 RNLKWC/B.Sc./CBCS/IVS/MTMHC402/22

2022

Mathematics

[Honours]

(B.Sc. Fourth Semester End Examination-2022)

PAPER-MTMH C402 (Ring Theory - I)

Full Marks: 60 ***Time: 03 Hrs***

The figures in the right hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

Illustrate the answers wherever necessary

Group-A

1. Answer any ten questions from the following: 10x2= 20

a) Find all the maximal and prime ideals of the ring $\mathbb{Z}_{12}$.

b) Find all the subring of $\mathbb{Z}_{15}$.

c) Obtain the units in the integral domain $\mathbb{Z}\left[\sqrt{-3}\right]$.

d) Show that the ring $\mathbb{Z}$ .and $2\mathbb{Z}$ .are not isomorphic.

e) Find all the irreducible polynomials of degree 3 in $\mathbb{Z}_2\left[x\right]$.

f) Find all the units and divisor of zero in the ring $\left(\mathbb{Z}_{40},+,\bullet\right)$.

g) R is a commutative ring of prime characteristic p. Show that $\left(a+b\right)^p = a^p + b^p$ for all $a,b \in R$

h) Show that the smallest non commutative ring is of order 4.

i) Show that x^2+x+4 is irreducible over $\mathbb{Z}_{11}$

j) Show that the set $S\{0,3,6,9,12\}$ is a subring of the ring $\mathbb{Z}_{15}$, Is it a field?

k) Is $\mathbb{Z}_6$ a subring of $\mathbb{Z}_{12}$

l) Let, R be finite non-zero commutative ring with unity. Prove that any non-zero element of R is either a unit or a zero divisor.

m) Test the reducibility of the polynomial $x^3-312312x+123123$

n) Let, $f(x)=x^7-105x+12$. Then show that there does not exist on integer m such that $f(m)=105$

o) Show that the element $3+\sqrt{-5}$ is irreducible in the ring $\mathbb{Z}\left[\sqrt{-5}\right]$

Group-B

2. Answer any four questions from the following: 4x5 = 20

a) If F be a field, then prove that $F[x]$ is a PID

b) Obtain all the nontrivial ring homomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{28}$

c) If p is an odd prime and if $1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{p-1}=\frac{a}{b}$, where a, b are positive integers, prove that a is divisible by p.

d) Show that the domain $\mathbb{Z}\left[\sqrt{-5}\right]$ is not UFD by showing that the element 21 has two different factorisation.

e) Show that the quotient ring

$$\frac{\mathbb{Z}_2[x]}{\langle x^2+x+1\rangle}$$ is a field of 4 elements.

f) If D_1 and D_2 be two isomorphic integral, domains then show that their respective fields of quotients F_1 and F_2 are also isomorphic.

Group -C

3. Answer any two questions: 10x2 = 20

a) i) Let, addition $\oplus$ and multiplication $\odot$ be defined on the ring $(\mathbb{Z},\oplus,\odot)$ by $a\oplus b=a+b-1, a\odot b=a+b-a.b$ for $a,b\in\mathbb{Z}$. Prove that $(\mathbb{Z},\oplus,\odot)$ is a ring with unity. 6

ii) In a cumulative ring with unity, an ideal P is a prime ideal if and only if the quotient ring R/P is an integral domain. 4

b) i) Let, $C[0,1]$ be the ring of all real valued continuous function on $[0,1]$ Suppose $A=\left\{f\in C[0,1]: f\left(\frac{1}{4}\right)=0, f(3/4)=0\right\}$.

Then show that A is an ideal in $C[0,1]$ but is not a prime ideal in $C[0,1]$.

ii) Show that the polynomial x^3+2x+1 is irreducible in $\mathbb{Z}_3[x]$ and use it to construct a field with 27 elements. Find the inverse of x^2+1 with that field $\left(\text{where } I=\langle x^3+2x+1\rangle\right)$

c) i) If d be a $g.c.d$ of three elements a, b, c in a principal ideal domain D, show that d be expressed as $d = au + bu + cw$ for u, v, w in D.

ii) Let D be Euclidean domain with a Euclidean valuation v. If b is unit in D, Prove that $v(ab) = v(a)$ for all non-zero $a \in D$