifying mathematical structures with isomorphism classes (Leach-Krouse 2017, 5–6).

that are isomorphic also share the same structure and vice versa (Leach-Krouse 2017, 9–10).

In contrast to Carnap's account of structural abstraction, the abstraction operator $\S_T$ does not give isomorphism classes as values here. Rather, $\S_T$ expresses a type-lowering function just as in the case of Hume's principle in the neo-logicist project. More precisely, it presents a function that assigns an object of the individual domain *dom* of the object language to each model of the theory *T*. The only constraint on the interpretation of $\S_T$ determined by the principle (A_T) is that the function will assign the same individual to isomorphic systems. Thus, unlike in Carnap's account, the structure of a given model is specified here in terms of "first-order representatives" from the domain of the object language.[48]

A third possible approach to structural abstraction is presented in Linnebo and Pettigrew (2014) as well as in Reck (2018). In both accounts, the abstraction operator in (SA) gives as values pure structures of relational systems (of a given mathematical signature) that are thought of neither as equivalence classes nor as first-order representatives, but rather as newly introduced abstracta or "*sui generis* objects" (Linnebo and Pettigrew 2014, 274). More specifically, structures are themselves structured systems consisting of a domain of pure positions (or placeholders) and pure relations that can be exemplified by concrete set-theoretic systems. I cannot enter here into a closer discussion of the different approaches to structural abstraction or their philosophical implications.[49] Instead, let us turn to a second point of contact between Carnap's early structuralism and the modern debate.

4.2. Invariance and Definability

The notion of structural properties of (objects in) mathematical systems plays a central role in modern structuralism. In fact, structuralism is often characterized by reference to this notion: it is the thesis that mathematical theories investigate only the structural or relational properties of the objects in their respective domains.[50] According to this view, mathematical systems such as groups or

[48] Leach-Krouse's approach to structural abstraction seems similar to Mancosu's first strategy of thinking of the values of a mathematical abstraction operator in terms of representatives of a given equivalence class. Notice, however, that in Leach-Krause's account it is not required that the structures conceived of as first-order representatives form elements of the relevant isomorphism classes they stand for.

[49] In this respect, it might be interesting to give a closer discussion of possible connections between an abstraction-based structuralism and debates on the metaontology and logic of abstraction principles in neo-logicism. See, e.g., Linnebo (2018).

[50] Compare, for instance, Hellman on this point: "On a structuralist view, . . . the mathematician claims knowledge of structural relationships on the basis of proofs from assumptions that are frequently taken as *stipulative of the sort of structure(s) one means to be investigating*" (Hellman 1989, 5).

number systems are usually specified axiomatically, that is, in terms of implicit definitions. The task of the mathematician is then to investigate the "structural relationships" between the objects in such systems based on deductive proofs.

As we saw, this account of modern axiomatics closely corresponds to Carnap's view. Compare again the passage from his lecture notes on geometry (already quoted in section 3.1):

> An AS determines (defines) one (or several) structure[s] of a relational system, the "theorems" [*Lehrsätze*] determine structural properties of that system that follow from this definition, the AS; therefore analytic. (RC 089-62-02)

In his work on type theory and general axiomatics, Carnap proposes two ways in which this notion of structural properties can be made logically precise. The first approach— presented in Carnap (1929) and Carnap (2000)— is to specify structural properties of relations (and henceforth also of models of axiomatics theories) in terms of the notion of invariance under isomorphic transformations. Carnap gives the following definition in his *Untersuchungen* manuscript:

Definition 1.7.1. *The property fP of relations is called a "structural property" if, in case it applies to a relation P, it also applies to any other relation isomorphic to P. . . .*

$$(P,Q)\left[\left(fP \,\&\, Ism(Q,P)\right) \rightarrow fQ\right]$$

> The structural properties are so to speak the invariants under isomorphic transformation. They are of central importance for axiomatics. (Carnap 2000, 74)

The structural properties of a relation are thus those properties left invariant or preserved under suitable isomorphisms. Typical examples of such properties mentioned by him concern the arity and type of relations, the cardinality of their fields, as well as properties such as the reflexivity, symmetry, and transitivity of a binary relation.

In addition to this invariance-based account, Carnap proposed a second way to think about structural or "formal" properties in his monograph *Der Logische Aufbau der Welt* of 1928. In the first section of the book, the notion of a relational structure is characterized in terms of "the totality [*Inbegriff*] of its formal properties" (Carnap 1928, 13). Put differently, the structure of a given relation can be determined by considering all formal properties that apply to it. Formal properties, in turn, are specified in the *Aufbau* as follows:

> By formal properties of a relation, we mean those that can be formulated without reference to the meaning of the relation and the type of objects between which it holds. They are the subject of the theory of relations. The

> formal properties of relations can be defined exclusively with the aid of logistic symbols, i.e., ultimately with the aid of the few fundamental symbols which form the basis of logistics (symbolic logic). (Carnap 1928, 21)

Such properties are thus determined by means of the notion of logical definability: a property of a relation is formal just in case it is definable in a pure type-theoretic language.

Given Carnap's suggestions on how to explicate the notion of structural properties, two further remarks should be made here. First, Carnap was clearly one of the first philosophers to reflect on a general *duality* between two conceptually distinct ways to specify the structural content of mathematics. This is the use of invariance criteria on the one hand and the method of logical definability on the other hand. This duality has also been discussed in more recent work on logic and model theory. Compare, for instance, Hodges's characterization:

> In a sense, structure is whatever is preserved by automorphisms. One consequence . . . is that a model-theoretic structure implicitly carries with it all the features which are set-theoretically definable in terms of it, since these features are preserved under all automorphisms of the structure. There is a rival model-theoretic slogan: structure is whatever is definable. Surprisingly, this slogan points in the same direction as the previous one. (Hodges 1997, 93)

The general observation stated here is clearly in line with Carnap's two attempts to specify the structural properties of mathematical relations.[51]

Second, Carnap's approach to structural properties is also closely related to recent work on mathematical structuralism. In particular, one can find both ways to think about such properties, namely in terms of isomorphism invariance and logical definability, also in the present literature. For instance, a definability-based account of structural properties of positions in abstract structures is discussed in detail in work on non-eliminative structuralism, e.g., in Keränen (2001) and Shapiro (2008). An invariance-based account of structural properties of such positions is presented in Linnebo and Pettigrew's (2014) work on structural abstraction.[52]

Moreover, the notion also plays a crucial role in several of the systematic debates in these fields, for instance, on identity criteria for positions in abstract structures (e.g., in work by Shapiro, Keränen, and Leitgeb). The central bone

[51] An important difference from Hodges's account concerns the logical framework in which structural properties are specified. Whereas Hodges's book is about first-order model theory, Carnap's focus is on the definability of properties in a higher-order language of logical type theory.

[52] See Korbmacher and Schiemer (2018) for a more systematic comparison of the two definitions of structural properties.

of contention here concerns the question whether structurally indiscernible positions in a given structure—that is, positions that share the same structural properties—should be identified. A related Leibnizian principle of structural indiscernibility is usually formulated as follows:

For all structural properties P and all objects a, b in the domain of a structure S:

$$(P(a) \Leftrightarrow P(b)) \Leftrightarrow a =_S b$$

The objects (conceived of as pure positions) in a structure are thus identified if there exists no structural property that allows one to discriminate between them. In this case, the objects can be said to play the same role in a given structure.[53]

Interestingly, Carnap developed a similar account of structuralist identity conditions in his work from the late 1920s. In the *Aufbau*, he first states the idea of a purely "structural description" of an object in a domain in terms of its formal properties. His example of the graphical representation of the European-Asian railway network is used as an illustration of how one can, in principle, discriminate between different objects (that is, train stations) by considering only such properties (Carnap 1928, 17–19).[54] Carnap adds that in the hypothetical case that two objects share exactly the same formal properties, they have to be "treated as identical in the strict sense" of the term (Carnap 1928, 19). A similar account of structural identity is expressed in his work on general axiomatics. The notes of the fragmented second part of the *Untersuchungen* contain a section titled "Reduction of the Primitive Concepts" (RC 081-01-12). Here Carnap addresses the question which objects of a given relation are identifiable purely in terms of the relation. He holds that "an R-element x is describable through R if there exists a formal property with respect to R that only applies to x and to no other R-element" (RC 081-01-12/1). Carnap's specification of this approach is based on the further distinction between two properties of pairs of elements of a relation, which he calls "homotopical" and "heterotopical." Roughly put, two objects x, x' are homotopical with respect to a relation R if there exists an automorphism $f : R \cong R$ such that $f(x) = x'$. Objects that are not homotopical with any other object in R are called heterotopical R-elements.[55]

[53] See Keränen (2001) and Shapiro (2008). Compare also Leitgeb and Ladyman (2008) for a critical discussion of such a "structuralist" identity criterion.

[54] In his concrete example, these are graph-theoretic properties of the nodes in the unlabeled graph representing the structure of the railway system.

[55] According to Carnap, systems consisting only of heterotopical objects are called heterotopical systems. This concept corresponds closely to the modern notion of rigid systems, i.e., systems like the natural number systems whose automorphism class contains only the trivial automorphism. Compare Leitgeb and Ladyman (2008).

The relevant result stated in Carnap's *Untersuchungen* is that it is precisely the heterotopical objects in a relation (or a model) that can be identified in terms of formal properties. In contrast, in models consisting only of pairwise homotopical objects, such a discrimination of individuals is not possible given that there are often infinitely many non-trivial automorphisms of the model. Carnap's observation is obviously connected to the modern debate on the principle of identity of structurally indiscernible objects. In particular, it has been pointed out by Keränen (among others) that adopting such a principle will force structuralists to identify objects in nonrigid structures that can be mapped to each other by nontrivial automorphisms.

The second point to be mentioned here concerns the notion of the structural identity of relations or relational systems. In *Untersuchungen*, structural properties are defined for models of a given axiomatic theory in terms of the notion of isomorphisms. As Carnap points out in the passage cited earlier, such properties present the "invariants" under isomorphic transformations. A point not discussed in his 1928 manuscript, but briefly addressed in the *Abriss*, is whether one can formulate a structural property for a given relation (or a system of relations) that allows one to discriminate it from all other non-isomorphic relations.

This directly relates to the question whether, for a given system, one can identify a complete invariant or, in Carnap's terminology, a complete "structure characteristic" of it. Put in modern terms, an invariant is simply a function f that assigns the same value to isomorphic systems, that is, for any two systems R, S, one has $f(R) = f(S) \Leftrightarrow R \cong S$. An invariant for a given type of systems is complete if it also allows one to discriminate between any two non-isomorphic systems.[56] Compare Carnap's characterization of such complete invariants in his *Abriss*:

> The task of presenting a "structure characteristic" . . . is to present a procedure by which one can assign a formula expression (for instance one consisting of numbers) to the given relations . . . in a way that two relations are assigned the same characteristic if and only if they are isomorphic. (Carnap 1929, 55)

Carnap made a rough suggestion in *Abriss* on how to formulate such a complete invariant for finite relations based on their graph-theoretical representations and the corresponding adjacency matrices. Unfortunately, he did not further develop the ideas sketched there (see Carnap 1929, §22e). The relevant point for us to

[56] Notice that the operators in the structural abstraction principle (SA) discussed in the previous section present complete invariants in this sense.

note is that Carnap's work already contains several of the key ideas—particularly, on the identity criteria for objects and systems—that are prominently discussed in contemporary debates on structuralism.

5. Conclusion

This chapter surveyed Carnap's contributions to a structuralist account of mathematics from the 1920s and early 1930s. As several other chapters in the present volume show, his early structuralism was by no means an isolated position but shared by several other philosophers working at the time. Carnap's contemporaries Ernst Cassirer, Bertrand Russell, and also Edmund Husserl can be mentioned in this respect. Characteristic of their respective work is the fact that it is based on a close philosophical reflection of several methodological developments in 19th- and early 20th-century mathematics.

As we saw, this also holds of Carnap's pre-*Syntax* philosophy of mathematics. In his contributions from the period in question, one can identify three ways to characterize the thesis that mathematical theories are about abstract structures. The first method concerns axiomatic definitions which, according to Carnap, can be both understood as implicit definitions of the primitive terms of a given theory as well as explicit definitions of its class of models. The second method is based on the notion of logical constructions, specified by him in terms of explicit definitions in a logical type theory. Finally, Carnap's work on general axiomatics depends crucially on the notion of model structures, characterized as isomorphism classes of models, specifiable in terms of definitions by abstraction.

The study of Carnap's logical analysis of these different approaches allowed us to highlight several aspects of his early structuralism. First, Carnap took the different ways to characterize structures to be essentially equivalent. In particular, it is clear from the discussion given in *Der Raum* and in later writings that he understood Hilbert's axiomatic approach and Russell's genetic approach as two alternative ways to characterize the structural content of a theory. Second, it was shown that there are close connections in Carnap's work between a structuralist account of mathematics and his understanding of the logicist thesis. More specifically, his proposal to treat the content of mathematical theories in terms of explicit concepts has direct ramifications for a *generalized* logicism: it allows one to treat also non-arithmetical theories as reducible to logic and directly motivates an "if-thenist" reconstruction of mathematical theorems.

Finally, I presented two points of contact between Carnap's early philosophy of mathematics and recent debates on structuralism. The first concerns the role of structural abstraction principles in the formulation of versions of *in re* structuralism. Carnap, closely following Russell in this respect, proposed to think of structures of relational systems in terms of equivalence classes. Alternative ways to treat the operators in structural abstraction principles have been developed in work by Linnebo, Pettigrew, Reck, and Leach-Krouse. The second point of contact with modern work concerns Carnap's suggestions on how to explicate the notion of structural properties, namely in terms of the notions of definability and invariance. As we saw, this proposal connects his early contributions to structuralism with debates on adequate structuralist identity conditions for positions in mathematical structures.

The focus of this chapter was on Carnap's early contributions to the philosophy of mathematics. It would be interesting to give a comparison between the structuralist thesis concerning mathematical knowledge developed there and Carnap's more general scientific structuralism in his later work on the logic of science. Specifically, the present literature on Carnap still lacks a closer analysis of how his early contributions to general axiomatics are related to his mature work on logical theory reconstruction, for instance, on the *ramsification* of theories. A comparative study of Carnap's structuralist ideas from different periods of his intellectual career will have to be developed elsewhere.

Acknowledgments

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No. 715222). Earlier versions of this chapter were presented at the conference "Early Carnap in Historical Context" at the University of Konstanz in 2017 as well as at the Sixth Annual Conference of the Society for the Study of the History of Analytical Philosophy at the University of Calgary in 2017. I would like to thank the respective audiences for their comments as well as Francesca Biagioli, John Wigglesworth, Christian Damböck, André Carus, Andrea Sereni, Christoph Limbeck-Lillienau, Hannes Leitgeb, Erich Reck, and Steve Awodey for helpful discussions on the topics addressed in this chapter.

References

Awodey, S. 2007. Carnap's Quest for Analyticity: The Studies in Semantics. In *The Cambridge Companion to Carnap*, edited by M. Friedman and R. Creath, pp. 226–247. Cambridge: Cambridge University Press.

Awodey, S. 2017. Carnap and the Invariance of Logical Truth. *Synthese*, special issue: Carnap on Logic (194), 67–78.

Awodey, S., and A. Carus. 2001. Carnap, Completeness, and Categoricity: The *Gabelbarkeitssatz* of 1928. *Erkenntnis* 54, 145–172.

Awodey, S., and A. Carus. 2007. Carnap's Dream: Godel, Wittgenstein, and *Logical Syntax*. *Synthese* 159(1), 23–45.

Awodey, S., and E. Reck. 2002. Completeness and Categoricity, Part 1: 19th Century Axiomatics to 20th Century Metalogic. *History and Philosophy of Logic* 23, 1–30.

Awodey, S., and E. Reck. 2004. *Frege's Lectures on Logic: Carnap's Student Notes 1910–1914*. Chicago: Open Court.

Benacerraf, P. 1965. What Numbers Could Not Be. *Philosophical Review* 74(1), 47–73.

Bernays, P. 1967. Hilbert, David. In *Encyclopedia of Philosophy*, edited by P. Edwards, vol. 3, pp. 496–405. New York: Macmillan.

Bohnert, H. G. 1975. Carnap's Logicism. In *Rudolf Carnap, Logical Empiricist*, edited by J. Hintikka, pp. 183–216. Dordrecht: Reidel.

Carnap, R. 1922. *Der Raum: Ein Beitrag zur Wissenschaftslehre*. Vol. 56 of *Kant-Studien Ergänzungshefte*. Berlin: Reuther & Reichard. English translation in Carnap (2019), 22–208

Carnap, R. 1927. Eigentliche und uneigentliche Begriffe. *Symposion* 1, 355–374.

Carnap, R. 1928. *Der logische Aufbau der Welt*. Springer: Berlin.

Carnap, R. 1929. *Abriss der Logistik*. Vienna: Springer.

Carnap, R. 1930. Die alte und die neue Logik. *Erkenntnis* 1, 12–26.

Carnap, R. 1931. Die logizistische Grundlegung der Mathematik. *Erkenntnis* 2, 91–105.

Carnap, R. 1934. *Logische Syntax der Sprache*. Vienna: Springer. translated as *The Logical Syntax of Language*. London: Open Court, 2002.

Carnap, R. 2000. *Untersuchungen zur allgemeinen Axiomatik*. Darmstadt: Wissenschaftliche Buchgesellschaft.

Carnap, R. 2019. *The Collected Works of Rudolf Carnap*. Vol. 1, *Early Writings, 1922–1927*. Edited by A. W. Carus et al. Oxford: Oxford University Press.

Carnap, R., and F. Bachmann. 1936. Uber Extremalaxiome. *Erkenntnis* 6, 166–188.

Carnap, R., and F. Bachmann. 1981. On Extremal Axioms. *History and Philosophy of Logic* 2, 67–85. Translation of Carnap and Bachmann 1936.

Carus, A. 2007. *Carnap and Twentieth-Century Thought*. Cambridge: Cambridge University Press.

Dedekind, R. 1888. *Was sind und was sollen die Zahlen?* Braunschweig: Vieweg.

Friedman, M. 1999. *Reconsidering Logical Positivism*. Cambridge: Cambridge University Press.

Gandon, S. 2009. Toward a Topic-Specific Logicism? Russell's Theory of Geometry in the *Principles of Mathematics*. *Philosophia Mathematica* 17(1), 35–72.

Gandon, S. 2012. *Russell's Unknown Logicism: A Study in the History and Philosophy of Mathematics*. New York: Palgrave Macmillan.

Goldfarb, W., and T. Ricketts. 1992. Carnap and the Philosophy of Mathematics. In *Science and Subjectivity. The Vienna Circle and 20th Century Philosophy*, edited by D. Bell and W. Vossenkuhl, pp. 61–78. Berlin: Akademieverlag.

Grattan-Guinness, I. 2000. *The Search for Mathematical Roots, 1870–1940: Logics, Set Theories and the Foundations of Mathematics from Cantor through Russell to Gödel*. Princeton, NJ: Princeton University Press.

Gray, J. 2008. *Plato's Ghost: The Modernist Transformation of Mathematics*. Princeton, NJ: Princeton University Press.

Hellman, G. 1989. *Mathematics without Numbers: Towards a Modal-Structural Interpretation*. Oxford: Oxford University Press.

Hilbert, D. 1899. *Grundlagen der Geometrie*. Leipzig: Teubner.

Hintikka, J. 2011. What Is the Axiomatic Method? *Synthese* 183(1), 69–85.

Hodges, W. 1997. *A Shorter Model Theory*. Cambridge: Cambridge University Press.

Keränen, J. 2001. The Identity Problem for Realist Structuralism. *Philosophia Mathematica* 9(3), 308–330.

Korbmacher, J., and G. Schiemer. 2018. What Are Structural Properties? *Philosophia Mathematica* 3(26), 295–323.

Leach-Krouse, G. 2017. Structural-Abstraction Principles. *Philosophia Mathematica* 25(1), 45–72.

Leitgeb, H., and L. Ladyman. 2008. Criteria of Identity and Structuralist Ontology. *Philosophia Mathematica* 16(3), 388–396.

Linnebo, O. 2018. *Thin Objects: An Abstractionist Account*. Oxford: Oxford University Press.

Linnebo, O., and R. Pettigrew. 2014. Two Types of Abstraction for Structuralism. *Philosophical Quarterly* 64(255), 267–283.

Mancosu, P. 2016. *Abstraction and Infinity*. Oxford: Oxford University Press.

Mormann, T. 2007. Geometrical Leitmotifs in Carnap's Early Philosophy. In *Cambridge Companion to Carnap*, edited by R. Creath and M. Friedman, pp. 57–75. Cambridge: Cambridge University Press.

Reck, E. 2004. From Frege and Russell to Carnap: Logic and Logicism in the 1920s. In *Carnap Brought Home: The View from Jena*, edited by S. Awodey and C. Klein, pp. 151–180. Chicago: Open Court.

Reck, E. 2007. Carnap and Modern Logic. In *The Cambridge Companion to Carnap*, edited by M. Friedman and R. Creath, pp. 176–199. Cambridge: Cambridge University Press.

Reck, E. 2018. On Reconstructing Dedekind Abstraction Logically. In *Logic, Philosophy of Mathematics, and Their History: Essays in Honor of W.W. Tait*, edited by E. Reck, pp. 113–137. London: College Publications.

Reck, E., and M. Price. 2000. Structures and Structuralism in Contemporary Philosophy of Mathematics. *Synthese* 125(3), 341–383.

Russell, B. 1903. *The Principles of Mathematics*. Cambridge: Cambridge University Press.

Russell, B. 1919. *Introduction to Mathematical Philosophy*. London: George Allen & Unwin.

Russell, B., and A. N. Whitehead. 1962. *Principia Mathematica*. 3 vols. Cambridge: Cambridge University Press.

Schiemer, G. 2012a. Carnap on Extremal Axioms, "Completeness of the Models," and Categoricity. *Review of Symbolic Logic* 5(4), 613–641.

Schiemer, G. 2012b. Carnap's *Untersuchungen*: Logicism, Formal Axiomatics, and Metatheory. In *Rudolf Carnap and the Legacy of Logical Empiricism*, edited by R. Creath. Berlin: Springer, 13–36.

Schiemer, G. 2013. Carnap's Early Semantics. *Erkenntnis* 78(3), 487–522.

Schiemer, G., R. Zach, and E. Reck. 2017. Carnap's Early Metatheory: Scope and Limits. *Erkenntnis* 194(1), 33–65.

Schlimm, D. 2013. Axioms in Mathematical Practice. *Philosophia Mathematica* 21(1), 37–92.

Shapiro, S. 1997. *Philosophy of Mathematics: Structure and Ontology*. New York: Oxford University Press.

Shapiro, S. 2008. Identity, Indiscernibility, and Ante Rem Structuralism: The Tale of i and –i. *Philosophia Mathematica* 3(16), 285–309.

Stein, H. 1988. *Logos*, Logic, and *Logistiké*: Some Philosophical Remarks on Nineteenth-Century Transformation of Mathematics. In *History and Philosophy of Modern Mathematics*, volume 11, edited by W. Aspray and P. Kitcher, pp. 238–259. Minneapolis: University of Minnesota Press.
Torretti, R. 1978. *Philosophy of Geometry from Riemann to Poincaré*. Boston: Reidel.
Wagner, P., ed. 2009. *Carnap's Logical Syntax of Language*. New York: Palgrave Macmillan.